Continuous example

Exponential law case: Let

- $X \sim \mathcal{E}(1)$ and $Y \sim \mathcal{E}(1)$
- $X \perp\!\!\!\perp Y$

We set $S = X + Y$.

Then

CRL of X given S is $\mathcal{U}([0, S])$.

Step 1: Computing $E[\psi(X) \mid S]$

We will use a formula from our set of examples: if f is the joint density of (X, S), then

$$E[\psi(X) \mid S] = g(S)$$

where

$$g(s) = \frac{\int \psi(x) f(x,s)\, dx}{\int f(x,s)\, dx}$$

Step 0 : Compute f. Here

$X \sim \mathcal{E}(1)$, $Y \sim \mathcal{E}(1)$, $X \perp\!\!\!\perp Y$

In order to compute f, we evaluate

$$\mathbb{E}[\overset{\text{generic } C_b}{h}(X,S)] = \mathbb{E}[h(X,X+Y)]$$

$$= \int_{\mathbb{R}^2} h(x,x+y)e^{-x} \mathbb{1}_{\mathbb{R}_+}(x) e^{-y} \mathbb{1}_{\mathbb{R}_+}(y)\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty h(x, x+y)\, e^{-(x+y)}\, dx\, dy$$

Recall

$$\mathbb{E}[h(x,s)] = \int_0^\infty \int_0^\infty h(x, x+y)\, e^{-\overbrace{(x+y)}^{s}}\, dx\, dy$$

CV: $\omega = x \qquad s = x+y \qquad \begin{array}{l} x = \omega \\ y = s - \omega \end{array}$

Domain: $0 \leq \omega \leq s < \infty$

Jacobian: $J = \left| \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right| = 1$

Thus

$$\mathbb{E}[h(x,s)] = \int_0^\infty ds\, e^{-s} \int_0^s h(\omega, s)\, d\omega$$

$$= \int_0^\infty \int_0^\infty h(\omega, s)\, \overbrace{e^{-s}\, \mathbb{1}(0 \leq \omega \leq s < \infty)}^{= f(\omega, s)}\, d\omega\, ds$$

$f(x,s) = e^{-s} \mathbb{1}_{(0 < x \le s < \infty)}$

For a test function, $E[\varphi(X)|S] = g(S)$ with

$$g(s) = \frac{\int \varphi(x) f(x,s)\, dx}{\int f(x,s)\, dx}$$

$$= \frac{\int_{\mathbb{R}} \varphi(x)\, e^{-s}\, \mathbb{1}_{(0 \le x \le s)}\, dx}{\int_{\mathbb{R}} e^{-s}\, \mathbb{1}_{(0 \le x \le s)}\, dx}$$

$$= \frac{\cancel{e^{-s}} \int_0^s \varphi(x)\, dx}{\cancel{e^{-s}}\, s} = \frac{1}{s} \int_0^s \varphi(x)\, dx$$

<u>Conclusion</u> : $E[\varphi(X) \mid S] \overset{a.s.}{=} \frac{1}{S} \int_0^S \varphi(x)\, dx$

<u>Question</u>: is this well defined? $\rightarrow$ Yes, since

(i) $\mathbb{P}(S=0) = \int_0^0 s e^{-s}\, ds = 0$ cdf of S

(ii) $\mathbb{P}(S=\infty) = 1 - \lim_{x \to \infty} F(x) = 0$

<u>Back to CRL</u>

(i) For fixed φ, the quantity

$$\omega \mapsto \frac{1}{S(\omega)} \int_0^\infty \varphi(x)\, \mathbb{1}(x \le S(\omega))\, dx$$

is measurable

(ii) If ω fixed, the function

$$\varphi \in C_b(\mathbb{R}) \longmapsto \frac{1}{S(\omega)} \int_0^{S(\omega)} \varphi(x)\, dx$$

defines a distribution with density

$$f(x) = \frac{1}{S(\omega)} \mathbb{1}_{[0, S(\omega)]}(x)$$

We get a $\mathcal{U}([0, S(\omega)])$ distribution

$\boxed{\text{Thus } \mathcal{U}([0,S]) \text{ defines a CRL for } \mathcal{L}(X \mid S).}$

Continuous example

Proof: The joint density of (X, S) is given by

$$f(x,s) = e^{-s}\mathbf{1}_{\{0 \le x \le s\}}.$$

Let then $\psi \in \mathcal{B}_b(\mathbb{R}_+)$. Thanks to Example 5, we have

$$\mathbf{E}[\psi(X)|S] = u(S),$$

with

$$u(s) = \frac{\int_{\mathbb{R}_+} \psi(x) f(x,s)dx}{\int_{\mathbb{R}_+^2} f(x,s)dx} = \frac{1}{s}\int_0^s \psi(x)dx.$$

Proof

In addition, $S \neq 0$ almost surely, and thus if $A \in \mathcal{B}(\mathbb{R})$ we have:

$$\mathbf{E}[\psi(X)|S] = \frac{\int_0^S \psi(x)dx}{S}.$$

Considering the state space as $= \mathbb{R}_+$, $\mathcal{S} = \mathcal{B}(\mathbb{R}_+)$ and setting

$$\mu(\omega, f) = \frac{1}{S(\omega)} \int_0^{S(\omega)} \psi(x)dx,$$

one can verify that we have defined a conditional regular law.

Existence of the CRL

Theorem 29.

Let

- X a random variable on $(\Omega, \mathcal{F}_0, P)$.
- Taking values in a space of the form $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.
- $\mathcal{G} \subset \mathcal{F}_0$ a σ-algebra.

Then the CRL of X given $\mathcal{G}$ exists.

Proof: nontrivial and omitted.

Computation rules for CRL

(1) If $\mathcal{G} = \sigma(Y)$, with Y random variable with values in $\mathbb{R}^m$, we have

$$\mu(\omega, f) = \tilde{\mu}(Y(\omega), f),$$

and one can define a CRL of X given Y as a family $\{\tilde{\mu}(y, .); y \in \mathbb{R}^m\}$ of probabilities on $\mathbb{R}^n$, such that for all $f \in C_b(\mathbb{R}^n)$ the function

Fact: $\mu(y,\cdot) = \mathcal{L}(X | Y=y)$

$$y \mapsto \mu(y, f)$$

is measurable.

(2) If Y is a discrete r.v, this can be reduced to:

$E[\mathbb{1}_A(X) | Y=y]$

$=$

$$\mu(y, A) = \mathbf{P}(X \in A | Y = y) = \frac{\mathbf{P}(X \in A, Y = y)}{\mathbf{P}(Y = y)}.$$

Computation rules for CRL (2)

(3) When one knows the CRL, quantities like the following (for $\phi \in \mathcal{B}(\mathbb{R}^n)$) can be computed:

$$\mathbf{E}\left[\phi(X)|\mathcal{G}\right] = \int_{\mathbb{R}^n} \phi(x)\, \mu(\omega, dx)$$
$$\mathbf{E}\left[\phi(X)|Y\right] = \int_{\mathbb{R}^n} \phi(x)\, \mu(Y, dx).$$

(4) ~~The CRL is not unique.~~
~~However~~ if N_1, N_2 are 2 CRL of X given $\mathcal{G}$
$\hookrightarrow$ we have ω-almost surely:

$$N_1(\omega, f) = N_2(\omega, f) \quad \text{for all} \quad f \in C_b(\mathbb{R}^n).$$