Outline

1. Definition
   - Baby conditional distributions: discrete case
   - Baby conditional distributions: continuous case
   - Definition with measure theory
2. Examples
3. Existence and uniqueness
4. Conditional expectation: properties
5. Conditional expectation as a projection
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General definition

$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Definition 1.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y pmf of X
- y such that $p_Y(y) > 0$

$\frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)} = \frac{p(x,y)}{p_Y(y)}$

Then the conditional pmf of X given $Y = y$ is defined by

$$p_{X|Y}(x|\, y) = \mathbf{P}\left(X = x|\, Y = y\right) = \frac{p(x, y)}{p_Y(y)}$$

Example ctd: tossing 3 coins (1)

Q: $A \perp\!\!\!\perp B$?

Experiment: for n coins, $A \not\perp\!\!\!\perp B$ unless $n=3$

Tossing a coin 3 times

Events: We consider

$$A = \text{"At most one Head"}$$
$$B = \text{"At least one Head and one Tail"}$$

Random variables: Set

$$X_1 = \mathbf{1}_A, \qquad X_2 = \mathbf{1}_B, \qquad X = (X_1, X_2)$$

Natural sample space : $\Omega = \{h, t\}^3$

Probability: Uniform on Ω

ω	(X_1, X_2)	ω	(X_1, X_2)
h h h	$(0, 0)$	t h h	$(0\ \ 1)$
h h t	$(0, 1)$	t h t	$(1\ \ 1)$
h t h	$(0, 1)$	t t h	$(1\ \ 1)$
h t t	$(1, 1)$	t t t	$(1\ \ 0)$

pmf $\quad \mathbb{P}(X_1 = 0, X_2 = 0) = \frac{1}{8}$

$\mathbb{P}(X_1 = 0, X_2 = 1) = \frac{3}{8} \quad , \ldots$

Table

$X_1 \backslash X_2$	0	1	Marg X_1
0	$1/8$	$3/8$	$1/2$
1	$1/8$	$3/8$	$1/2$
Marg X_2	$1/4$	$3/4$	

$$\boxed{X_1 \sim B(\tfrac{1}{2}) \qquad X_2 \sim B(3/4)}$$

$X_1 \setminus X_2$	0	1	Marg X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg X_2	1/4	3/4	

Rmk we also have

$P_{X_2|X_1}(0|1) = 1/4$

$P_{X_2|X_1}(1|1) = 3/4$

$\Rightarrow X_1 \perp\!\!\!\perp X_2$

$\Rightarrow A \perp\!\!\!\perp B$

Cond. prob. given $X_1 = 0$

$$\boxed{\begin{aligned} P_{X_2|X_1}(0 \mid 0) &= \frac{P_{X_1 X_2}(0,0)}{P_{X_1}(0)} = \frac{1/8}{1/2} = \frac{1}{4} \\ P_{X_2|X_1}(1 \mid 0) &= \frac{P_{X_1 X_2}(0,1)}{P_{X_1}(0)} = \frac{3/8}{1/2} = \frac{3}{4} \end{aligned}}$$

Conditional pmf

Example ctd: tossing 3 coins (2)

Joint distribution of (X_1, X_2):

$X_1 \backslash X_2$	0	1	Marg. X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg. X_2	1/4	3/4	1

Conditional probabilities given $X_1 = 0$:

$$p_{X_2|X_1}(0|\,0) = \frac{1/8}{1/2} = \frac{1}{4}, \quad p_{X_2|X_1}(1|\,0) = \frac{3/8}{1/2} = \frac{3}{4}$$

Conditional probabilities given $X_2 = 1$:

$$p_{X_1|X_2}(0|\,1) = \frac{3/8}{3/4} = \frac{1}{2}, \quad p_{X_1|X_2}(1|\,1) = \frac{3/8}{3/4} = \frac{1}{2}$$

Conditioning Poisson random variables

Proposition 2.

Let

- $X \sim \mathcal{P}(\lambda_1)$, $Y \sim \mathcal{P}(\lambda_2)$
- $X \perp\!\!\!\perp Y$
- $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$

Then

$$\mathcal{L}\left(X \middle| X+Y=n\right) = \mathrm{Bin}(n,p)$$

Situation : $X \sim \mathcal{P}(\lambda_1)$, $Y \sim \mathcal{P}(\lambda_2)$, $X \perp\!\!\!\perp Y$

Aim : Compute the conditional pmf

$0 \leq k \leq n$

$$p_{X|X+Y}(k \mid n) = \mathbb{P}(X = k \mid X+Y = n)$$

$$= \frac{\mathbb{P}(X=k, X+Y=n)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\mathbb{P}(X=k, Y=n-k)}{\mathbb{P}(X+Y=n)}$$

$$\overset{\perp\!\!\!\perp}{=} \frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)}$$

Recall : $X+Y \sim \mathcal{P}(\lambda_1+\lambda_2)$

Summary

$$\boxed{p_{X|X+Y}(k \mid n)} = \frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\cancel{e^{-\lambda_1}}\,\dfrac{\lambda_1^k}{k!}\;\cancel{e^{-\lambda_2}}\,\dfrac{\lambda_2^{n-k}}{(n-k)!}}{\cancel{e^{-(\lambda_1+\lambda_2)}}\;\dfrac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$\rightarrow (\lambda_1+\lambda_2)^k(\lambda_1+\lambda_2)^{n-k}$$

$$= \binom{n}{k}\left(\overbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}}^{p}\right)^k\left(\overbrace{\frac{\lambda_2}{\lambda_1+\lambda_2}}^{1-p}\right)^{n-k}$$

$$\boxed{= \binom{n}{k}\,p^k\,(1-p)^{n-k}}$$

we have obtained

$p_{X|X+Y}^{(\cdot|n)} \equiv$ pmf of a $\mathrm{Bin}(n,p)$

law (or distribution)

$\Rightarrow \boxed{\mathcal{L}(X \mid X+Y=n) = \mathrm{Bin}(n,p)}$

(identity between 2 probability measures)

Proof (1)

Expression for the conditional probabilities:
Let $0 \leq k \leq n$. Then invoking $X \perp\!\!\!\perp Y$,

$$\mathbf{P}\left(X = k \middle| X + Y = n\right) = \frac{\mathbf{P}\left(X = k\right)\mathbf{P}\left(Y = n - k\right)}{\mathbf{P}\left(X + Y = n\right)}$$

Law of $X + Y$: One can prove that

$$X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$$

Proof (2)

Computation of the conditional probabilities:

$$\begin{aligned}
&\mathbf{P}\left(X = k \middle| X + Y = n\right) \\
&= e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}\left[e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}\right]^{-1} \\
&= \binom{n}{k}p^k(1-p)^{n-k}
\end{aligned}$$

Conclusion:

$$\mathcal{L}\left(X \middle| X + Y = n\right) = \text{Bin}(n, p)$$

Cond. expectation in the discrete case

Definition 3.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y, y such that $p_Y(y) > 0$
- $p_{X|Y}(x|\,y)$ conditional distribution

Then the conditional exp. of X given $Y = y$ is defined by

$$\mathbf{E}[X|\,Y = y] = \sum_{x \in \mathcal{E}} x\, p_{X|Y}(x|\,y)$$

Example 1

$X \sim \mathcal{P}(\lambda_1)$
$Y \sim \mathcal{P}(\lambda_2)$
$X \perp\!\!\!\perp Y$

Then $\mathcal{L}(X \mid X+Y=n) = \mathrm{Bin}(n,p)$

$$\Rightarrow \boxed{\mathbb{E}[X \mid X+Y=n] = np}$$

$$= \sum_{k=0}^{n} k \; \mathbb{P}(X=k \mid X+Y=n)$$

Binomial example (1)

Situation: Let

- $X, Y \sim \text{Bin}(n, p)$
- $X \perp\!\!\!\perp Y$
- $Z = X + Y$

Problem: We wish to compute

$$\mathbf{E}\left[X \mid Z = m\right]$$

$X, Y \sim \mathrm{Bin}(n,p) \quad X \perp\!\!\!\perp Y$

Conditional pmf $\qquad Z = X+Y \sim \mathrm{Bin}(2n,p)$

$$\boxed{\mathbb{P}(X=k \mid Z=m)} \overset{\perp\!\!\!\perp}{=} \frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=m-k)}{\mathbb{P}(Z=m)}$$

$$= \frac{\binom{n}{k} p^{k} (1-p)^{n-k} \binom{n}{m-k} p^{m-k} (1-p)^{n-(m-k)}}{\binom{2n}{m} p^{m} (1-p)^{2n-m}}$$

$$= \boxed{\frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}}$$

Binomial example (2)

Distribution for Z:

$$Z = \sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j \sim \text{Bin}(2n, p)$$

Computation for conditional pmf: For $k \le \min(n, m)$ we have

$$\begin{aligned}
\mathbf{P}\left(X = k \middle| Z = m\right) &= \frac{\mathbf{P}(X = k,\ X + Y = m)}{\mathbf{P}(Z = m)} \\
&= \frac{\mathbf{P}(X = k,\ Y = m - k)}{\mathbf{P}(Z = m)} \\
&= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}
\end{aligned}$$

Hypergeometric random variable (1)

Use: Consider the experiment

- Urn containing N balls
- m white balls, $N - m$ black balls
- Sample of size n is drawn without replacement
- Set $X = \#$ white balls drawn

Then

$$X \sim \text{HypG}(n, N, m)$$

Hypergeometric random variable (2)

Notation:

$$X \sim \mathrm{HypG}(n, N, m), \quad \text{for } N \in \mathbb{N}^*, \ m, n \le N$$

State space:

$$\{0, \ldots, n\}$$

Pmf:

$$\mathbf{P}(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, \quad 0 \le k \le n$$

Expected value and variance: Set $p = \frac{m}{N}$. Then

$$\mathbf{E}[X] = np, \qquad \mathbf{Var}(X) = np(1-p)\left(\frac{N-n}{N-1}\right)$$

Binomial example (3)

Conditional pmf: For $k \leq \min(n, m)$ we have seen

$$p_{X|Z}(k|\, m) = \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

Recall: If $V \sim \text{HypG}(n, N, m)$ then

$$\mathbf{P}(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

$\mathbb{E}[X|Z=m] = m \times \frac{n}{2n}$

$= \frac{m}{2}$

Identification of the conditional pmf: We have

$$p_{X|Z}(k|\, m) = \text{Pmf of HypG}(m, 2n, n)$$

Binomial example (4)

Conditional expectation: Let $V \sim \mathrm{HypG}(\cancel{2n, m}, n)$. Then

$m \quad 2n \quad n$

$$\mathbf{E}[X \,|\, Z = m] = \mathbf{E}[V]$$

Numerical value:

According to the values for hypergeometric distributions,

$$\mathbf{E}[X \,|\, Z = m] = m \times \frac{n}{2n} = \frac{m}{2}$$

Outline

General definition

$f_X(x) = \int f(x,y)\,dy$
$f_Y(y) = \int f(x,y)\,dx$

Definition 4.

Let

- (X, Y) couple of continuous random variables
- Joint density f
- Marginal densities f_X, f_Y
- y such that $f_Y(y) > 0$

Then the conditional density of X given $Y = y$ is defined by

$$f_{X|Y}(x|\,y) = \frac{f(x,y)}{f_Y(y)}$$

Intuition

$$f_{X|Y}(x|y)\,dx \overset{\text{def}}{=} \frac{f(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

$$\text{"="} \quad \frac{\mathbb{P}(x \le X \le x+dx,\ y \le Y \le y+dy)}{\mathbb{P}(y \le Y \le y+dy)}$$

$$= \mathbb{P}(x \le X \le x+dx \mid y \le Y \le y+dy)$$

not rigorous

Note : $\boxed{\mathbb{E}[X|Y=y] = \int x\, f_{X|Y}(x|y)\,dx}$

Justification of the definition

Heuristics: $f_{X|Y}(x|\,y)$ can be interpreted as

$$\begin{aligned} f_{X|Y}(x|\,y)\,dx &= \frac{f(x,y)\,dxdy}{f_Y(y)\,dy} \\ &\simeq \frac{\mathbf{P}\left(x \le X \le x+dx,\; y \le Y \le y+dy\right)}{\mathbf{P}\left(y \le Y \le y+dy\right)} \\ &= \mathbf{P}\left(x \le X \le x+dx|\, y \le Y \le y+dy\right) \end{aligned}$$

Use of the conditional probability: compute probabilities like

$$\mathbf{P}\left(X \in A|\, Y = y\right) = \int_A f_{X|Y}(x|\,y)\,dx$$

Rigorous definition: see next sections