Conditional expectation and independence

Theorem 22.

Let

- X, Y two independent random variables
- $\alpha : \mathbb{R}^2 \to \mathbb{R}$ such that $\alpha(X, Y) \in L^1(\Omega)$

We set, for $x \in \mathbb{R}$,

$$g(x) = \mathbf{E}[\alpha(x, Y)].$$

$\mathcal{F} = \sigma(X)$

Then $Y \perp\!\!\!\perp X$

$$\mathbf{E}[\alpha(X, Y)|X] = g(X).$$

Proof: with 4 steps method applied to α. Step 1: Take $\alpha(x,y) = \mathbb{1}_A(x)\, \mathbb{1}_B(y)$

Generalization of the previous theorem

Theorem 23.

Let

- $\mathcal{F} \subset \mathcal{F}_0$
- $X \in \mathcal{F}$ and $Y \perp\!\!\!\perp \mathcal{F}$ two random variables
- $\alpha : \mathbb{R}^2 \to \mathbb{R}$ such that $\alpha(X, Y) \in L^1(\Omega)$

We set, for $x \in \mathbb{R}$,

$$g(x) = \mathbf{E}[\alpha(x, Y)].$$

Then

$$\mathbf{E}[\alpha(X, Y)|\mathcal{F}] = g(X).$$

Outline

Orthogonal projection

Application: $H = L^2(\Omega)$

In a problem: $\langle X, Y \rangle = E[XY]$ is an inner product

Definition: Let

Conclusion: $L^2(\Omega)$ is a Hilbert space

- H Hilbert space
  $\hookrightarrow$ complete vectorial space equipped with inner product.
- F closed subspace of H.

Then, for all $x \in H$

- There exists a unique $y \in F$, denoted by $y = \pi_F(x)$

Satisfying one of the equivalent conditions (i) or (ii).

(i) For all $z \in F$, we have $\langle x - y, z \rangle = 0$.

(ii) For all $z \in F$, we have $\|x - y\|_H \leqslant \|x - z\|_H$.

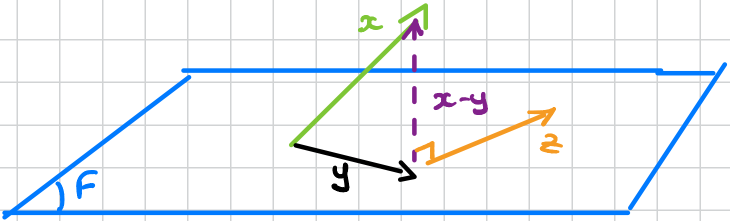
$\pi_F(x)$ is denoted orthogonal projection of x onto F.

Question: If H is a Hilbert space, what is a closed subspace F of H?

Subspace: F is stable by $+$ and multiplication by a scalar

Closed: If $x_n \in F$ and $x_n \to x$ in H

then $x \in F$



$y = \pi_F(x)$ satisfies ((i) $\Leftrightarrow$ (ii))

(i) $\langle x-y, z \rangle = 0 \quad \forall z \in F$

(ii) $\|x-y\|_H = \min\{\|x-z\|;\ z \in F\}$

$\hookrightarrow$ y is the best approximation of x in F

Conditional expectation and projection

Theorem 24.

Consider

- The space $L^2(\mathcal{F}_0) \equiv \{Y \in \mathcal{F}_0;\, \mathbf{E}[Y^2] < \infty\}$.
- $X \in L^2(\mathcal{F}_0)$.
- $\mathcal{F} \subset \mathcal{F}_0$

Then

1. $L^2(\mathcal{F}_0)$ is a Hilbert space
   $\hookrightarrow$ Inner product $\langle X, Y \rangle = \mathbf{E}[XY]$.
2. $L^2(\mathcal{F})$ is a closed subspace of $L^2(\mathcal{F}_0)$.
3. $\pi_{L^2(\mathcal{F})}(X) = \mathbf{E}[X|\mathcal{F}]$. $\Rightarrow$ best approximation of X in $L^2(\mathcal{F})$

② $L^2(\mathbb{F})$ is a closed subspace of $L^2(\mathbb{G})$

Subspace: If W, Z are two r.v. in $L^2(\mathbb{F})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then

$\alpha_1 W + \alpha_2 Z$ is a continuous function of (W, Z) $\Rightarrow$ it is $\mathbb{F}$-meas.

It is also in L^2, since L^2 is a vector space

Closed: Take $X_n \in L^2(\mathcal{F})$ s.t. $X_n \to X$ in $L^2(\mathcal{F}_0)$

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0$$

In this case we know that $X \in L^2(\mathcal{F}_0)$

In addition, if $X_n \xrightarrow{L^2} X$, $\exists\, n_k$ s.t. $X_{n_k} \to X$ a.s.

Since $X_{n_k} \in \mathcal{F}$, we have $X \in \mathcal{F}$.

Conclusion: $\boxed{L^2(\mathcal{F}) \text{ closed subspace}}$

$\mathbb{P}$ is a pb meas on both $(\Omega, \mathcal{F}_0)$ and $(\Omega, \mathcal{F})$

③ $\mathbb{E}[X \mid \mathcal{F}] = \pi_F(X)$, $F = L^2(\mathcal{F})$

For this, we need to prove that for all $z \in L^2(\mathcal{F})$, we have

$$\langle X - \mathbb{E}[X \mid \mathcal{F}], z \rangle = 0$$

$$\Leftrightarrow \quad \mathbb{E}\{(X - \mathbb{E}[X \mid \mathcal{F}])\, z\} = 0$$

We have

$$\mathbb{E}\{ (X - \mathbb{E}[X \mid \mathcal{F}]) \, Z \} = 0 \qquad \in L^2(\mathcal{F})$$

$$= \mathbb{E}[X Z] - \mathbb{E}\{ \mathbb{E}[X \mid \mathcal{F}] \, Z \}$$

$$= \mathbb{E}[X Z] - \mathbb{E}\{ \mathbb{E}[X Z \mid \mathcal{F}] \}$$

$$= \mathbb{E}[X Z] - \mathbb{E}[X Z]$$

$$= 0$$

$$\Longrightarrow \boxed{\mathbb{E}[X \mid \mathcal{F}] = \pi_{L^2(\mathcal{F})}(X)}$$

Proof

Proof of 2:
If $X_n \to X$ in $L^2 \Rightarrow$ There exists a subsequence $X_{n_k} \to X$ a.s.
Thus, if $X_n \in \mathcal{F}$, we also have $X \in \mathcal{F}$.

Proof of 3: Let us check (i) in our definition of projection

Let $Z \in L^2(\mathcal{F})$.
$\hookrightarrow$ We have $\mathbf{E}[Z\,X|\mathcal{F}] = Z\,\mathbf{E}[X|\mathcal{F}]$, and thus

$$\mathbf{E}\{Z\,\mathbf{E}[X|\mathcal{F}]\} = \mathbf{E}\{\mathbf{E}[X\,Z|\mathcal{F}]\} = \mathbf{E}[X\,Z]\,,$$

which ensures (i) and $\mathbf{E}[X|\mathcal{F}] = \pi_{L^2(\mathcal{F})}(X)$.

Application to Gaussian vectors

Example: Let

- (X, Y) centered Gaussian vector in $\mathbb{R}^2$
- Hypothesis: $V(Y) > 0$.

Then

$$\mathbf{E}[X|Y] = \alpha Y, \quad \text{with} \quad \alpha = \frac{\mathbf{E}[X\,Y]}{V(Y)}.$$

Def (X,Y) is a centered Gaussian vector if $\forall\ \alpha, \beta \in \mathbb{R}$, the r.v.

$$(\alpha X + \beta Y) \sim \mathcal{N}(0, \sigma^2_{\alpha,\beta})$$

Example 1: $X \sim \mathcal{N}(0,\sigma_1^2)$, $Y \sim \mathcal{N}(0,\sigma_2^2)$ and $X \perp\!\!\!\perp Y$. Then

$$\alpha X + \beta Y \sim \mathcal{N}(0, \alpha^2\sigma_1^2 + \beta^2\sigma_2^2)$$

$\Rightarrow$ $\boxed{(X,Y) \text{ Gaussian vector}}$

Example 2 $X \sim \mathcal{N}(0,1)$

ε s.t. $\mathbb{P}(\varepsilon = \pm 1) = \frac{1}{2}$, $\varepsilon \perp\!\!\!\perp X$

and $Y = \varepsilon X$. Then

check

(i) $X \sim \mathcal{N}(0,1)$ (ii) $Y \sim \mathcal{N}(0,1)$

(iii) $\boxed{(X,Y) \text{ not a Gaussian vector:}}$

Take $\alpha = 1$, $\beta = +1$. Then

$$\alpha X + \beta Y = X + Y = (1+\varepsilon) X$$

Indeed $\mathbb{P}((1+\varepsilon)X = 0) \geq \mathbb{P}(1+\varepsilon = 0)$

$= \frac{1}{2} \Rightarrow$ $\boxed{X+Y \text{ not Gauss.}}$

In our example, we assume (X, Y) is Gauss. vector

Claim: $\mathbb{E}[X \mid Y] = \alpha Y$
with $\alpha = \dfrac{\mathbb{E}[XY]}{V(Y)}$

We will prove that by proving that

$$\alpha Y = \Pi_{L^2(\sigma(Y))}(X)$$

i-e $\boxed{\forall z \in L^2(\sigma(Y)) \text{ we have } \mathbb{E}[(X - \alpha Y) z] = 0}$

$$\alpha = \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]}$$

Aim: prove

$\forall z \in L^2(\sigma(Y))$ we have

$$\mathbb{E}[(X - \alpha Y) z] = 0$$

Rmk: Any $z \in L^2(\sigma(Y))$ can be written as $z = \varphi(Y)$

Take $\varphi = \mathrm{Id}$. We wish

$$\mathbb{E}[(X - \alpha Y) Y] \overset{?}{=} 0$$

$$= \mathbb{E}[XY] - \alpha \, \mathbb{E}[Y^2] = 0$$

Proof

Step 1: We look for α such that

$$Z = X - \alpha Y \quad \Longrightarrow \quad Z \perp\!\!\!\perp Y.$$

Recall: If (Z, Y) is a Gaussian vector
$\hookrightarrow$ $Z \perp\!\!\!\perp Y$ iff $\mathrm{cov}(Z, Y) = 0$

Application: $\mathrm{cov}(Z, Y) = \mathbf{E}[Z\, Y]$. Thus

$$\mathrm{cov}(Z, Y) = \mathbf{E}[(X - \alpha Y)\, Y] = \mathbf{E}[X\, Y] - \alpha V(Y),$$

et

$$\mathrm{cov}(Z, Y) = 0 \quad \text{iff} \quad \alpha = \frac{\mathbf{E}[XY]}{V(Y)}.$$

Proof (2)

Step 2: We invoke (i) in the definition of π.
$\hookrightarrow$ Let $V \in L^2(\sigma(Y))$. Then

$$Y \perp\!\!\!\perp (X - \alpha Y) \quad \Longrightarrow \quad V \perp\!\!\!\perp (X - \alpha Y)$$

and

$$\mathbf{E}[(X - \alpha Y)\, V] = \mathbf{E}[X - \alpha Y]\, \mathbf{E}[V] = 0.$$

Thus

$$\alpha Y = \pi_{\sigma(Y)}(X) = \mathbf{E}[X|\, Y].$$