Filtration

$$\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$$

$\equiv$ information up to the present

Next aim: Define a σ-algebra of the future

We set $\mathcal{F}_n' = \sigma(X_k ; k > n)$

This $\mathcal{F}_n'$ is in fact a limit. That is set

finite # of r.v.

$$\mathcal{K}_{n,j} = \sigma(\overbrace{X_{n+1}, \dots, X_{n+j}})$$

Then

not a σ-algebra in general

$$\boxed{\mathcal{F}_n' = \sigma\Big(\overbrace{\bigcup_{j=1}^{\infty} \mathcal{K}_{n,j}}\Big)}$$

Side remark.

$\mathcal{F}_1, \mathcal{F}_2$ σ-algebras $\not\Rightarrow$ $\mathcal{F}_1 \cup \mathcal{F}_2$ σ-algebra

Counter example. Take $\Omega = \{a, b, c\}$ and

$$\mathcal{F}_1 = \{\emptyset, \{a\}, \{b,c\}, \Omega\} = \sigma(\{a\})$$

$$\mathcal{F}_2 = \{\emptyset, \{b\}, \{a,c\}, \Omega\}$$

Then $\cap = \{c\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$

$$\mathcal{F}_1 \cup \mathcal{F}_2 = \{\emptyset, \{a\}, \{b\}, \{a,c\}, \{b,c\}, \Omega\}$$

Tail σ-field

Definition 9.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables
- $\mathcal{F}'_n = \sigma\left(X_k;\, k \geq n\right)$

We set

$$\mathcal{T} = \bigcap_{n \geq 1} \mathcal{F}'_n$$

The σ-field $\mathcal{T}$ is called Tail σ-field

Interpretation: We have

$A \in \mathcal{T}$ if changing a finite number of X_n's
does not change the occurence of A.

Examples of events in $\mathcal{T}$

General setting: We consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables
- $S_n = \sum_{k=1}^{n} X_i$

$= \limsup A_n$, with $A_n = (X_n > 0)$

Then we have

1. $(X_n > 0 \text{ i.o}) \in \mathcal{T}$
2. $(\lim_{n\to\infty} S_n \text{ exists}) \in \mathcal{T}$
3. $(\limsup_{n\to\infty} X_n > 0) \in \mathcal{T}$
4. $(\limsup_{n\to\infty} S_n > 0) \notin \mathcal{T}$
5. $(\limsup_{n\to\infty} \frac{1}{a_n} S_n > 0) \in \mathcal{T}$ if $\lim_{n\to\infty} a_n = \infty$

Claim : $\boxed{\text{Set } A = (\limsup X_n > 0) . \text{ Then } A \in \mathscr{G}}$

Proof

$$A = \bigcup_{m \geq 1} (\limsup X_n \geq \tfrac{1}{m} |$$

$$= \bigcup_{m \geq 1} \underbrace{\bigcap_{n \geq 1} \underbrace{\bigcup_{k > n} (X_k \geq \tfrac{1}{m})}_{\in \mathcal{F}'_n}}_{\in \mathscr{G}}$$

Claim : Then $\boxed{\text{Set } B = (\limsup S_n > 0) . \\ B \notin \mathcal{T}}$

Setting : $X_1 \sim \mathcal{B}(\tfrac{1}{2})$, then $X_n = 0$ for $n \geq 2$

Then if $S_n = \sum_{k=1}^{n} X_k$ $(= X_1)$,

$(\limsup S_n > 0) = (X_1 > 0) \in \mathcal{F}_1$

$\Downarrow$

$\notin \mathcal{T}$

Note : For $\frac{S_n}{n}$, $(\limsup \frac{S_n}{n} > 0) \in \mathcal{T}$

Kolmogorov's 0-1 law

Theorem 10.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of independent random variables
- The tail σ-field $\mathcal{T}$

Rmk : Reversed B-C is a special case, with $A = \limsup A_n$

Then $\mathcal{T}$ is trivial, that is:

1. If $A \in \mathcal{T}$ we have

$$\mathbf{P}(A) \in \{0, 1\}$$

2. If $Y \in \mathcal{T}$, there exists $k \in [-\infty, \infty]$ such that

$$\mathbf{P}(Y = k) = 1$$

Strategy: Based on the $\perp\!\!\!\perp$ of the sequence X_n.

Claim: If $A \in \mathscr{C}$, then

$$A \perp\!\!\!\perp A$$

Then

$$\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)^2$$

$$\Rightarrow \mathbb{P}(A) = (\mathbb{P}(A))^2$$

$$\Rightarrow \mathbb{P}(A) \in \{0, 1\}$$

Step 1: Prove that $\mathcal{F}_k \perp\!\!\!\perp \mathcal{F}_k'$,

that is if $A \in \mathcal{F}_k$, $B \in \mathcal{F}_k'$, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$

Easy preliminary step.

If $A \in \mathcal{F}_k = \sigma(X_1, \dots, X_k)$

$B \in \mathcal{K}_{k,j} = \sigma(X_{k+1}, \dots, X_{k+j})$

then since X_n's are $\perp\!\!\!\perp$ we have

$$\mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A \cap B)$$

$A \in \mathcal{F}_k = \sigma(X_1, \dots, X_k)$

$\Rightarrow A \perp\!\!\!\perp B$

$B \in \mathcal{T}_{k,j} = \sigma(X_{k+1}, \dots, X_{k+j})$

limits?

We have $A \perp\!\!\!\perp B$ if $B \in \bigcup_{j \geq 1} \mathcal{T}_{k,j}$

Recall

π-system $\mathcal{P}$: If $A, B \in \mathcal{P}$, $A \cap B \in \mathcal{P}$

λ-system $\mathcal{L}$: $\Omega \in \mathcal{L}$, $A \in \mathcal{L} \Rightarrow A^c \in \mathcal{L}$

$A_i \in \mathcal{L}$, If $A_i \cap A_j = \emptyset$ when $i \neq j$ then

$$\bigcup_{i \geq 1} A_i \in \mathcal{L}$$

$A \in \mathcal{F}_k = \sigma(X_1, \dots, X_k)$

$B \in \mathcal{K}_{k,j} = \sigma(X_{k+1}, \dots, X_{k+j})$ $\Rightarrow A \perp\!\!\!\perp B$

limits?

<u>We have</u> $A \perp\!\!\!\perp B$ if $B \in \bigcup_{j \geq 1} \mathcal{K}_{k,j}$

<u>Take</u> $\boxed{\mathcal{P} = \bigcup_{j \geq 1} \mathcal{K}_{k,j}}$.

If $A_1 \in \mathcal{K}_{k,j_1}$, $A_2 \in \mathcal{K}_{k,j_2}$, $j_2 > j_1$

then

$\in \mathcal{K}_{k,j_2}$

$A_1 \cap A_2 \in \mathcal{K}_{k,j_2}$

Thus $\mathcal{P}$ stable by $\cap$

$\Rightarrow$ $\boxed{\pi\text{-system}}$

$A \in \mathcal{F}_k = \sigma(X_1, \dots, X_k)$
$B \in \mathcal{K}_{k,j} = \sigma(X_{k+1}, \dots, X_{k+j})$ $\Rightarrow A \perp\!\!\!\perp B$ (*)

limits?

We have $A \perp\!\!\!\perp B$ if $B \in \bigcup_{j \geq 1} \mathcal{K}_{k,j}$

$\mathcal{L}$ - system : Take $A \in \mathcal{F}_k$ and

$$\mathcal{L} = \{ B \, ; \, \mathbb{P}(A \cap B) = \mathbb{P}(A)\, \mathbb{P}(B) \}$$

↳ check that this is a $\mathcal{L}$- system

and (*) $\Rightarrow$ any $B \in \bigcup_{j \geq 1} \mathcal{K}_{k,j}$ is in $\mathcal{L}$

Dynkin $\Rightarrow$ $\mathcal{L} \supset \sigma\left(\bigcup_{j \geq 1} \mathcal{K}_{k,j} \right) = \mathcal{F}'_k$

Partial conclusion: $\boxed{\mathcal{F}_k \perp\!\!\!\perp \mathcal{F}'_k}$

Claim 2 : If $A \in \sigma(X_1, \ldots,)$, $B \in \mathcal{C}$ $\Rightarrow A \perp\!\!\!\perp B$

If this is true then take $A \in \mathcal{C}$. We have

$$A \in \sigma(X_1, \ldots,) \quad , \quad A \in \mathcal{C}$$

$$\Rightarrow \quad A \perp\!\!\!\perp A$$

$$\Rightarrow \quad \boxed{\mathbb{P}(A) \in \{0,1\}}$$

Easy to prove: If $A \in \mathcal{F}_k$

$B \in \mathcal{T} = \bigcap_{n \geq 1} \mathcal{F}'_n$,

in particular, $B \in \mathcal{F}'_k$ $\overset{\mathcal{F}_k \perp\!\!\!\perp \mathcal{F}'_k}{\Longrightarrow}$ $A \perp\!\!\!\perp B$

We obtain: $A \perp\!\!\!\perp B$ if

$A \in \bigcup_{k \geq 1} \mathcal{F}_k$ $\qquad B \in \mathcal{T}$

If we want to have $A \perp\!\!\!\perp B$ with

$A \in \sigma(X_1, X_2, \dots)$, $B \in \mathcal{T}$

we use $\boxed{\pi\text{-systems and } \lambda\text{-systems}}$

Recalling π-systems and λ-systems

π-system: Let $\mathcal{P}$ family of subsets of Ω. $\mathcal{P}$ is a π-system if:

$$A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}$$

λ-system: Let $\mathcal{L}$ family of subsets of Ω. $\mathcal{L}$ is a λ-system if:

1. $\Omega \in \mathcal{L}$
2. If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$
3. If for $j \geq 1$ we have:
   - $A_j \in \mathcal{L}$
   - $A_j \cap A_i = \emptyset$ if $j \neq i$

   Then $\cup_{j\geq 1} A_j \in \mathcal{L}$

Recalling Dynkin's π-λ lemma

Proposition 11.

Let $\mathcal{P}$ et $\mathcal{L}$ such that:

- $\mathcal{P}$ is a π-system
- $\mathcal{L}$ is a λ-system
- $\mathcal{P} \subset \mathcal{L}$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$

Proof of Theorem 10 (1)

Strategy: For $A \in \mathcal{T}$,

1. We will prove $A \perp\!\!\!\perp A$
2. If $A \perp\!\!\!\perp A$, then

$$\mathbf{P}(A)^2 = \mathbf{P}(A), \quad \text{thus} \quad \mathbf{P}(A) \in \{0, 1\}$$

Proof of Theorem 10 (2)

Step 1: We will prove that

$$A \in \sigma(X_1, \ldots, X_k),\ B \in \sigma(X_{k+1}, \ldots) \quad \Longrightarrow \quad A \perp\!\!\!\perp B$$

Proof of Theorem 10 (3)

Proof of Step 1: We have

- Let $\mathcal{K}_{k,j} = \sigma(X_{k+1}, \ldots, X_{k+j})$. Then $\cup_{j\geq 0}\mathcal{K}_{k,j}$ is a π-system
- Let $A \in \sigma(X_1, \ldots, X_k)$ and

$$\mathcal{L} = \{B;\ \mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)\}$$

Then $\mathcal{L}$ is a λ-system such that $\mathcal{L} \supset (\cup_{j\geq 0}\mathcal{K}_{k,j})$

Thus

$$\mathcal{L} \supset \sigma\left(\cup_{j\geq 0}\mathcal{K}_{k,j}\right) = \sigma(X_{k+1}, \ldots)$$

Proof of Theorem 10 (4)

Step 2: We will prove that

$$A \in \sigma(X_1, \ldots), \quad \text{and} \quad B \in \mathcal{T} \quad \Longrightarrow \quad A \perp\!\!\!\perp B$$

Conclusion: If $A \in \mathcal{T}$ we have

$$A \in \sigma(X_1, \ldots), \quad \text{and} \quad A \in \mathcal{T}. \quad \text{Thus} \quad A \perp\!\!\!\perp A$$

Proof of Theorem 10 (5)

Proof of Step 2: We have

- Let $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$. Then $\cup_{k\geq 1}\mathcal{F}_k$ is a π-system
- Let $A \in \mathcal{T}$ and

$$\mathcal{L} = \{B;\, \mathbf{P}(A \cap B) = \mathbf{P}(A)\, \mathbf{P}(B)\}$$

Then $\mathcal{L}$ is a λ-system such that $\mathcal{L} \supset (\cup_{k\geq 1}\mathcal{F}_k)$

Thus

$$\mathcal{L} \supset \sigma\left(\cup_{j\geq 0}\mathcal{K}_j\right) = \sigma(X_1, \ldots)$$

Proof that $\mathcal{L} \supset (\cup_{k\geq 1}\mathcal{F}_k)$: If $B \in \mathcal{F}_k$ and $A \in \mathcal{T}$, then

$$A \in \mathcal{K}_{k+1}, \quad \text{and thus} \quad A \perp\!\!\!\perp B$$

Application to law of large numbers

Theorem 12.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of independent random variables
- $S_n = \sum_{i=1}^n X_i$ $= \bar{X}_n$
- $Z_1 = \liminf_{n\to\infty} \frac{1}{n} S_n$ and $Z_2 = \overbrace{\limsup_{n\to\infty} \frac{1}{n} S_n}^{\mathcal{T}\text{- measurable}}$

Then the following holds true:

1. There exists $k_1, k_2 \in [-\infty, \infty]$ such that
$$Z_1 = k_1, \quad \text{and} \quad Z_2 = k_2 \quad \text{a.s}$$
2. If $A \equiv (\lim_{n\to\infty} \frac{1}{n} S_n \text{ exists})$, we have
$$\mathbf{P}(A) \in \{0, 1\}$$

Outline

Statement of the problem

General problem: We consider

- $\{X_n;\ n \geq 1\}$ sequence of random variables
- $S_n = \sum_{i=1}^{n} X_i$

Then we wish to investigate a convergence of the form

$$\frac{S_n}{n} - a_n \longrightarrow S$$

To be specified:

1. Constants a_n, b_n
2. Random variable S
3. Mode of convergence

Reviewing old results

Proposition 13.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- $\mathbf{E}[X_1] = \mu$ and $\mathbf{Var}(X_1) = \sigma^2$
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{(d)} \mu, \quad \text{and} \quad \sqrt{n}\,\frac{(\bar{X}_n - \mu)}{\sigma} \xrightarrow{(d)} \mathcal{N}(0,1)$$

Proof of CLT : with characteristic functions.

LLN can also be obtained using char. functions

Compute

$$\phi_n(t) \equiv \mathbb{E}[e^{it\bar{X}_n}]$$

$$= \mathbb{E}[e^{i\frac{t}{n}\sum_{i=1}^{n} X_i}]$$

$$\overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{n} \mathbb{E}[e^{i\frac{t}{n}X_i}]$$

$$\overset{i.d}{=} (\mathbb{E}[e^{i\frac{t}{n}X_1}])^n$$

$= (\phi(\frac{t}{n}))^n$ with

$\phi(u) = \mathbb{E}[e^{iuX_1}]$

→ Taylor expansion