Outline

1. Definitions and first properties
2. Strategies and stopped martingales
3. Convergence
4. Convergence in L^p
5. Optional stopping theorems

Adaptation

Interpretation:

$\mathcal{F}_n \equiv$ " information up to time n "

Context: We are given

- A probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- A filtration $\{\mathcal{F}_n;\ n \geq 0\}$
  $\hookrightarrow$ Sequence of σ-algebras such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

Definition 1.

A sequence of random variables $\{X_n;\ n \geq 0\}$ is adapted if:

$$X_n \in \mathcal{F}_n.$$

$X_n \in \mathcal{F}_n$: $X_n \equiv$ function of the information we have today. Does not anticipate.

Martingales, Supermartingales, Submartingales

Definition 2.

We consider a sequence of random variables $X = \{X_n;\ n \geq 0\}$ such that

1. $\{X_n;\ n \geq 0\}$ is adapted.
2. $X_n \in L^1(\Omega)$ for all $n \geq 0$.

Then

- X is a martingale if $X_n = \mathbf{E}[X_{n+1} | \mathcal{F}_n]$.
- X is a supermartingale if $X_n \geq \mathbf{E}[X_{n+1} | \mathcal{F}_n]$.
- X is a submartingale if $X_n \leq \mathbf{E}[X_{n+1} | \mathcal{F}_n]$.

Interpretation

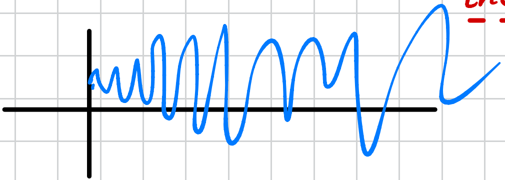
Sub $\geq$

Martingale: $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$

Best approx of X_{n+1} given the information we have at time n is X_n

→ We expect X_n = constant + fluctuations

increasing + fluctuation



Interpretation

Adaptation: The data X_n only depends on information until instant n.

Martingale: $n \mapsto X_n$ constant plus fluctuations.

Submartingale: $n \mapsto X_n$ increasing plus fluctuations.

Supermartingale: $n \mapsto X_n$ decreasing plus fluctuations.

Random walk

Definition: Let

- $\{Z_i;\ i \geq 1\}$ independent Rademacher r.v
  $\hookrightarrow \mathbf{P}(Z_i = -1) = \mathbf{P}(Z_i = 1) = 1/2$
- We set $X_0 = 0$, and for $n \geq 1$,

$$X_n = \sum_{i=1}^{n} Z_i.$$

X is called random walk in $\mathbb{Z}$.

Property: X is a martingale.

Proof $\quad X_n = \sum_{i=1}^{n} z_i$. Then

(i) If we take $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$,
then $X_n \in \mathcal{F}_n$

We can also take
$\mathcal{F}_n' = \sigma(z_1, \dots, z_n)$. Then

$$X_n = \sum_{i=1}^{n} z_i = \varphi(z_1, \dots, z_n) \Rightarrow X_n \in \mathcal{F}_n'$$

In fact we have $\mathcal{F}_n = \mathcal{F}_n'$

(ii) $X_n = \sum_{i=1}^{n} Z_i$, and $X_n \in L^1(\Omega)$:

$$\mathbb{E}[|X_n|] \leq \sum_{i=1}^{n} \mathbb{E}[|Z_i|] = n < \infty$$

(iii) $\mathbb{E}[X_{n+1} | \mathcal{F}_n]$

$$= \mathbb{E}\Big[\sum_{i=1}^{n} Z_i \,\Big|\, \mathcal{F}_n\Big] = \mathbb{E}[X_n + Z_{n+1} | \mathcal{F}_n]$$

$$\overset{\text{linearity}}{=} \mathbb{E}[\overbrace{X_n}^{\in \mathcal{F}_n} | \mathcal{F}_n] + \mathbb{E}[\overbrace{Z_{n+1}}^{\perp\!\!\!\perp \mathcal{F}_n} | \mathcal{F}_n]$$

$$= X_n + \overbrace{\mathbb{E}[Z_{n+1}]}^{0}$$

$$= X_n$$

$\boxed{(X_n) \text{ is a martingale}}$

Rmk We have seen 3 types of sequences

(i) independent r.v.

(ii) Markov chains

$$\mathbb{E}[\varphi(X_{n+1}) \mid \mathcal{F}_n] = \mathbb{E}[\varphi(X_{n+1}) \mid X_n]$$

(iii) Martingales

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$$