- Prob space $(\Omega, \mathcal{F}, \mathbb{P})$
- Filtration $\{\mathcal{F}_n ; n \geq 0\}$, $\mathcal{F}_n \subset \mathcal{F}_{n+1}$
- Take $T : \Omega \longrightarrow \{0, 1, \dots\} \cup \{\infty\}$
- We say that T is a stopping time if

$$\{T = n\} \in \mathcal{F}_n \qquad \forall n \geq 0$$

- Equivalently (check), T stopping time if

$$\{T \leq n\} \in \mathcal{F}_n \qquad \forall n$$

Example of stopping time

- $(X_n)_{n\geq 1}$ adapted $(X_n \in \mathcal{F}_n)$
- $T = \inf\{n \geq 1;\ X_n = 2\}$

Second example

$X_n = 0$ and

$T_2 = \inf\{n \geq 1;\ X_n \notin [-3, 4]\}$

Stopped martingales

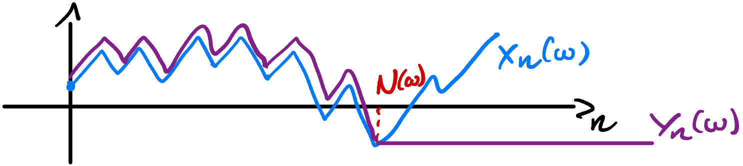
$n \wedge N = \inf\{n, N\}$

Theorem 8.

Let

- X martingale
- N stopping time

We set $Y_n = X_{n \wedge N}$. Then Y is a martingale.



Recall: If X is a martingale
H is predictable $H_n \in \mathcal{F}_{n-1}$

$\Rightarrow$ $H \cdot X$ is a martingale, where

$$[H \cdot X]_n = \sum_{k=1}^{n} H_k \, \Delta X_k + a$$

Here, if $Y_n = X_{n \wedge N}$ we have

$$\Delta Y_k = Y_k - Y_{k-1} = \Delta X_k \, \mathbb{1}(k-1 < N)$$

$$\Rightarrow Y_n = Y_0 + \sum_{k=1}^{n} \Delta Y_k$$

$$= Y_0 + \sum_{k=1}^{n} \overbrace{\mathbb{1}(N > k-1)}^{H_k} \Delta X_k$$

$\mathcal{F}_n \subset \mathcal{F}_{n+1}$

Summary $\quad Y_n = Y_0 + \sum_{k=1}^{n} H_k \, \Delta X_k$

with $\quad H_k = \mathbb{1}_{(N > k-1)}$) $\in \mathcal{F}_{k-1}$?

This is a mart. transform if $H_k \in \mathcal{F}_{k-1}$

$\in \mathcal{F}_{k-1}$ since N stopping time

We have $(N > k-1) = (N \leq k-1)^c$

$$\Rightarrow \mathbb{1}_{(N>k-1)} \in \mathcal{F}_{k-1}$$

Conclusion : $\boxed{Y_n = X_{n \wedge N} \text{ is a martingale}}$

Stopping time : if $(N \leq k) \in \mathcal{F}_k$

Proof

Decomposition of Y: We have

$$Y_j - Y_{j-1} = (X_j - X_{j-1})\, \mathbf{1}_{(j-1<N)}.$$

Expression as transformed martingale: Set $H_j = \mathbf{1}_{(j-1<N)}$. Then

$$\begin{aligned} Y_n &= Y_0 + \sum_{j=1}^{n} (Y_j - Y_{j-1}) \\ &= Y_0 + \sum_{j=1}^{n} (X_j - X_{j-1})\, \mathbf{1}_{(j-1<N)} \\ &= Y_0 + \sum_{j=1}^{n} H_j\, \Delta X_j \end{aligned}$$

In addition H is predictable. Thus Y is a martingale.

Outline

Convergence philosophy:

- Submartingales are $\nearrow$ (on average)
- If we have a proper bound on the sequence, we will get convergence

Convergence in L^2

Theorem 9.

Let X such that

- $\{X_n;\ n \geq 1\}$ is a martingale.
- For all n we have $X_n \in L^2(\Omega)$ and

$$\sup\left\{\mathbf{E}[X_n^2];\ n \geq 0\right\} \equiv M < \infty. \tag{2}$$

Then

1. $L^2 - \lim_{n\to\infty} X_n = X_\infty$.
2. For all $n \geq 0$, we have $X_n = \mathbf{E}[X_\infty |\, \mathcal{F}_n]$.

First aim: Prove that $(X_n)_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega)$.
That is

$$\lim_{\substack{m\to\infty \\ n\geq m}} \mathbb{E}[(X_n - X_m)^2] = 0$$

$$= \mathbb{E}\{\mathbb{E}\{X_n X_m \mid \mathcal{F}_m\}\}$$
$$= \mathbb{E}\{X_m \, \mathbb{E}[X_n \mid \mathcal{F}_m]\}$$
$$= \mathbb{E}[X_m^2]$$

We have

$$\mathbb{E}[(X_n - X_m)^2] = \mathbb{E}[X_n^2] - 2\mathbb{E}[X_n X_m] + \mathbb{E}[X_m^2]$$

$$= \mathbb{E}[X_n^2] - \mathbb{E}[X_m^2]$$

Summary For $n \geq m$,

$$\mathbb{E}[(X_n - X_m)^2] = \overbrace{\mathbb{E}[X_n^2]}^{a_n} - \mathbb{E}[X_m^2]$$

About sequence $(a_n)_{n \geq 1}$. We have

(i) a_n is $\nearrow$, since $a_n - a_m = \mathbb{E}[\text{square}]$

(ii) a_n is bounded   Hyp: $\mathbb{E}[X_n^2] \leq M < \infty$

$\Rightarrow$ (a_n) is convergent, thus Cauchy

Conclusion: (X_n) is Cauchy in $L^2(\Omega)$

and $\boxed{X_n \longrightarrow X_\infty \text{ in } L^2(\Omega)}$

Second aim: prove $X_n = \mathbb{E}[X_\infty \mid \mathcal{F}_n]$

For this, set $\boxed{V = |\mathbb{E}[X_\infty \mid \mathcal{F}_n] - X_n|}$

Since $V \geq 0$, we have $V = 0$ a.s. iff $\mathbb{E}[V] = 0$. Now

$$\mathbb{E}[V] = \mathbb{E}\{ |\mathbb{E}[X_\infty \mid \mathcal{F}_n] - X_n| \}$$ large

$$= \mathbb{E}\{ |\mathbb{E}[X_\infty \mid \mathcal{F}_n] - \mathbb{E}[X_{n+k} \mid \mathcal{F}_n]| \}$$

$$= \mathbb{E}\{ |\mathbb{E}[X_\infty - X_{n+k} \mid \mathcal{F}_n]| \}$$ $|\cdot|$ convex function

Jensen

$$\leq \mathbb{E}\{ \mathbb{E}[|X_\infty - X_{n+k}| \mid \mathcal{F}_n] \}$$

$$= \mathbb{E}[|X_\infty - X_{n+k}|]$$

$\mathbb{E}[\mathbb{E}[Z \mid \mathcal{F}]] = \mathbb{E}[Z]$

Recall: $X_n \to X_\infty$ in $L^2(\Omega)$

<u>Conclusion</u> We have seen

$$\mathbb{E}[V] \leq \mathbb{E}[\,|X_\infty - X_{n+k}|\,]$$

$$\leq \mathbb{E}^{\frac{1}{2}}[\,|X_\infty - X_{n+k}|^2\,] \xrightarrow{k\to\infty} 0$$

We get $\mathbb{E}[V]=0 \Rightarrow V=0$ a.s.

$$\Rightarrow \boxed{\mathbb{E}[X_\infty \mid \mathcal{F}_n] = X_n \quad \text{a.s.}}$$

Proof

Step 1: We set $a_n = \mathbf{E}[X_n^2]$. We will show that if $n \geq m$, then

$$\mathbf{E}\left[(X_n - X_m)^2\right] = a_n - a_m.$$

Indeed,

$$\mathbf{E}[X_m X_n] = \mathbf{E}\left\{X_m \, \mathbf{E}[X_n | \mathcal{F}_m]\right\} = \mathbf{E}\left[X_m^2\right].$$

Therefore

$$\begin{aligned}
\mathbf{E}\left[(X_n - X_m)^2\right] &= \mathbf{E}\left[X_n^2\right] + \mathbf{E}\left[X_m^2\right] - 2\,\mathbf{E}[X_m X_n] \\
&= \mathbf{E}\left[X_n^2\right] - \mathbf{E}\left[X_m^2\right] \\
&= a_n - a_m.
\end{aligned}$$

Proof (2)

Step 2: Convergence in L^2.

- $a_{n+1} - a_n = \mathbf{E}[(X_{n+1} - X_n)^2] \Longrightarrow n \mapsto a_n$ increasing.
- Inequality (2) $\Longrightarrow (a_n)_{n\geq 0}$ bounded $\Longrightarrow (a_n)_{n\geq 0}$ convergent.
- $\mathbf{E}\left[(X_n - X_m)^2\right] = a_n - a_m \Longrightarrow (X_n)_{n\geq 0}$ Cauchy in $L^2(\Omega)$

Conclusion: $(X_n)_{n\geq 0}$ converges in $L^2(\Omega)$ towards X_∞.

Proof (3)

Step 3: We have $X_n = \mathbf{E}[X_\infty | \mathcal{F}_n]$.

Set

$$V = \left|\mathbf{E}[X_\infty | \mathcal{F}_n] - X_n\right| .$$

We are reduced to show that $\mathbf{E}[V] = 0$.

Computation: For $n, k \geq 0$,

$$\begin{aligned} V &= \left|\mathbf{E}[X_\infty | \mathcal{F}_n] - \mathbf{E}[X_{n+k} | \mathcal{F}_n]\right| \\ &= \left|\mathbf{E}[X_\infty - X_{n+k} | \mathcal{F}_n]\right| \leq \mathbf{E}\left[|X_\infty - X_{n+k}| \,|\, \mathcal{F}_n\right] \end{aligned}$$

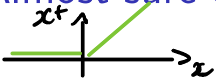
Hence

$$\mathbf{E}[V] \leq \mathbf{E}\left[|X_\infty - X_{n+k}|\right] \leq \mathbf{E}^{1/2}\left[(X_\infty - X_{n+k})^2\right]$$

We get $\mathbf{E}[V] = 0$ whenever $k \to \infty$ above.

Almost sure convergence

$x_n^+ = \max(0, x_n)$



Theorem 10.

Let X satisfying

- $\{X_n;\ n \geq 0\}$ is a martingale or a submartingale.

supermartingale

- We have

$\sup\{E[X_n^-];\ n\geq 0\} = M < \infty$

$$\sup\left\{\mathbf{E}[X_n^+];\ n \geq 0\right\} \equiv M < \infty. \tag{3}$$

Then

1. $\text{a.s} - \lim_{n\to\infty} X_n = X_\infty$.
2. We have $\mathbf{E}[|X_\infty|] < \infty$.

Application 1. Take X_n martingale with $X_n \geq 0$. Then

(i) X_n is a supermartingale

(ii) $X_n^- = 0 \Rightarrow X_n^- \leq M$ a.s.

Thus $\boxed{X_n \longrightarrow (X_\infty \in L^1) \text{ a.s.}}$

Note $X_\infty \in L^1$, but we don't have necessarily $X_n \xrightarrow{L^1} X_\infty$

Application 2

Take

(i) X_n martingale

(ii) $\left(\mathbb{E}[(X_n)^2]\right)^{\frac{1}{2}} \leq M < \infty$

Then

$$\mathbb{E}[X_n^+] \leq \mathbb{E}[|X_n|] \leq \mathbb{E}^{\frac{1}{2}}[X_n^2] \leq M$$

Thus

AND

$$\boxed{\begin{array}{l} X_n \xrightarrow{a.s.} X_\infty \\ X_n \xrightarrow{L^2} X_\infty \end{array}}$$

Particular cases

Particular case 1:

$(X_n)_{n\geq 0}$ positive martingale $\Longrightarrow \mathrm{a.s} - \lim_{n\to\infty} X_n = X_\infty$.

Particular case 2:

$\sup\{\mathbf{E}[X_n^2];\, n \geq 0\} \equiv M < \infty \Longrightarrow \mathrm{a.s} - \lim_{n\to\infty} X_n = X_\infty$.

$\hookrightarrow$ We have both a.s and L^2 convergence.

Convergence counterexample

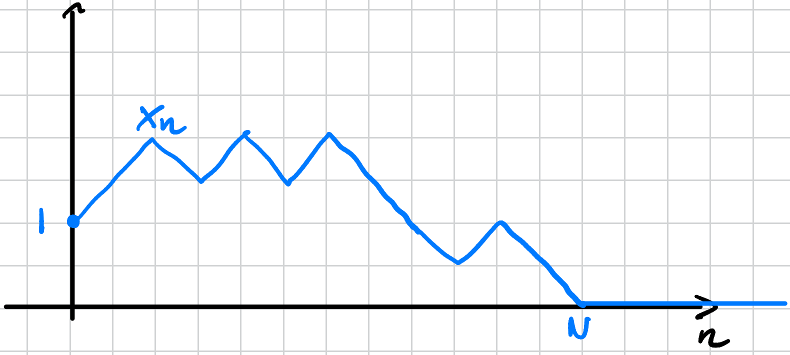
Example 11.

Let

- $\{\xi_n;\ n \geq 1\}$ i.i.d Rademacher sequence
- $\{S_n;\ n \geq 0\}$ defined by
  - $S_0 = 1$
  - $S_n = S_{n-1} + \xi_n$ for $n \geq 1$
- $N = \inf\{n \geq 1;\ S_n = 0\}$
- $X_n = S_{n \wedge N}$ → X_n stopped martingale

Then the following holds true:

1. X_n converges almost surely to 0
2. X_n does not converge in $L^1(\Omega)$



x_n
1
N
n