Claim If $\{X_n ; n \geq 1\}$ sequence of iid $\boxed{\text{Cauchy}}$ r.v. then

?

$\bar{X}_n \xrightarrow{(d)} Z$

But

$\bar{X}_n \not\xrightarrow{P}$

$\forall n, \ X_n \sim$ Cauchy

Density

$$f(x) = \frac{1}{\pi(1+x^2)}$$

Cauchy random variable (1)

Notation:

$$\mathrm{Cauchy}(\alpha), \text{ with } \alpha \in \mathbb{R}$$

State space:

$$\mathbb{R}$$

Density:

$$f(x) = \frac{1}{\pi} \frac{1}{1 + (x - \alpha)^2}$$

Expected value and variance:

Not defined (divergent integrals)!

Fact If $X \sim$ Cauchy, $X \notin L^1$:

$$\mathbb{E}[X] = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{1+x^2}\, dx$$ → not 0 divergent

Recall $X \in L^1(\Omega) \Leftrightarrow \mathbb{E}[|X|] < \infty$

Here

$g(x) \overset{\infty}{\sim} \frac{1}{x}$

$$\boxed{\mathbb{E}[|X|]} = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}\, dx \boxed{= \infty}$$

Riemann's rule. If at ∞ we have

$g(x) \sim \frac{1}{x^\alpha}$, then $\int^\infty g(x)\,dx < \infty$ iff $\alpha > 1$

$\lim_{x \to \infty} \frac{g(x)}{x^{-\alpha}} = 1$

Cauchy random variable (2)

Use 1: Trigonometric function of a uniform r.v
Namely if

- $X \sim \mathcal{U}([-\frac{\pi}{2}, \frac{\pi}{2}])$
- $Y = \tan(X)$

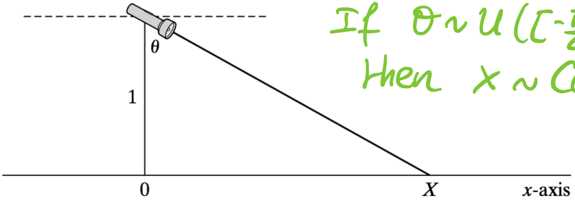
Then $Y \sim \text{Cauchy} \equiv \text{Cauchy}(0)$ check at home

Use 2:
Typical example of r.v with no mean (not in $L^1(\Omega)$)

Example: beam (1)

Experiment:

- Narrow-beam flashlight spun around its center
- Center located a unit distance from the x-axis
- $X =$ point at which the beam intersects the x-axis
  when the flashlight has stopped spinning



$X = \tan(\theta)$

If $\theta \sim \mathcal{U}([-\frac{\pi}{2}, \frac{\pi}{2}])$

then $X \sim$ Cauchy

Characteristic function of $X \sim$ Cauchy

Define $\phi(t) = \mathbb{E}[e^{itX}]$

Here

$$\phi(t) = \int_{\mathbb{R}} \frac{e^{itx}}{1+x^2}\, dx$$

$= \cos(tx) + i\sin(tx)$

Method to compute ϕ

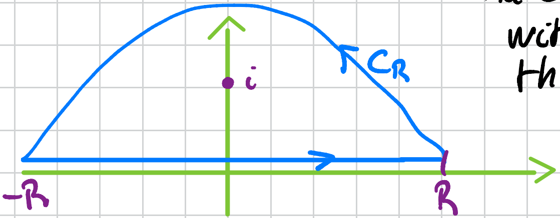
$\hookrightarrow$ Residues

$$\phi(t) = \int_{\mathbb{R}} \frac{e^{itx}}{\pi(1+x^2)}\, dx$$

Set $$g_t(z) = \frac{e^{itz}}{\pi(1+z^2)}$$

$g_t(z)$ has 2 poles: $z = \pm i$

Natural contour:



Rule: If $|g_t(z)| \le \frac{c}{|z|^p}$ with $p > 1$, then

$$\lim_{R \to \infty} \int_{C_R} g_t(z)\, dz = 0$$

If $t > 0$, $|g_t(z)| \underset{C_R}{\le} \frac{1}{|1+z^2|} \le \frac{c}{|z|^2}$

$\Rightarrow \lim_R \int_{C_R} g_t(z) = 0$

Thus for $t>0$

$$\phi(t) = \lim_{R\to\infty} \int_{-R}^{R} g_t(x)\,dx = 2\pi i \operatorname{Res}(g_t, i)$$

$$= 2\pi i \lim_{z\to i} (z-i)\, g_t(z) = e^{-t}$$

$$g_t(z) = \frac{e^{itz}}{\pi(1+z^2)} = \frac{e^{itz}}{\pi(z-i)(z+i)}$$

$$\lim_{z\to i} (z-i) g_t(z) = \lim_{z\to i} \frac{e^{itz}}{\pi(z+i)}$$

$$= \frac{e^{-t}}{2i\pi}$$

For $t < 0$, we get

$$\phi(t) = e^{t}$$

Conclusion : if $X \sim$ Cauchy

$$\boxed{\phi(t) = e^{-|t|}}$$

Example: beam (2)

Model:

- We assume $\theta \sim \mathcal{U}([-\frac{\pi}{2}, \frac{\pi}{2}])$
- We have $X \sim \tan(\theta)$

Conclusion:

$$X \sim \text{Cauchy}$$

Example with no WLLN

Rmk: This an example of $X_1 \notin L^1$, for which LLN is far from being satisfied

Proposition 19.

We consider

- $\{X_n;\ n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$
- $X_1 \sim$ Cauchy

Then

$$\bar{X}_n \xrightarrow{(d)} \text{Cauchy}, \quad \text{but} \quad \bar{X}_n \text{ does not converge in P}$$

Char. function of $\bar{X}_n$

$$\phi_n(t) = E[e^{it\bar{X}_n}] = E[e^{i\frac{t}{n}\sum_{i=1}^{n} X_i}]$$

$$\overset{\perp\!\!\!\perp}{=} \prod_{i=1}^{n} \underbrace{E[e^{i\frac{t}{n}X_i}]}_{= e^{-\frac{|t|}{n}}}$$

$$= e^{-|t|}$$

This is the char. function of Cauchy.

Thus $\bar{X}_n \sim$ Cauchy $\forall n$

$$\Rightarrow \boxed{\bar{X}_n \xrightarrow{(d)} \text{Cauchy}}$$

Why don't we have $\bar{X}_n \xrightarrow{P}$?

For this we have seen $\bar{X}_n \xrightarrow{P} \mu$ iff either

① $n\,\mathbb{P}(|X_1|>n) \longrightarrow 0$ + other condition

Here $\mathbb{P}(|X_1|>n) = \frac{2}{\pi}\int_n^\infty \frac{1}{1+x^2}\,dx$

$$\geq c_1 \int_n^\infty \frac{1}{x^2}\,dx \geq c_2 \frac{1}{n}$$

Thus $\boxed{n\, P(|X_1|>n) \not\longrightarrow 0}$

2. $\phi(t)$ is differentiable at 0
and $\phi'(0) = i\mu$

Here $\boxed{\phi(t) = e^{-|t|}\text{, not differentiable at } 0}$

Thus

$$\boxed{\bar{X}_n \overset{P}{\not\Rightarrow}}$$

Next step Prove that

$$X_1 \in L^1(\Omega) \implies \bar{X}_n \xrightarrow{a.s.} \mu = E[X_1]$$

Strategy : We already know

$$X_1 \in L^2(\Omega) \implies \bar{X}_n \xrightarrow{a.s.} \mu$$

We will thus truncate the X_j's and then take limits

Truncation. Assume $X_j \geq 0$. Then define

Here truncation depends on n

$$Y_n = X_n \mathbb{1}(X_n < n)$$

Fact. Set

$$A_n = (X_n \neq Y_n)$$

Then

True if $\sum \mathbb{P}(A_n) < \infty$ by B-C

$$\mathbb{P}(A_n \text{ occurs i.o})$$

$$= \mathbb{P}(\limsup A_n) = 0$$

Here

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \mathbb{P}(X_n \geq n)$$

$$\overset{\text{iid}}{=} \sum_{n=1}^{\infty} \mathbb{P}(X_1 \geq n)$$

$$\overset{\text{Pb3}}{\leq} \mathbb{E}[X_1] \overset{L^1}{<} \infty$$

Conclusion

$$\boxed{\mathbb{P}(X_n \neq Y_n \text{ i.o}) = 0}$$