<u>Aim</u>: If $X \in \mathcal{L}^1(\mathbb{P})$, $\mathcal{F}$ σ-algebra

existence + $\boxed{\text{uniqueness}}$ of $\mathbb{E}[X \mid \mathcal{F}]$?

$\hookrightarrow$ proved

<u>Tool</u>: Consider $\nu \ll \mu$. Then

$\exists f \geq 0$, measurable such that for all $g \in \mathcal{B}_b$

$$\nu(g) = \mu(f g)$$

(In particular $\nu(\mathbb{1}_A) = \mu(f \mathbb{1}_A)$)

Conditional expectation: existence

Proposition 12.

On the probability space $(\Omega, \mathcal{F}_0, \mathbf{P})$ consider

- A σ-algebra $\mathcal{F} \subset \mathcal{F}_0$.
- $X \in \mathcal{F}_0$ such that $\mathbf{E}[|X|] < \infty$.

Then the random variable

$\mathbf{E}[X|\mathcal{F}]$ satisfies

(i) $\overbrace{E[X|F]}^{Y} \in L^1$

(ii) $E[Y 1_A] = E[X 1_A]$ $\forall A \in F$

exists and is uniquely defined.

Proof Hyp: $X \geq 0$

(1) Define a measure ν on $(\Omega, \mathcal{F})$ by setting

$$\nu(A) = \mathbb{E}[X \mathbb{1}_A], \quad \forall A \in \mathcal{F}$$

Claim: ν is a measure, i.e

(a) $\nu(\emptyset) = 0$ disjoint union

(b) $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu(A_i)$

(c) $\nu(A) \geq 0 \quad \forall A \in \mathcal{F}$

Here (a) and (c) are clear.
Moreover, if A_i's are disjoint

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{E}\left[X \, \mathbb{1}_{\bigcup_{i=1}^{\infty} A_i}\right]$$

$$\overset{\text{disjoint}}{=} \mathbb{E}\left[X \sum_{i=1}^{\infty} \mathbb{1}_{A_i}\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{\infty} X \, \mathbb{1}_{A_i}\right]$$

$$\overset{\text{Fubini}}{=} \sum_{i=1}^{\infty} \mathbb{E}\left[X \, \mathbb{1}_{A_i}\right]$$

$$= \sum_{i=1}^{\infty} \nu(A_i) \quad \Rightarrow \text{(b) verified}$$

Thus $\boxed{\nu \text{ is a measure}}$

Recall : $\nu(A) = \mathbb{E}[X \mathbb{1}_A]$

Take $\mu = \mathbb{P}$. μ is a probability on $(\Omega, \mathcal{F}_0)$, but also on $(\Omega, \mathcal{F})$

We can write $\mu(A) = \mathbb{E}[\mathbb{1}_A]$

Question : Do we have $\boxed{\nu \ll \mu}$?

If $\mu(A) = 0$, then $\mathbb{1}_A = 0$ a.s.

$\Rightarrow X \mathbb{1}_A = 0$ a.s.

Yes!

$\Rightarrow \nu(A) = \mathbb{E}[X \mathbb{1}_A] = 0$

Radon-Nykodym: Since $\nu \ll \mu$,
$\exists\, Y \in \mathcal{F}$ s.t. $\forall A \in \mathcal{F}$

$$\nu(\mathbb{1}_A) = \mathbb{E}[Y \mathbb{1}_A]$$

Thus $Y \in \mathcal{F}$ and for all $A \in \mathcal{F}$

$$\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[Y \mathbb{1}_A]$$

Conclusion: $\boxed{Y = \mathbb{E}[X \mid \mathcal{F}]}$

Proof of existence

Hypothesis: We have

- A σ-algebra $\mathcal{F} \subset \mathcal{F}_0$.
- $X \in \mathcal{F}_0$ such that $\mathbf{E}[|X|] < \infty$.
- $X \geqslant 0$.

Defining two measures: we set

1. $\mu = P$, measure on $(\Omega, \mathcal{F})$.
2. $\nu(A) \equiv \mathbf{E}[X \mathbf{1}_A] = \int_A X \, d\mathbf{P}$.

Then ν is a measure (owing to Beppo-Levi).

Proof of existence (2)

Absolute continuity: we have

$$\begin{aligned} \mathbf{P}(A) = 0 &\Rightarrow \mathbf{1}_A = 0 \quad P\text{-a.s.} \\ &\Rightarrow X\,\mathbf{1}_A = 0 \quad P\text{-a.s.} \\ &\Rightarrow \nu(A) = 0 \end{aligned}$$

Thus $\nu \ll P$

Conclusion: invoking Radon-Nykodym, there exists $f \in \mathcal{F}$ such that, for all $A \in \mathcal{F}$, we have $\nu(A) = \int_A f \, d\mathbf{P}$.
$\hookrightarrow$ We set $f = \mathbf{E}[X|\mathcal{F}]$.

Outline

1. Definition
   - Baby conditional distributions: discrete case
   - Baby conditional distributions: continuous case
   - Definition with measure theory
2. Examples
3. Existence and uniqueness
4. Conditional expectation: properties
5. Conditional expectation as a projection
6. Conditional regular laws

Linearity, expectation

Proposition 13.

Let $X \in L^1(\Omega)$. Then

$$\mathbf{E}\left\{\overbrace{\mathbf{E}[X|\mathcal{F}]}^{\text{still a r.v.}}\right\} = \mathbf{E}[X].$$

Proposition 14.

Let $\alpha \in \mathbb{R}$, and $X, Y \in L^1(\Omega)$. Then

$$\mathbf{E}[\alpha X + Y|\mathcal{F}] = \alpha\,\mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}] \quad \text{a.s.}$$

Proof of linearity. Set

$$z = \alpha \mathbb{E}[X \mid \mathcal{F}] + \mathbb{E}[Y \mid \mathcal{F}]$$

We want to prove (i) and (ii) for z

(i) $\boxed{z \in \mathcal{F}}$, since

$$\mathbb{E}[X \mid \mathcal{F}] \in \mathcal{F}, \quad \mathbb{E}[Y \mid \mathcal{F}] \in \mathcal{F}$$

and z is a linear combination of the two.

$$Z = \alpha\, \mathbb{E}[X \mid \mathcal{F}] + \mathbb{E}[Y \mid \mathcal{F}]$$

(ii) Take $A \in \mathcal{F}$. Then

$$\mathbb{E}[Z \mathbb{1}_A]$$

$$= \mathbb{E}\Big\{\big(\alpha\, \overbrace{\mathbb{E}[X \mid \mathcal{F}]}^{\text{r.v.}} + \overbrace{\mathbb{E}[Y \mid \mathcal{F}]}^{\text{r.v.}}\big)\mathbb{1}_A\Big\}$$

$\mathbb{E}$ linear

$$= \alpha\, \mathbb{E}\{\mathbb{E}[X \mid \mathcal{F}]\mathbb{1}_A\} + \mathbb{E}\{\mathbb{E}[Y \mid \mathcal{F}]\mathbb{1}_A\}$$

(ii)

$$= \alpha\, \mathbb{E}[X \mathbb{1}_A] + \mathbb{E}[Y \mathbb{1}_A]$$

linearity

$$= \mathbb{E}[(\alpha X + Y)\mathbb{1}_A]$$

$\Rightarrow$ $\boxed{\text{(ii) verified}}$

Proof that $\mathbb{E}\{\underbrace{\mathbb{E}[X \mid \mathcal{F}]}_{Y}\} = \mathbb{E}[X]$

We have that $\forall A \in \mathcal{F}$,

$$\mathbb{E}[Y \mathbb{1}_A] = \mathbb{E}[X \mathbb{1}_A]$$

In particular, $\Omega \in \mathcal{F}$. We get

$$\mathbb{E}[Y \mathbb{1}_\Omega] = \mathbb{E}[X \mathbb{1}_\Omega]$$

$$\Rightarrow \quad \boxed{\mathbb{E}[Y] = \mathbb{E}[X]}$$

Proof

Strategy: Check (i) and (ii) in the definition for the r.v

$$Z \equiv \alpha\, \mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}].$$

Verification: we have

(i) Z is a linear combination of $\mathbf{E}[X|\mathcal{F}]$ and $\mathbf{E}[Y|\mathcal{F}]$
$\hookrightarrow Z \in \mathcal{F}$.

(ii) For all $A \in \mathcal{F}$, we have

$$\begin{aligned}
\mathbf{E}[Z\,\mathbf{1}_A] &= E\Big\{(\alpha \mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}])\,\mathbf{1}_A\Big\} \\
&= \alpha E\Big\{\mathbf{E}[X|\mathcal{F}]\,\mathbf{1}_A\Big\} + E\Big\{\mathbf{E}[Y|\mathcal{F}]\,\mathbf{1}_A\Big\} \\
&= \alpha \mathbf{E}[X\,\mathbf{1}_A] + \mathbf{E}[Y\,\mathbf{1}_A] \\
&= \mathbf{E}[(\alpha X + Y)\,\mathbf{1}_A].
\end{aligned}$$

Monotonicity

Proposition 15.

Let $X, Y \in L^1(\Omega)$ such that $X \leqslant Y$ almost surely. We have

$$\mathbf{E}[X|\mathcal{F}] \leqslant \mathbf{E}[Y|\mathcal{F}],$$

almost surely.

Proof: Along the same lines as proof of uniqueness for the conditional expectation.For instance if we set

$$A_\varepsilon = \{\mathbf{E}[X|\mathcal{F}] - \mathbf{E}[Y|\mathcal{F}] \geqslant \varepsilon > 0\},$$

then it is readily checked that

$$\mathbf{P}(A_\varepsilon) = 0.$$

Proof. Take $X \leq Y$. We want to prove that

$$\boxed{\mathbb{E}[X \mid \mathcal{F}] \leq \mathbb{E}[Y \mid \mathcal{F}]}$$

It is enough to prove that $\forall \varepsilon > 0$, if

$$A_\varepsilon = \left(\mathbb{E}[X \mid \mathcal{F}] - \mathbb{E}[Y \mid \mathcal{F}] \geq \varepsilon\right),$$

then

$$\boxed{\mathbb{P}(A_\varepsilon) = 0}$$

$A_\varepsilon = \left(\mathbb{E}[X \mid \mathcal{F}] - \mathbb{E}[Y \mid \mathcal{F}] \geq \varepsilon \right) \in \mathcal{F}$

We have

$$\varepsilon \, \mathbb{P}(A_\varepsilon) \leq \mathbb{E}\left\{ \overbrace{\left(\mathbb{E}[X \mid \mathcal{F}] - \mathbb{E}[Y \mid \mathcal{F}] \right)}^{\geq \varepsilon \text{ on } A_\varepsilon} \mathbb{1}_{A_\varepsilon} \right\}$$

$$\overset{\text{Linearity of } \mathbb{E}[\cdot \mid \mathcal{F}]}{=} \mathbb{E}\left\{ \mathbb{E}[(X-Y) \mid \mathcal{F}] \, \mathbb{1}_{A_\varepsilon} \right\}$$

$$\overset{(ii)}{=} \mathbb{E}\Big[\overbrace{(X-Y)}^{\leq 0 \text{ since } X \leq Y} \mathbb{1}_{A_\varepsilon} \Big]$$

$$\leq 0$$

Thus $\varepsilon \, \mathbb{P}(A_\varepsilon) \leq 0$

$$\Longrightarrow \boxed{\mathbb{P}(A_\varepsilon) = 0}$$

Monotone convergence

Proposition 16.

Let $\{X_n; n \geq 1\}$ be a sequence of random variables such that

- $X_n \geqslant 0$ $\quad X_{n+1} \geqslant X_n$ a.s. and $X_n \to X$ a.s.
- $\boxed{X_n \nearrow X}$ almost surely
- $\mathbf{E}[X] < \infty$.

Then

$$\mathbf{E}[X_n|\mathcal{F}] \nearrow \mathbf{E}[X|\mathcal{F}]. \quad \text{a.s.}$$

Recall: $X_n \nearrow X$ a.s. We set $Y_n = X - X_n \geq 0$. We have $Y_n \searrow 0$ a.s.

Set $Z_n = \mathbb{E}[Y_n \mid \mathcal{F}]$. We have

(i) Since $\mathbb{E}[\cdot \mid \mathcal{F}]$ is monotone and $Y_n \searrow$, we have

$Z_n \searrow$ ($Z_{n+1} \leq Z_n$ a.s.)

(ii) $Y_n \geq 0 \Rightarrow \overbrace{\mathbb{E}[Y_n \mid \mathcal{F}]}^{Z_n} \geq 0$

Thus $\boxed{\exists\, Z_\infty \geq 0 \text{ s.t. } Z_n \searrow Z_\infty}$

Summary : $Y_n \equiv X - X_n$
$Z_n \equiv \mathbb{E}[\,Y_n \mid F\,]$
$Z_n \searrow Z_\infty$ a.s.
$Z_\infty \geq 0$ a.s.

We wish to prove : $Z_\infty = 0$

Rmk: If $Z_\infty \geq 0$, in order to prove that $Z_\infty = 0$, it is enough to prove $\mathbb{E}[\,Z_\infty\,] = 0$

However $\mathbb{E}[\,Z_n\,] = \mathbb{E}\{\,\mathbb{E}[\,Y_n \mid F\,]\,\}$
$= \mathbb{E}[\,Y_n\,]$

Beppo-Levi

In addition, $Y_n \searrow 0 \Rightarrow \mathbb{E}[\,Y_n\,] \searrow 0$

We have obtained

$$\mathbb{E}[Z_n] \searrow 0$$

In addition,

$$Z_n \searrow Z_\infty \quad a.s.$$

Beppo-Levi

$$\Rightarrow \quad \mathbb{E}[Z_n] \searrow \mathbb{E}[Z_\infty]$$

Thus

$$\mathbb{E}[Z_\infty] = 0 \Rightarrow Z_\infty = 0 \quad a.s.$$

$$\Rightarrow \boxed{\text{Beppo-Levi for } \mathbb{E}[\,\cdot \mid \mathcal{F}]}$$

Proof

Strategy: Set $Y_n \equiv X - X_n$. We are reduced to show $Z_n \equiv \mathbf{E}[Y_n|\mathcal{F}] \searrow 0$.

Existence of a limit: $n \mapsto Y_n$ is decreasing, and $Y_n \geqslant 0$
$\hookrightarrow$ Z_n is decreasing and $Z_n \geqslant 0$.
$\hookrightarrow$ Z_n admits a limit a.s, denoted by Z_∞.

Aim: Show that $Z_\infty = 0$.

Proof (2)

Expectation of Z_∞: we will show that $\mathbf{E}[Z_\infty] = 0$. Indeed

- X_n converges a.s. to X.
- $0 \leqslant X_n \leqslant X \in L^1(\Omega)$.

Thus, by dominated convergence, $\mathbf{E}[X_n] \to \mathbf{E}[X]$.

We deduce:

- $\mathbf{E}[Y_n] \to 0$
- Since $\mathbf{E}[Y_n] = \mathbf{E}[Z_n]$, we also have $\mathbf{E}[Z_n] \to 0$.
- By monotone convergence, we have $\mathbf{E}[Z_n] \to \mathbf{E}[Z_\infty]$

This yields $\mathbf{E}[Z_\infty] = 0$.

Conclusion: $Z_\infty \geq 0$ and $\mathbf{E}[Z_\infty] = 0$
$\hookrightarrow Z_\infty = 0$ almost surely.