

Extreme example !

If XEF, then E[XIF] = X

Extreme example 2

If X II F, then E[XIF] = E[X]

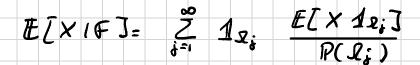
In-between, the recipe is

(i) Freeze what you know

(ic) Average on what you don't know

In-berueen example 1

If & generated by a purtition ↓ R; i ≥ 15, then



Particular case . If F= o(B),

RAIF) = E[1AIF]

= P(AIB) 13 + R(AIB) 13c

Dice throwing

$$F_{S} = P(\mathcal{Q}), P = \mathcal{U}(\langle 1, ..., 6 \rangle)$$

Example: We consider $\int \mathcal{F} = \sigma(B)$
 \wedge
 $\Omega = \{1, 2, 3, 4, 5, 6\}, A = \{4\}, B = "even number".$
Then
$$P(A|\mathcal{F}) = \frac{1}{3}\mathbf{1}_{B}.$$

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 $\mathcal{D} = \langle 1, .., 6 \rangle$, $\mathcal{F}_{0} = \mathcal{P}(\mathcal{D})$

F= (B)= {\$, B, B, 2}

B= even number = { 2, 4, 6}

Computation

According to the general formula P(AIF) = P(AIB) 1 B + P(AIB) 1 BC

A= {44

$=\frac{1}{3}$ 1B

Conditioning a r.v by another r.v

Definition 8.

Let

- X random variable such that $X \in L^1(\Omega)$
- Y random variable

We set

 $\mathbf{E}[X|Y] = \mathbf{E}[X|\sigma(Y)].$

J(Y)= {Y-'(B); BEB(R)} = smallest J-algebra which makes > measurable

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Characterizing $\sigma(Y)$

How to know if $A \in \sigma(Y)$: We have $A \in \sigma(Y)$ iff $A = \{\omega; Y(\omega) \in B\}, \text{ or } \mathbf{1}_A = \mathbf{1}_B(Y)$

How to know if $Z \in \sigma(Y)$: Let Z and Y be two random variables. Then

 $Z \in \sigma(Y)$ iff we can write Z = U(Y), with $U \in \mathcal{B}(\mathbb{R})$. Note: U: $\mathbb{R} \to \mathbb{N}$ is a generic measurable function

Conditioning a r.v by a discrete r.v $\sigma(\gamma)$ is generated by $\{x_i, j \ge i\}$ where $\mathcal{L}_{i=}(\gamma = y_i)$

Example 4: Whenever X and Y are discrete random variables \hookrightarrow Computation of $\mathbf{E}[X|Y]$ can be handled as in example 3.

More specifically:

- Assume $Y \in E$ with $E = \{y_i; i \ge 1\}$
- Hypothesis: $\mathbf{P}(Y = y_i) > 0$ for all $i \ge 1$.

Then $\mathbf{E}[X|Y] = h(Y)$ with $h: E \to \mathbb{R}$ defined by:

$$h(y) = \frac{\mathsf{E}[X \mathbf{1}_{(Y=y)}]}{\mathsf{P}(Y=y)}.$$

Typical example of E: E = {1,...,6}

 $r.v. X: \mathcal{D} \longrightarrow E$

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clain: E[XIY]= h(Y) where $h(y) = \frac{E[X \ 1_{X=y}]}{P(Y=y)} = E[X|Y=y]$ baby anditionina we check $z = h(Y) \in \sigma(Y)$ (*i*) >True since 2 is form h(x) for of the measurable Note h: F -> R

(ic) we should check $2 = h(\gamma)$ $E[2 \mathbf{1}_{A}] = E[\times \mathbf{1}_{A}]$ for $\forall A \in \sigma(Y) = \sigma(\Omega_i, j \ge 1)$ where $\Omega_{i} = (Y = Y_{i})$ we are thus reduced to prove $E[2 1(y=y_{s})] = E[X 1(y=y_{s})]$ 4 5 21

 $E[2 1(y_{\pm}y_{i})] = E[X 1(y_{\pm}y_{i})]$ $\sum_{i=1}^{\infty} h(y_i) \mathbf{1}(y_i)$ z = h(Y) =Recall EIX 1(x=yi)] 1(/=y:) P(Y=y:) E[2 1(Y=y;)] = 0 if i + s Then E[×1(+++++)] = 2 $E\left[1(y=y_{i}) \quad 1(y=y_{i})\right]$ $P(Y = y_i)$ $P(Y=y_i)$ 1 (i=i) $E[X = (y_{z})]$ $\mathbb{P}(Y=y_i) = \mathbb{E}[X \mathbf{1}_{Y=y_i}]$ P(Y= y;)" (ic) Gerified

Conditioning a r.v by a continuous r.v

Example 5: Let (X, Y) couple of real random variables with measurable density $f : \mathbb{R}^2 \to \mathbb{R}_+$. We assume that

$$\int_{\mathbb{R}} f(x,y) dx > 0, \quad \text{for all } y \in \mathbb{R}.$$

Let $g : \mathbb{R} \to \mathbb{R}$ a measurable function such that $g(X) \in L^1(\Omega)$. Then $\mathbf{E}[g(X)|Y] = h(Y)$, with $h : \mathbb{R} \to \mathbb{R}$ defined by:

$$h(y) = \frac{\int_{\mathbb{R}} g(x) f(x, y) dx}{\int_{\mathbb{R}} f(x, y) dx}.$$

= $E[g(X)|Y=y]$ from baby conditioning

Heuristic proof

Formally one can use a conditional density:

$$\mathbf{P}(X = x | Y = y)'' = "\frac{\mathbf{P}(X = x, Y = y)}{\mathbf{P}(Y = y)} = \frac{f(x, y)}{\int f(x, y) dx},$$

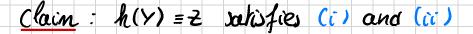
Integrating against this density we get:

$$\mathbf{E}[g(X)|Y = y] = \int g(x)\mathbf{P}(X = x|Y = y) dx$$
$$= \frac{\int g(x)f(x,y)dx}{\int f(x,y)dx}.$$

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Recall: h(y1= Jag(z) f(z,y) dz $J_{R} \neq (4y) dz$



(c) $z \in \sigma(y)$, since z = h(y)with h measurable

(ii) We should prove

$E[21_A] = E[X1_A] \forall A \in \sigma(Y)$

Generic A: 1A = 1B(Y)

In fact it is enough to prove E[Z Y(Y)] = E[X Y(Y)] meaningle for any $\psi \in \mathcal{B}_{\delta}(\mathbb{R})$ $E[Z \psi(Y)] = E[h(Y) \psi(Y)]$ = $\int_{\mathbb{R}^2} h(y) \psi(y) f(z,y) dz dy$ \downarrow Enough: EZZ $I_{B}(\gamma)$ = E[× $I_{B}(\gamma)$]

 $E[Z \psi(Y)]$ $= \int_{\mathbb{R}^2} h(y) \, \psi(y)$

4(z,y) dz dy $\int_{\mathbb{R}^{2}} \int_{\mathbb{R}} g(z,y) f(z,y) dz = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2}} \frac{1$

Fillowi = $\int dy dz f(z,y) g(z,y) \psi(y) \int f(z,y) dz$ joint density $\int f(u,y) du^{2}$

$= \int g(z, y) \psi(y) f(z, y) dz dy$

= E[g(x) +(x)] -> (ii) verified

Rigorous proof Strategy: Check (i) and (ii) in the definition for the r.v h(Y). (i) If $h \in \mathbb{B}(\mathbb{R})$, we have seen that $h(Y) \in \sigma(Y)$. (ii) Let $A \in \sigma(Y)$ Then

$$A = \left\{ \omega; Y(\omega) \in B \right\} \implies \mathbf{1}_A = \mathbf{1}_B(Y)$$

Thus

$$\begin{aligned} \mathbf{E}[h(Y)\mathbf{1}_{A}] &= \mathbf{E}[h(Y)\mathbf{1}_{B}(Y)] \\ &= \int_{B} \int_{\mathbb{R}} h(y)f(x,y)dxdy \\ &= \int_{B} dy \int_{\mathbb{R}} \left\{ \frac{\int g(z)f(z,y)dz}{\int f(z,y)dz} \right\} f(x,y)dx \\ &= \int_{B} dy \int g(z)f(z,y)dz = \mathbf{E}[g(X)\mathbf{1}_{B}(Y)]. \end{aligned}$$

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