# Reviewing old results

Interpretation of CLT: $\bar{X}_n \simeq \mu + \frac{\sigma Z}{\sqrt{n}}$

**Proposition 13.**

We consider

- $\{X_n;\ n \geq 1\}$ sequence of i.i.d random variables
- $\mathbf{E}[X_1] = \mu$ and $\mathbf{Var}(X_1) = \sigma^2$
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{(d)} \mu, \quad \text{and} \quad \sqrt{n}\,\frac{(\bar{X}_n - \mu)}{\sigma} \xrightarrow{(d)} \mathcal{N}(0,1)$$

Proof of LLN in (d). Consider

$$\phi(u) = \mathbb{E}[e^{iuX_1}]$$

Then

$$\phi_n(t) \equiv \mathbb{E}[e^{it\bar{X}_n}] = \mathbb{E}[e^{i\frac{t}{n}\sum_{i=1}^{n} X_i}]$$

$$\overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{n} \mathbb{E}[e^{i\frac{t}{n}X_i}] = \left(\phi\left(\tfrac{t}{n}\right)\right)^n$$

Aim: Take limits in $\phi_n(t)$

Then apply convergence of c.f

$\Rightarrow$ convergence in (d)

**Recall:** $\phi_n(t) = \left(\phi\left(\frac{t}{n}\right)\right)^n$

$$= \exp\left(n \ln\left(\phi\left(\frac{t}{n}\right)\right)\right)$$

**Recall:** If $X_1 \in L^1(\Omega)$, then for small $u$, $\mu = \mathbb{E}[X_1]$

$$\phi(u) = 1 + i\mu u + o(u)$$

$$\Rightarrow \ln\left(\phi\left(\frac{t}{n}\right)\right) = i\mu\frac{t}{n} + o\left(\frac{1}{n}\right)$$

$$\Rightarrow n \ln\left(\phi\left(\frac{t}{n}\right)\right) = i\mu t + o(1)$$

$$\Rightarrow \lim_{n\to\infty} \exp\left(n \ln \phi\left(\frac{t}{n}\right)\right) = e^{i\mu t}$$

$\phi_n(t) = \mathbb{E}[e^{iut \bar{X}_n}]$

Summary : We have proved that

(i) $\lim_{n\to\infty} \phi_n(t) = e^{i\mu t} \quad \forall t \in \mathbb{R}$

Recall . If $(Z_n)_{n\geq 1}$ sequence of r.v. s.t.

c.f. of $Z_n$

(i) $\phi_n(t) \xrightarrow{n\to\infty} \phi(t)$ pointwise

(ii) $\phi$ is continuous at $0$

Then $Z_n \xrightarrow{(d)} Z$, where $Z$ has c.f. $\phi$.

$\mathcal{N}(\mu, \sigma^2) \rightarrow \phi(t) = e^{it\mu} e^{-\frac{\sigma^2}{2}t^2}$

**Here we get:**

$\bar{X}_n \xrightarrow{(d)} Y$, where

$\mathbb{E}[e^{itY}] = e^{it\mu}$

$\Rightarrow Y \sim \delta_\mu \quad (Y = \mu \text{ a.s.})$

**Conclusion**

$$\boxed{\bar{X}_n \xrightarrow{(d)} \mu}$$

**CLT**: With Taylor expansion of order 2

# Proof of Proposition 13 (1)

Characteristic functions: For $t, u \in \mathbb{R}$ set

$$\phi(u) = \mathbf{E}\left[\exp\left(\imath u X_1\right)\right], \quad \text{and} \quad \phi_n(t) = \mathbf{E}\left[\exp\left(\imath t \bar{X}_n\right)\right],$$

Then we have

$$\phi_n(t) = \left[\phi\left(\frac{t}{n}\right)\right]^n$$

Expansion for $\phi_n$: We get

$$\phi_n(t) = \left(1 + \imath\frac{\mu\, t}{n} + o\left(\frac{1}{n}\right)\right)^n$$

# Proof of Proposition 13 (2)

Limit for $\phi_n$: By Taylor expansions arguments, for all $t \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \phi_n(t) = \exp(\imath \mu t)$$

Conclusion: By characteristic function method,

$$\bar{X}_n \xrightarrow{(d)} \mu$$

Method for CLT part:
$\hookrightarrow$ Expansions of order 2 for characteristic functions

# A first improvement: weak LLN

**Proposition 14.**

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- Hyp: $X_1 \in L^1(\Omega)$ and $\mathbf{E}[X_1] = \mu$
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\mathrm{P}} \mu,$$

Recall : We have seen

$$(Z_n \xrightarrow{d} c) \Rightarrow (Z_n \xrightarrow{P} c)$$

Here

$$\bar{X}_n \xrightarrow{d} \mu \text{ (constant)}$$

$$\Rightarrow \boxed{\bar{X}_n \xrightarrow{P} \mu}$$

# Proof of Proposition 14

Quick proof: The result stems from

- $\bar{X}_n \xrightarrow{(d)} \mu$
- $\mu$ is a constant

# Strong LLN under $L^2$ conditions

**Proposition 15.**

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- Hyp: $X_1 \in L^2(\Omega)$ and $\mathbf{E}[X_1] = \mu$, $\mathbf{Var}(X_1) = \sigma^2$
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu, \quad \text{and} \quad \bar{X}_n \xrightarrow{L^2} \mu$$

**$L^2$-convergence** : we want to prove

$$\mathbb{E}[(\overline{X}_n-\mu)^2] \xrightarrow{n\to\infty} 0$$

**Computation** $\mathbb{E}[(\overline{X}_n-\mu)^2]$

$$= \mathbb{E}\Big[\Big(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)\Big)^2\Big]$$

$\perp\!\!\!\perp$ centered r.v.

$$= \frac{1}{n^2}\,\mathbb{E}\Big[\Big(\sum_{i=1}^{n}(X_i-\mu)\Big)^2\Big]$$

$$= \frac{1}{n^2}\,\mathrm{Var}\Big(\sum_{i=1}^{n}X_i\Big)$$

$$\overset{\perp\!\!\!\perp}{=} \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i)$$

$$= \frac{1}{n}\mathrm{Var}(X_1) = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0$$

$\boxed{\text{Conclusion: } \overline{X}_n \xrightarrow{L^2} \mu}$

Recall : We have seen that

$$\bar{X}_n \xrightarrow{P} \mu$$

$\exists n_k$

$$\bar{X}_{n_k} \xrightarrow{a.s.} \mu$$

Here, using $L^2$-norms, we can construct a specific $n_k$ s.t. $\forall \varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}(A_{n_k}(\varepsilon)) < \infty \qquad A_{n_k}(\varepsilon) = (|\bar{X}_{n_k} - \mu| > \varepsilon)$$

This implies $\boxed{\bar{X}_{n_k} \xrightarrow{a.s.} \mu}$

## Specific $n_k$

$A_{n_k}(\varepsilon) = (|\overline{X}_{n_k} - \mu| > \varepsilon)$

$$\mathbb{P}(A_{n_k}(\varepsilon)) = \mathbb{P}(|\overline{X}_{n_k} - \mu| > \varepsilon)$$

Chevychell

$$\leq \frac{\mathbb{E}[|\overline{X}_{n_k} - \mu|^2]}{\varepsilon^2}$$

$$= \frac{\sigma^2}{n_k \varepsilon^2}$$

If we want $\sum \mathbb{P}(A_{n_k}(\varepsilon)) < \infty$, it is sufficient to take

$$n_k = k^2$$

$$\Rightarrow \boxed{\sum_{k=1}^{\infty} \mathbb{P}(A_{n_k}(\varepsilon))} \leq \frac{\sigma^2}{\varepsilon^2} \sum \frac{1}{k^2} \boxed{< \infty}$$

We have obtained:

$$\boxed{\bar{X}_{k^2} \xrightarrow{a.s.} \mu}$$

From $\bar{X}_{k^2}$ to $\bar{X}_n$. Assume $X_n \geq 0$ a.s.

Take $n$. Then $k^2 \leq n \leq (k+1)^2$ for a given $k$. Thus

$$S_{k^2} \leq S_n \leq S_{(k+1)^2} \quad (X_j\text{'s are } \geq 0)$$

$$\frac{S_{k^2}}{(k+1)^2} \leq \frac{S_n}{n} \leq \frac{S_{(k+1)^2}}{k^2}$$

## Summary

$$\frac{S_{k^2}}{(k+1)^2} \leq \frac{S_n}{n} \leq \frac{S_{(k+1)^2}}{k^2}$$

$$\Rightarrow \quad \underset{\substack{\downarrow\, k\to\infty \\ 1}}{\frac{k^2}{(k+1)^2}} \; \underset{\substack{\downarrow\, \text{a.s.}\; k\to\infty \\ \mu}}{\overbrace{\frac{S_{k^2}}{k^2}}^{\bar{X}_{k^2}}} \leq \frac{S_n}{n} \leq \underset{\substack{\downarrow\, \text{a.s.} \\ \mu}}{\frac{S_{(k+1)^2}}{(k+1)^2}} \; \underset{\substack{\downarrow \\ 1}}{\frac{(k+1)^2}{k^2}}$$

## Conclusion

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mu$$

Case of a signed $X_n$ :

Write $X_n = X_n^+ - X_n^-$ , then apply the previous result for both $X_n^+$ and $X_n^-$

# Proof of Proposition 15 (1)

$L^2$ convergence: We compute

$$\begin{aligned}
\mathbf{E}\left[\left(\bar{X}_n-\mu\right)^2\right] &= \frac{1}{n^2}\,\mathbf{E}\left[\left(\sum_{i=1}^{n}(X_i-\mu)\right)^2\right] \\
&= \frac{1}{n^2}\,\mathbf{Var}\left(\sum_{i=1}^{n}(X_i-\mu)\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{Var}\left(X_i\right) \\
&= \frac{1}{n}\,\mathbf{Var}\left(X_1\right)
\end{aligned}$$

Conclusion:

$$\lim_{n\to\infty}\mathbf{E}\left[\left(\bar{X}_n-\mu\right)^2\right]=0$$

# Proof of Proposition 15 (2)

General result for a subsequence: Since $\bar{X}_n \xrightarrow{\mathrm{P}} \mu$, we have:

There exists a subsequence $\{n_k;\, k \geq 1\}$ such that $\bar{X}_{n_k} \xrightarrow{\text{a.s}} \mu$

# Proof of Proposition 15 (3)

A more concrete subsequence: Set $n_k = k^2$ and

$$A_k(\varepsilon) = \left\{ |\bar{X}_{n_k} - \mu| > \varepsilon \right\}$$

Then by Chebyshev,

$$\mathbf{P}\left(A_k(\varepsilon)\right) \leq \frac{\mathbf{E}\left[\left(\bar{X}_{k^2} - \mu\right)^2\right]}{\varepsilon^2} \leq \frac{\mathbf{Var}(X_1)}{k^2 \varepsilon^2}$$

Almost sure convergence: We have

$$\sum_{k=1}^{\infty} \mathbf{P}\left(A_k(\varepsilon)\right) < \infty \text{ for all } \varepsilon > 0, \quad \text{and thus} \quad \bar{X}_{k^2} \xrightarrow{\text{a.s}} \mu$$

# Proof of Proposition 15 (4)

Case of a positive sequence: If $X_n \geq 0$, then if $k^2 \leq n \leq (k+1)^2$

$$\begin{aligned} S_{k^2} &\leq S_n \leq & S_{(k+1)^2} \\ \frac{S_{k^2}}{(k+1)^2} &\leq \tfrac{S_n}{n} \leq & \frac{S_{(k+1)^2}}{k^2} \end{aligned}$$

Taking $n \to \infty$ we get

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu$$

# Proof of Proposition 15 (5)

Signed sequence case: For a general $X_n$ we argue as follows:

1. Write $X_n = X_n^+ - X_n^-$
2. Apply positive sequence case to both $X_n^+$ and $X_n^-$
3. This is allowed since $X_n^+$ i.i.d with $\mathbf{Var}(X_1^+) < \infty$

Conclusion: We still have

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu$$

# Outline

# The strong law

**Theorem 16.**

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu, \quad \Longleftrightarrow \quad X_1 \in L^1(\Omega)$$

# Nsc for weak convergence

**Theorem 17.**

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\mathbf{P}} \mu \quad \Longleftrightarrow \quad \text{Condition (2) or (3) holds,}$$

with $\phi =$ c.f. of $X_1$

$$\lim_{n\to\infty} n\,\mathbf{P}\left(|X_1| > n\right) = 0, \text{ and } \lim_{n\to\infty} \mathbf{E}\left[X_1\,\mathbf{1}_{(|X_1|\leq n)}\right] = \mu \tag{2}$$

$$\phi \text{ differentiable at } 0, \quad \text{and} \quad \phi'(0) = \imath\,\mu \tag{3}$$

# Example of WLLN without SLLN

example: $X_1 \sim \mathcal{N}(0,\sigma^2)$

**Proposition 18.**

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$
- $X_1$ symmetric random variable
- Common cdf satisfies $1 - F(x) \sim \frac{1}{x \ln(x)}$ as $x \to \infty$

$X_1$ symmetric if $\mathcal{L}(-X_1) = \mathcal{L}(X_1)$
$X_1 \overset{(d)}{=} -X_1$

Then

$$\bar{X}_n \xrightarrow{\mathrm{P}} 0, \quad \text{but} \quad \bar{X}_n \text{ does not converge a.s}$$

Why don't we have $\bar{X}_n \xrightarrow{a.s.} \mu$ ?

It is due to the fact that $X_1 \notin \mathcal{L}^1(\Omega)$

Proof that $X_1 \notin \mathcal{L}^1(\Omega)$: we have

$$\mathbb{P}(X_1 \geq x) = 1 - F(x) \sim \frac{1}{x \ln(x)} \quad (\text{if } X_1 \text{ contin.})$$

Thus, if

$$\mathbb{E}[X_1^+] = \int_0^\infty \overbrace{\mathbb{P}(X_1 \geq x)}^{\sim \frac{1}{x \ln(x)}} dx = \infty$$

$$\Rightarrow \boxed{X_1 \notin \mathcal{L}^1(\Omega)}$$

Why do we have $\bar{X}_n \xrightarrow{P} \mu$ ? We should verify

$$n\,\mathbb{P}(|X_1| > n) \xrightarrow{n\to\infty} 0 \qquad (1)$$

$$\lim_{n\to\infty} E[X_1 \mathbb{1}_{(|X_1| \le n)}] = \mu \qquad (2)$$

Proof of (1)

$$n\,\mathbb{P}(|X_1| > n) \sim \frac{n}{n \ln n} \xrightarrow{n\to\infty} 0$$

Proof of (2) Since $X_1$ symmetric

$$E[X_1 \mathbb{1}_{(|X_1| \le n)}] \overset{\text{symm.}}{=} 0 \xrightarrow{n\to\infty} 0$$