Probability space

Probability space: $(\Omega, \mathcal{F}, \mathbf{P})$ with

- Ω set
- $\mathcal{F}$ a σ-algebra
- $\mathbf{P}$ probability measure

Complete probability space

Idea: Any set which has "$\mathbb{P}(A)=0$" is declared measurable, so that $\mathbb{P}(A)$ can be computed

Hypothesis: We assume that **P** is complete, i.e we don't know that $B \in \mathcal{F}$

$$A \in \mathcal{F} \text{ such that } \mathbf{P}(A) = 0, \text{ and } B \subset A$$
$$\Longrightarrow$$
$$B \in \mathcal{F} \text{ and } \mathbf{P}(B) = 0.$$

Remark: A probability can always be completed

Simple examples (1)

Tossing 2 dice:

- $\Omega = \{1, 2, 3, 4, 5, 6\}^2$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathbf{P}(A) = \frac{|A|}{36}$

Uniform distribution on $[0, 1]$:

- $\Omega = [0, 1]$
- $\mathcal{F} = \mathcal{B}([0, 1])$
- $\mathbf{P} = \lambda$, Lebesgue measure

Simple examples (2)

Probability given by $\mathcal{N}(\mu, \sigma^2)$ with density $f(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Gaussian law on $\mathbb{R}$:

- $\Omega = \mathbb{R}$
- $\mathcal{F} = \mathcal{B}(\mathbb{R})$
- $\mathbf{P}(A) = \frac{1}{(\cancel{2\pi})^{1/2}} \int_A e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$, for $A \in \mathcal{F}$

$(2\pi\sigma^2)^{\frac{1}{2}}$

Rmk In the 3 examples we have seen, the natural Ω was simple enough ($\{1,..,6\}^2$, $[0,1]$, $\mathbb{R}$)

Generally speaking, Ω can be very complicated and it is unspecified.

Example : For experiment given by rolling 2 dice, we have taken $\Omega = \{1, .., 6\}^2$. However, Ω should be (maybe)

$$\Omega = \{ \text{path of 2 diced rolled} \}$$

"Typical" example for this course

$\ell^p = \{$ sequences $(u_n)_{n\geq 1}$; $\sum_{n=1}^{\infty} |u_n|^p < \infty \}$

Proposition 1.

Let $\Omega = \ell^p$ with $p \in (1, \infty)$. We set:

$$d(u, v) = \left(\sum_{n\geq 1} |u_n - v_n|^p \right)^{1/p} .$$

Then Ω is a complete metric separable space.

complete: If v^n Cauchy sequence in ℓ^p, then $v^n \to v$ and $v \in \ell^p$

separable: $\exists \{ v^n ; n \geq 1 \}$ dense in ℓ^p

Random variables

Definition 2.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ complete probability space
- A function $X : \Omega \to \mathbb{R}$

Then

X is said to be a random variable if X is measurable

Def: If $X:(\Omega,\mathcal{F}) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$, we say that X is measurable if

$\forall A \in \mathcal{B}(\mathbb{R})$ (or $\forall A$ open in $\mathbb{R}$),

$$X^{-1}(A) \in \mathcal{F}$$

$$= \{\omega \in \Omega \,;\, X(\omega) \in A\}$$

Rmk Usually we write $\mathbb{P}(X \in A)$ instead of

we need this to be in $\mathcal{F}$ if we want the probability to make sense

$$\mathbb{P}(X \in A) = \mathbb{P}(\{\omega \in \Omega \,;\, X(\omega) \in A\})$$

Rmk Related to the notion of measurability, we have

- Simple random variables
- $L^p(\Omega)$ spaces
- Monotone convergence
- Dominated convergence

Independence (1)

Hyp: J is countable

Independence of r.v: Let $(X_j)_{j\in J}$ r.v in $\mathbb{R}^n$.
Those r.v are said to be independent if for all $m \geq 2$:

- For every $j_1, \ldots, j_m \in J$, the r.v $(X_{j_1}, \ldots, X_{j_m})$ are $\perp\!\!\!\perp$
- Otherwise stated: for all $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}^n)$ we have

$$\mathbf{P}\left(X_{j_1} \in A_1, \ldots, X_{j_m} \in A_m\right) = \prod_{k=1}^{m} \mathbf{P}\left(X_{j_k} \in A_k\right)$$

Independence (2)

If $B \in \mathcal{F}_j$, then $B \in \mathcal{F}$

Independence of σ-algebras: Let $(\mathcal{F}_j)_{j\in J}$ σ-algebras, $\mathcal{F}_j \subset \mathcal{F}$.
Those σ-algebras are said to be independent if for all $m \geq 2$:

- For all $j_1, \ldots, j_m \in J$, the σ-algebras $(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_m})$ are $\perp\!\!\!\perp$
- Otherwise stated: for all $B_1 \in \mathcal{F}_{j_1}, \ldots, B_m \in \mathcal{F}_{j_m}$ we have

$$\underbrace{\mathbf{P}\left(\bigcap_{k=1}^{m} B_k\right) = \prod_{k=1}^{m} \mathbf{P}\left(B_k\right)}$$

The B_k's should be $\perp\!\!\!\perp$ as events

π-systems and λ-systems

π-system: Let $\mathcal{P}$ family of subsets of Ω. $\mathcal{P}$ is a π-system if:

$$A, B \in \mathcal{P} \quad \Longrightarrow \quad A \cap B \in \mathcal{P}$$

λ-system: Let $\mathcal{L}$ family of subsets of Ω. $\mathcal{L}$ is a λ-system if:

1. $\Omega \in \mathcal{L}$
2. If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$
3. If for $j \geq 1$ we have:
   - $A_j \in \mathcal{L}$
   - $A_j \cap A_i = \emptyset$ if $j \neq i$

   Then $\cup_{j\geq 1} A_j \in \mathcal{L}$

In applications:

(i) $\mathcal{L}$ is a family of sets which satisfies a "nice" property

(ii) $\mathcal{P}$ is a smaller set for which the property is easy to prove

Game: extend the property from $\mathcal{P}$ to $\mathcal{L}$

Dynkin's π-λ lemma

Proposition 3.

Let $\mathcal{P}$ et $\mathcal{L}$ such that:

- $\mathcal{P}$ is a π-system
- $\mathcal{L}$ is a λ-system
- $\mathcal{P} \subset \mathcal{L}$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$

$\sigma(\mathcal{P}) \equiv$ σ- algebra generated by $\mathcal{P}$

Application of Dynkin's π-λ lemma

μ_{X_j} is a probab. measure on $\mathbb{R}^m$, which is the distribution 'or law of X_j

$\mu_{X_j}(A) \equiv \mathbb{P}(X_j \in A)$

Proposition 4.

Let:

- $X_1, \dots, X_n$ r.v with values in $\mathbb{R}^m$.
- $X \equiv (X_1, \dots, X_n) \in \mathbb{R}^{m \times n}$.
- $\boxed{\mu_{X_j} = \mathcal{L}(X_j)}$ and $\mu_X = \mathcal{L}(X)$.

Then the two following assertions are equivalent:

1. $X_1, \dots, X_n$ are independent
2. $\mu_X = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$ on $\mathcal{B}(\mathbb{R}^{m \times n})$

The law of X is the product of the marginal laws

Def Let μ, ν be measures on $\mathbb{R}^m$

Then $\mu \otimes \nu$ is a measure on

$\mathbb{R}^m \times \mathbb{R}^m$ such that if A, B are

in $\mathcal{B}(\mathbb{R}^m)$, we have

$$\mu \otimes \nu (A \times B) = \mu(A) \, \nu(B)$$

$\hookrightarrow$ Then extension to $C \in \mathcal{B}(\mathbb{R}^m \times \mathbb{R}^m)$

$X = (X_1, \dots, X_n)$

Aim Consider two measures on $\mathbb{R}^{n \times n}$:

$$\mu_1 = \mu_X \quad , \quad \mu_2 = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$$

we want to prove that $\mu_1 = \mu_2$, i.e

$$\mu_1(B) = \mu_2(B) \quad \forall\, B \in \mathcal{B}(\mathbb{R}^{n \times m})$$

Thus natural candidate for $\mathcal{L}$

$$\mathcal{L} = \{ B \in \mathcal{B}(\mathbb{R}^{n \times m}) ; \ \mu_1(B) = \mu_2(B) \}$$

Natural candidate for $\mathcal{P}$

$$\mathcal{P} = \{ \text{product sets} \}$$

$$= \{ A \in \mathcal{B}(\mathbb{R}^{m \times n}) ;\ A = \overbrace{A_1}^{\text{all in } \mathcal{B}(\mathbb{R}^m)} \times \cdots \times A_n \}$$

If A is a product set, we have

$$\mu_1(A) = \mu_X(A) = \mathbb{P}(X \in A)$$

$$= \mathbb{P}(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n)$$

$$\overset{\perp\!\!\!\perp}{=} \mathbb{P}(X_1 \in A_1) \cdots \mathbb{P}(X_n \in A_n)$$

$$= \mu_{X_1}(A_1) \cdots \mu_{X_n}(A_n) = \mu_2(A)$$

Now we prove

(i) $\mathcal{P}$ is a π-system

(ii) $\mathcal{L}$ is a d-system

check at home

We conclude

$$\mathcal{L} \supset \sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}^{n \times m})$$

We thus get

$$\mu_1(B) = \mu_2(B) \quad \forall\, B \in \mathcal{B}(\mathbb{R}^{n \times m})$$

Proof (1)

Definition of two systems: We set

$$\mu_1 = \mu_X, \quad \text{and} \quad \mu_2 = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n},$$

and

$$\begin{aligned}
\mathcal{P} &\equiv \left\{ A \in \mathcal{B}(\mathbb{R}^{m\times n});\, A = A_1 \times \cdots \times A_n, \text{ where } A_j \in \mathcal{B}(\mathbb{R}^m) \right\} \\
\mathcal{L} &\equiv \left\{ B \in \mathcal{B}(\mathbb{R}^{m\times n});\, \mu_1(B) = \mu_2(B) \right\}.
\end{aligned}$$

Proof (2)

Application of Dynkin's lemma: We have

- $\mathcal{P}$ is a π-system
- $\mathcal{L}$ is a λ-system
- $\mu_1(C) = \mu_2(C)$ for all $C \in \mathcal{P}$

Thus $\sigma(\mathcal{P}) \subset \mathcal{L}$, and $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}^{m\times n})$

Conclusion:

$$\mu_1(A) = \mu_2(A) \text{ for all } A \in \mathcal{B}(\mathbb{R}^{m\times n})$$