Reversed Borel-Cantelli lemma

Theorem 8.

Conclusion: If the A_n's are $\perp\!\!\!\perp$, then

$\mathbf{P}(\limsup A_n) \in \{0,1\}$

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We assume

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty, \quad \text{and} \quad A_n\text{'s independent}$$

Then we have

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 1$$

Proof

$$A \equiv \limsup A_n = \bigcap_{n=1}^{\infty} \overbrace{\bigcup_{k \geq n} A_k}^{B_n \to \lim \mathbb{P}(B_n)}$$

Rmk If the A_k's are $\perp\!\!\!\perp$, it is easier to compute $\mathbb{P}(\cap A_k^c)$ rather than $\mathbb{P}(\cup A_k)$

$\Rightarrow$ look at A^c instead of A, where

$$\boxed{A^c = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k^c}$$

$A^c = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k^c$. Define

$D_n = \bigcap_{k \geq n} A_k^c$. Then $n \mapsto D_n$ is $\nearrow$

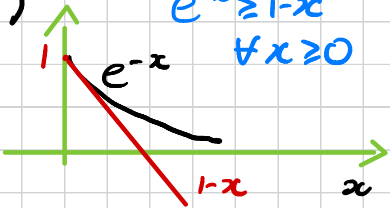
Thus

$$\boxed{\mathbb{P}(A^c)} = \mathbb{P}\left(\bigcup_{n=1}^{\infty} D_n\right) = \boxed{\lim_{n \to \infty} \mathbb{P}(D_n)}$$

Recall: $D_n = \bigcap_{k \geq n} A_k^c$. Thus

$$\mathbb{P}(D_n) = \mathbb{P}\Big(\bigcap_{k \geq n} A_k^c\Big)$$
$$\overset{\perp\!\!\!\perp}{=} \prod_{k=n}^{\infty} \mathbb{P}(A_k^c)$$
$$= \prod_{k=n}^{\infty} \big(1 - \mathbb{P}(A_k)\big)$$
$$\leq \prod_{k=n}^{\infty} e^{-\mathbb{P}(A_k)}$$
$$= \exp\Big(-\sum_{k=n}^{\infty} \mathbb{P}(A_k)\Big)$$
$$= 0 \quad \forall n$$

$e^{-x} \geq 1-x$ $\forall x \geq 0$



Hyp $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$

$\mathbb{P}(\limsup A_n) = 1$

Thus $\boxed{\mathbb{P}(A^c) = \lim_{n \to \infty} \mathbb{P}(D_n) = 0}$

$\boxed{\text{Digression on Markov chains}}$

Def A sequence $\{X_n ; n \geq 0\}$ with values in $\mathbb{Z}$ is a Markov chain if

$$\mathbb{P}(X_n = j \mid X_{n-1} = i, \ldots, X_1 = i_1, X_0 = i_0)$$

$$= \mathbb{P}(X_n = j \mid X_{n-1} = i) \equiv p(n, i, j)$$

If X_n is homogeneous,

$$\boxed{\mathbb{P}(X_n = j \mid X_{n-1} = i) = p(i, j)}$$

Basic question

$\lim_{n\to\infty} X_n$? If (X_n) is "stable" enough, then $X_n \xrightarrow{(d)} \pi$, where π is a probability on $\mathbb{Z}$.

This π is an invariant measure:

$$\boxed{\text{if } X_n \sim \pi, \text{ then } X_{n+1} \sim \pi}$$

Crucial step to know if X is stable enough: do we have "recurrent states"?

Def $i \in \mathbb{Z}$ is recurrent if

$$\mathbb{P}(X_n \text{ visits } i \text{ i.o } \mid X_0 = i) = 1$$

$$\Leftrightarrow \mathbb{P}(\limsup A_n \mid X_0 = i) = 1$$

with $A_n = (X_n = i)$

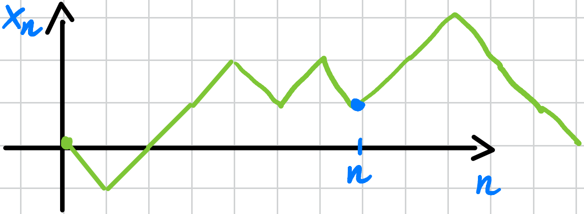
$z_i = 2Y_i - 1$ with $Y_i \sim B(\frac{1}{2})$

Example : Take z_i iid with

$$\mathbb{P}(z_i = -1) = \mathbb{P}(z_i = 1) = \frac{1}{2}$$

Set $X_n = \sum_{k=1}^{n} z_i$, with $X_0 = 0$

X_n is called random walk in $\mathbb{Z}$



$\mathbb{P}(A_{2n})$

$\|$ (*)

Here $A_{2n} = \{X_{2n} = 0\} \Rightarrow \mathbb{P}(Bin(2n, \frac{1}{2}) = n)$

Proof of (*). Since $z_i = 2Y_i - 1$, we have

$$X_n = \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} (2Y_i - 1)$$

$$= 2\sum_{i=1}^{n} Y_i - n$$

Thus

$$\boxed{\mathbb{P}(X_{2n} = 0)} = \mathbb{P}\left(2\sum_{i=1}^{2n} Y_i - 2n = 0\right)$$

$$= \mathbb{P}\left(\sum_{i=1}^{2n} Y_i = n\right)$$

$$= \boxed{\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}}$$

Criteria to know if i is recurrent:
we have seen

$$\mathbb{P}(\limsup A_n \mid X_0 = i) = 1$$

potential of the Markov Chain

$$\Leftrightarrow \sum_{n=0}^{\infty} p^n(i,i) = \infty \, ,$$

where $p(i,j) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$
is seen as a matrix and p^n
is the n-th power.

Not reversed B-C, because A_n's are not $\perp\!\!\!\perp$

Here we can only apply Borel-Cantelli, which gives

$$\left(\sum p^n(i,i) < \infty \Leftrightarrow \sum \mathbb{P}(A_n) < \infty\right)$$

$$\Rightarrow \mathbb{P}(\limsup A_n) = 0$$

$\Rightarrow$ we never visit i i.o

$\Rightarrow$ $\boxed{i \text{ is said to be transient}}$

For random walks: 0 is transient if X_n is a rw on $\mathbb{Z}^d$ and

$$\boxed{d \geq 3}$$

But we don't get a recurrence statement for $d = 1, 2$

↳ Those can be obtained using generating functions

$\boxed{\text{Filtrations}}$ Consider a sequence $\{X_n ; n \geq 1\}$. It is convenient to define a filtration $\{\mathcal{F}_n ; n \geq 1\}$, where $\mathcal{F}_n$ is the σ-algebra

$$\boxed{\mathcal{F}_n = \sigma(X_1, \dots, X_n)}$$

= smallest σ-algebra which makes $X_1, \dots, X_n$ measurable

$$= \sigma\left(\{\omega ; X_1(\omega) \in A_1, \dots, X_n(\omega) \in A_n, \text{ with } A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})\}\right)$$

Another point of view:

Y is $\mathcal{F}_n$-measurable ($Y \in \mathcal{F}_n$) iff

$$\boxed{Y = f(X_1, \dots, X_n)}$$

with $f : \mathbb{R}^n \to \mathbb{R}$ measurable

Interpretation: $\mathcal{F}_n$ carries "all the information" contained in $X_1, \dots, X_n$

Relation: $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ increasing sequence of σ-algebras

<u>Next aim</u>: Define a σ-algebra of the future

We set $\mathcal{F}_n' = \sigma(X_k ; k > n)$

This $\mathcal{F}_n'$ is in fact a limit. That is set

finite # of r.v.

$$\mathcal{K}_{n,j} = \sigma(\overbrace{X_{n+1}, \dots, X_{n+j}})$$

Then

not a σ-algebra in general

$$\boxed{\mathcal{F}_n' = \sigma\Big(\overbrace{\bigcup_{j=1}^{\infty} \mathcal{K}_{n,j}}\Big)}$$