Prob space $(\Omega, \mathcal{F}_0, \mathbb{P})$, $\mathcal{F} \subset \mathcal{F}_0$

Def of $\mathbb{E}[X \mid \mathcal{F}] = Y$, such that

"Best possible approx"

(i) $Y \in \mathcal{F}$

(ii) $\mathbb{E}[Y \mathbb{1}_A] = \mathbb{E}[X \mathbb{1}_A]$

$\forall A \in \mathcal{F}$

Extreme example 1

If $X \in \mathcal{F}$, then $\mathbb{E}[X|\mathcal{F}] = X$

Extreme example 2

If $X \perp\!\!\!\perp \mathcal{F}$, then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$

In-between, the recipe is

(i) Freeze what you know

(ii) Average on what you don't know

In-between example!

If $\mathcal{F}$ generated by a partition $\{\Omega_j ; j \geq 1\}$, then

$$\mathbb{E}[X \mid \mathcal{F}] = \sum_{j=1}^{\infty} \mathbb{1}_{\Omega_j} \frac{\mathbb{E}[X \mathbb{1}_{\Omega_j}]}{\mathbb{P}(\Omega_j)}$$

Particular case. If $\mathcal{F} = \sigma(B)$,

$$\mathbb{P}(A \mid \mathcal{F}) = \mathbb{E}[\mathbb{1}_A \mid \mathcal{F}]$$

$$= \mathbb{P}(A \mid B) \mathbb{1}_B + \mathbb{P}(A \mid B^c) \mathbb{1}_{B^c}$$

Dice throwing

$\mathcal{F}_0 = \mathcal{P}(\Omega)$, $\mathbb{P} = \mathcal{U}(\{1, .., 6\})$

$\mathcal{F} = \sigma(B)$

Example: We consider

- $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{4\}$, $B =$ "even number".

Then

$$\mathbf{P}(A|\mathcal{F}) = \frac{1}{3}\,\mathbf{1}_B .$$

Setting

$\Omega = \{1, \dots, 6\}$, $\mathcal{F}_0 = \mathcal{P}(\Omega)$

$A = \{4\}$

$\mathcal{F} = \sigma(B) = \{\emptyset, B, B^c, \Omega\}$

$B =$ even number $= \{2, 4, 6\}$

Computation

According to the general formula

$$\mathbb{P}(A \mid \mathcal{F}) = \overbrace{\mathbb{P}(A \mid B)}^{1/3} \mathbb{1}_B + \overbrace{\mathbb{P}(A \mid B^c)}^{0} \mathbb{1}_{B^c}$$

$$= \frac{1}{3} \mathbb{1}_B$$

Conditioning a r.v by another r.v

Definition 8.

Let

- X random variable such that $X \in L^1(\Omega)$
- Y random variable

We set

$$\mathbf{E}[X|Y] = \mathbf{E}[X|\sigma(Y)].$$

$\sigma(Y) = \{ Y^{-1}(B);\ B \in \mathcal{B}(\mathbb{R}) \}$

$=$ smallest σ-algebra which makes Y measurable

Characterizing $\sigma(Y)$

How to know if $A \in \sigma(Y)$:

for $B \in \mathcal{B}(\mathbb{R})$

We have $A \in \sigma(Y)$ iff

$$A = \{\omega;\ Y(\omega) \in B\}, \quad \text{or} \quad \mathbf{1}_A = \mathbf{1}_B(Y)$$

How to know if $Z \in \sigma(Y)$:

Let Z and Y be two random variables. Then

$$Z \in \sigma(Y) \quad \text{iff we can write} \quad Z = U(Y), \quad \text{with} \quad U \in \mathcal{B}(\mathbb{R}).$$

Note: $U : \mathbb{R} \to \mathbb{R}$ is a generic measurable function

Conditioning a r.v by a discrete r.v

$\sigma(Y)$ is generated by $\{\Omega_j;\, j \geq 1\}$ where $\Omega_j = (Y = y_j)$

Example 4: Whenever X and Y are discrete random variables
$\hookrightarrow$ Computation of $\mathbf{E}[X|Y]$ can be handled as in example 3.

More specifically:

Typical example of E: $E = \{1, \ldots, 6\}$
$\{0, \ldots, n\}$
$\mathbb{N}$

- Assume $Y \in E$ with $E = \{y_i;\, i \geq 1\}$
- Hypothesis: $\mathbf{P}(Y = y_i) > 0$ for all $i \geq 1$.

Then $\mathbf{E}[X|Y] = h(Y)$ with $h : E \to \mathbb{R}$ defined by:

$$h(y) = \frac{\mathbf{E}[X\, \mathbf{1}_{(Y=y)}]}{\mathbf{P}(Y = y)}.$$

r.v. $X : \Omega \longrightarrow E$

Claim: $\mathbb{E}[X|Y] = h(Y)$

where $h(y) = \dfrac{\mathbb{E}[X \, \mathbb{1}_{(Y=y)}]}{\mathbb{P}(Y=y)}$ $= \mathbb{E}[X | Y=y]$ from baby conditioning

We check

(i) $\boxed{Z \equiv h(Y) \in \sigma(Y)}$

$\hookrightarrow$ True since Z is of the form $h(Y)$ for h measurable

Note $h : E \longrightarrow \mathbb{R}$

(ii) We should check $\qquad Z \equiv h(Y)$

$$\mathbb{E}[Z \, \mathbb{1}_A] = \mathbb{E}[X \, \mathbb{1}_A]$$

for $\forall A \in \sigma(Y) = \sigma(\Omega_j \, ; \, j \geq 1)$

where $\Omega_j = (Y = y_j)$

We are thus reduced to prove

$$\boxed{\mathbb{E}[Z \, \mathbb{1}_{(Y=y_j)}] = \mathbb{E}[X \, \mathbb{1}_{(Y=y_j)}] \quad \forall j \geq 1}$$

$\mathbb{E}[Z\, \mathbb{1}_{(Y=y_j)}] = \mathbb{E}[X\, \mathbb{1}_{(Y=y_j)}]$

Recall $Z = h(Y) = \sum_{i=1}^{\infty} h(y_i)\, \mathbb{1}_{(Y=y_i)}$

$$= \sum_{i=1}^{\infty} \frac{\mathbb{E}[X\, \mathbb{1}_{(Y=y_i)}]}{\mathbb{P}(Y=y_i)}\, \mathbb{1}_{(Y=y_i)}$$

Then $\boxed{\mathbb{E}[Z\, \mathbb{1}_{(Y=y_j)}]}$

$$= \sum_{i=1}^{\infty} \frac{\mathbb{E}[X\, \mathbb{1}_{(Y=y_i)}]}{\mathbb{P}(Y=y_i)}\, \underbrace{\overbrace{\mathbb{E}[\mathbb{1}_{(Y=y_i)}\, \mathbb{1}_{(Y=y_j)}]}^{=0 \text{ if } i \neq j}}_{\mathbb{P}(Y=y_j)\, \mathbb{1}_{(i=j)}}$$

$$= \frac{\mathbb{E}[X\, \mathbb{1}_{(Y=y_j)}]}{\mathbb{P}(Y=y_j)}\, \mathbb{P}(Y=y_j) \boxed{= \mathbb{E}[X\, \mathbb{1}_{(Y=y_j)}]}$$

(ii) verified

Conditioning a r.v by a continuous r.v

Example 5: Let (X, Y) couple of real random variables with measurable density $f : \mathbb{R}^2 \to \mathbb{R}_+$. We assume that

$$\int_{\mathbb{R}} f(x, y)dx > 0, \quad \text{for all } y \in \mathbb{R}.$$

Let $g : \mathbb{R} \to \mathbb{R}$ a measurable function such that $g(X) \in L^1(\Omega)$. Then $\mathbf{E}[g(X)|Y] = h(Y)$, with $h : \mathbb{R} \to \mathbb{R}$ defined by:

$$h(y) = \frac{\int_{\mathbb{R}} g(x)f(x, y)dx}{\int_{\mathbb{R}} f(x, y)dx}.$$

$= \mathbb{E}[g(X) | Y=y]$ from baby conditioning

Heuristic proof

Formally one can use a conditional density:

$$\mathbf{P}(X = x|Y = y)\text{''} = \text{''}\frac{\mathbf{P}(X = x, Y = y)}{\mathbf{P}(Y = y)} = \frac{f(x, y)}{\int f(x, y)dx},$$

Integrating against this density we get:

$$\begin{aligned}\mathbf{E}[g(X)|Y = y] &= \int g(x)\mathbf{P}(X = x|Y = y)\,dx \\ &= \frac{\int g(x)f(x, y)dx}{\int f(x, y)dx}.\end{aligned}$$

Recall: $h(y) = \dfrac{\int_{\mathbb{R}} g(x) f(x,y)\,dx}{\int_{\mathbb{R}} f(x,y)\,dx}$

Claim: $h(Y) \equiv Z$ satisfies (i) and (ii)

(i) $Z \in \sigma(Y)$, since $Z = h(Y)$ with h measurable

(ii) We should prove

$$E[Z\, \mathbb{1}_A] = E[X\, \mathbb{1}_A] \quad \forall A \in \sigma(Y)$$

Generic A: $\mathbb{1}_A = \mathbb{1}_B(Y)$

In fact it is enough to prove

$$\boxed{\begin{array}{l}\mathbb{E}[Z\ \psi(Y)] = \mathbb{E}[X\ \psi(Y)] \\ \text{for any } \psi \in \underbrace{\mathcal{B}}_{\text{measurable}}\underbrace{{}_b}_{\text{bounded}}(\mathbb{R})\end{array}}$$

$$\mathbb{E}[Z\ \psi(Y)] = \mathbb{E}[h(Y)\ \psi(Y)]$$

$$= \int_{\mathbb{R}^2} h(y)\ \psi(y)\ f(x,y)\, dx\, dy$$

Enough: $\mathbb{E}[Z\ \mathbb{1}_B(Y)] = \mathbb{E}[X\ \mathbb{1}_B(Y)]$

$$\boxed{\mathbb{E}[Z\,\psi(Y)]}$$

$$= \int_{\mathbb{R}^2} h(y)\,\psi(y)\,f(x,y)\,dx\,dy$$

$$= \int_{\mathbb{R}^2} \frac{\int_{\mathbb{R}} g(z,\cancel{y})\,f(z,y)\,dz}{\int_{\mathbb{R}} f(u,y)\,du}\,\psi(y)\,f(x,y)\,dx\,dy$$

Fubini

$$= \int dy\,dz\; f(z,y)\,g(z,\cancel{y})\,\psi(y)\,\frac{\int f(x,y)\,dx}{\int f(u,y)\,du}$$

$$= \int g(z,\cancel{y})\,\psi(y)\,\overbrace{f(z,y)}^{\text{joint density}}\,dz\,dy$$

$$\boxed{= \mathbb{E}[g(X)\,\psi(Y)]} \longrightarrow$$

(ii) verified

Rigorous proof

Strategy: Check (i) and (ii) in the definition for the r.v $h(Y)$.

(i) If $h \in \mathbb{B}(\mathbb{R})$, we have seen that $h(Y) \in \sigma(Y)$.

(ii) Let $A \in \sigma(Y)$ Then

$$A = \left\{\omega;\ Y(\omega) \in B\right\} \quad \Longrightarrow \quad \mathbf{1}_A = \mathbf{1}_B(Y)$$

Thus

$$\begin{aligned}
\mathbf{E}[h(Y)\mathbf{1}_A] &= \mathbf{E}[h(Y)\mathbf{1}_B(Y)] \\
&= \int_B \int_{\mathbb{R}} h(y) f(x,y)\,dx\,dy \\
&= \int_B dy \int_{\mathbb{R}} \left\{ \frac{\int g(z) f(z,y)\,dz}{\int f(z,y)\,dz} \right\} f(x,y)\,dx \\
&= \int_B dy \int g(z) f(z,y)\,dz = \mathbf{E}[g(X)\mathbf{1}_B(Y)].
\end{aligned}$$