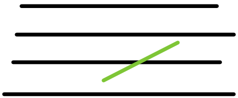
Outline

Experiment



Procedure:

- Consider a plane ruled by lines $y = k$, with $k \in \mathbb{Z}$
- Take a needle with length 1
- Fling the needle n times on the plane

$$X_i = \mathbb{1}_{A_i} = \begin{cases} 1 & \text{if } A_i \text{ realized} \\ 0 & \text{otherwise} \end{cases}$$

Outcome: We record, for $i = 1, \ldots, n$,

- $X_i \equiv \mathbf{1}_{A_i}$, where $A_i = (i\text{-th needle intersect a line})$
- $S_n \equiv \#$ times the needle intersects the line

Simulation:

`This website from UIUC`

Rmk We will see that

$$\frac{S_n}{n} = \text{average \# needles intersecting}$$

$$\xrightarrow{n\to\infty} \frac{2}{\pi}$$

This gives a way to approximate π

This can be seen as one of first examples of Monte Carlo method

use of probability to compute an integral

Limiting result

A_i = " ith needle hits one line "

Proposition 5.

Under the above conditions we have

$$\mathbf{P}(A_i) = \frac{2}{\pi}$$

$$\frac{S_n}{n} \longrightarrow \frac{2}{\pi}$$

Bernoulli r.v

$X \sim \mathcal{B}(p) \quad , \quad p \in (0,1)$

$X : \Omega \xrightarrow{\text{measurable}} \{0,1\}$

Pmf : (probability mass function)

$$p_X(0) = \mathbb{P}(X=0) = 1-p$$

$$p_X(1) = \underbrace{\mathbb{P}(X=1)}_{\mathbb{P}\left(\underbrace{\{\omega\in\Omega \,;\, X(\omega)=1\}}_{\in\, \mathcal{F}, \text{ since } \{1\} \text{ is measurable in } \mathbb{R}}\right)} = p$$

Bernoulli random variable (1)

Notation:

$$X \sim \mathcal{B}(p) \text{ with } p \in (0,1)$$

State space:

$$\{0,1\}$$

Pmf:

$$\mathbf{P}(X=0) = 1-p, \quad \mathbf{P}(X=1) = p$$

Expected value, variance, generating function:

$$\mathbf{E}[X] = p, \qquad \mathbf{Var}(X) = p(1-p), \qquad G_X(s) = (1-p) + p\,s$$

Bernoulli random variable (2)

Bernoulli

17th

18th

19th

Use 1, success in a binary game:

- Example 1: coin tossing
  - $X = 1$ if H, $X = 0$ if T
  - We get $X \sim \mathcal{B}(1/2)$
- Example 2: dice rolling
  - $X = 1$ if outcome $= 3$, $X = 0$ otherwise
  - We get $X \sim \mathcal{B}(1/6)$

Use 2, answer yes/no in a poll

- $X = 1$ if a person feels optimistic about the future
- $X = 0$ otherwise
- We get $X \sim \mathcal{B}(p)$, with unknown p

Jacob Bernoulli

Some facts about Bernoulli:

- Lifespan: 1654-1705, in Switzerland
- Discovers constant e
- Establishes divergence of $\sum \frac{1}{n}$
- Contributions in diff. eq
- First law of large numbers
- Bernoulli:
  family of 8 prominent mathematicians
- Fierce math fights between brothers

Binomial r.v

$X \sim \mathrm{Bin}(n, p)$

$X : \Omega \to \{0, 1, \dots, n\}$

pmf: $k = 0, 1, \dots, n$

$$\boxed{\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Binomial random variable (1)

Notation:

$$X \sim \text{Bin}(n, p), \text{ for } n \geq 1, \; p \in (0, 1)$$

State space:

$$\{0, 1, \ldots, n\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad 0 \leq k \leq n$$

Expected value, variance and generating function:

$$\mathbf{E}[X] = np, \qquad \mathbf{Var}(X) = np(1 - p), \qquad G_X(s) = [(1 - p) + p\, s]^n$$

Binomial random variable (2)

Use 1, Number of successes in a Bernoulli trial:

- Example: Roll a dice 9 times.
- $X = \#$ of 3 obtained
- We get $X \sim \text{Bin}(9, 1/6)$
- $\mathbf{P}(X = 2) = 0.28$ $= \binom{9}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^7$

Use 2: Counting a feature in a repeated trial:

- Example: stock of 1000 pants with 10% defects
- Draw 15 times a pant at random
- $X = \#$ of pants with a defect
- We get $X \sim \text{Bin}(15, 1/10)$

Binomial random variable (3)

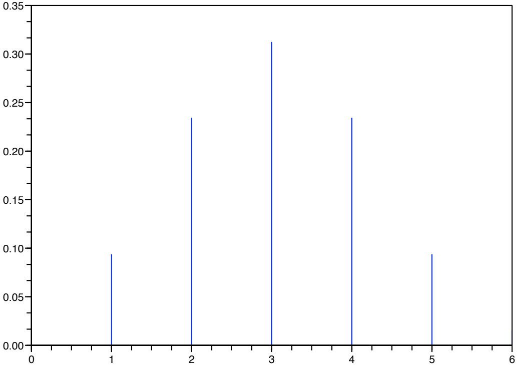


Figure: Pmf for Bin(6; 0.5). x-axis: k. y-axis: $\mathbf{P}(X = k)$

Binomial random variable (4)

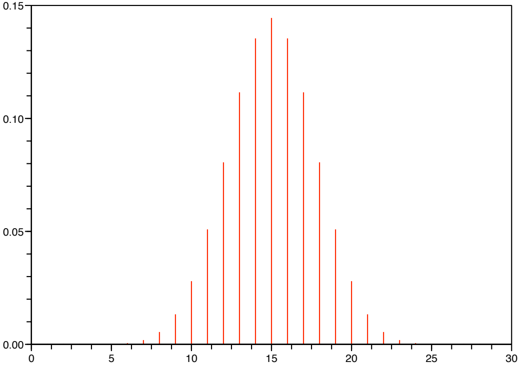


Figure: Pmf for Bin$(30; 0.5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Uniform random variable (1)

Notation:

$$X \sim \mathcal{U}([\alpha, \beta]), \text{ with } \alpha < \beta$$

State space:

$$[\alpha, \beta]$$

Density:

$$f(x) = \frac{1}{\beta - \alpha}\, \mathbf{1}_{[\alpha,\beta]}(x)$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{\alpha + \beta}{2}, \qquad \mathbf{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

Classical r.v

Discrete	Continuous
$\mathcal{B}(p)$	$\mathcal{U}([\alpha,\beta])$
$Bin(n,p)$	$\mathcal{E}(\lambda)$ exponential r.v
$\mathcal{P}(\lambda)$	$\mathcal{N}(\mu,\sigma^2)$
$\mathcal{G}(p)$	Cauchy

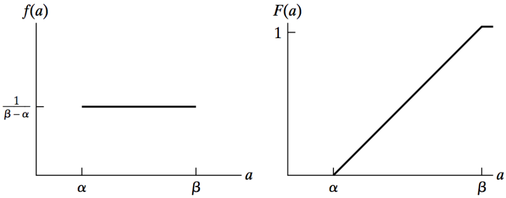
Uniform random variable (2)

Use:

- $\mathcal{U}([0,1])$ only r.v directly accessible on a computer
  $\hookrightarrow$ `rand` function

Example of computation: if $X \sim \mathcal{U}([8,10])$, then

$$\mathbf{P}(7.5 < X < 9.5) = \frac{1}{2}\int_8^{9.5} dx = \frac{9.5-8}{2} = \frac{3}{4}$$



Experiment (repeated)

Procedure:

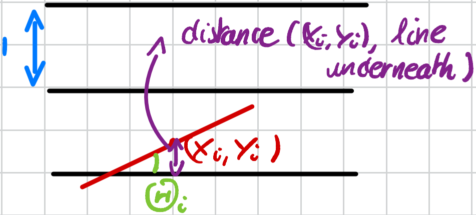
- Consider a plane ruled by lines $y = k$, with $k \in \mathbb{Z}$
- Take a needle with length 1
- Fling the needle n times on the plane

Outcome: We record, for $i = 1, \ldots, n$,

- $X_i \equiv \mathbf{1}_{A_i}$, where $A_i = (i\text{-th needle intersects a line})$
- $S_n \equiv \#$ times the needle intersects the line

Model : Call

(X_i, Y_i) = center of i-th needle



Θ_i = angle (i-th needle, x axis)

Z_i = distance ((X_i, Y_i), line underneath)

We assume

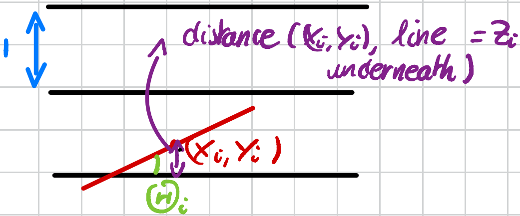
independent and identically distributed

- $Z_i \sim U([0, l])$
- $\Theta_i \sim U([0, \pi])$
- $Z_i \perp\!\!\!\perp \Theta_i$
- $\{Z_i ; i \geq 1\}$ i.i.d

  $\{\Theta_i ; i \geq 1\}$ i.i.d

Recall:

A_i = "i-th needle hits a line"



We decompose

$$A_i = A_i^- \cup A_i^+ \qquad \text{where}$$

$$A_i^- = \{ z_i \leq \tfrac{1}{2} \} \cap \{ z_i \leq \tfrac{1}{2} \sin \Theta_i \}$$

$$A_i^+ = \{ z_i > \tfrac{1}{2} \} \cap \{ 1 - z_i \leq \tfrac{1}{2} \sin \Theta_i \}$$

$$\Rightarrow \boxed{\mathbb{P}(A_i) = \mathbb{P}(A_i^+) + \mathbb{P}(A_i^-) = 2\,\mathbb{P}(A_i^-)}$$

$$A_i^- = \{ z_i \leq \tfrac{1}{2} \} \cap \{ z_i \leq \tfrac{1}{2} \sin \Theta_i \}$$

$$z_i \sim \mathcal{U}([0,1]) \qquad \Theta_i \sim \mathcal{U}([0,\pi]) \qquad \underbrace{z_i \perp\!\!\!\perp \Theta_i}$$

$$\mu_{(z_i, \Theta_i)} = \mu_{z_i} \otimes \mu_{\Theta_i}$$

Thus

$$\boxed{\mathbb{P}(A_i^-)} = \int_0^\pi d\theta \, \frac{1}{\pi} \int_0^1 dz \; \mathbb{1}\left(z \leq \tfrac{1}{2} \sin\theta \right)$$

$$= \frac{1}{\pi} \int_0^\pi d\theta \int_0^{\frac{1}{2}\sin\theta} dz$$

$$= \frac{1}{\pi} \int_0^\pi \frac{1}{2} \sin\theta \, d\theta = \frac{1}{2\pi} \; -\cos\theta \Big|_0^\pi$$

$$= \frac{1}{\pi} \quad \Rightarrow \mathbb{P}(A_i) = 2\,\mathbb{P}(A_i^-) \boxed{= \frac{2}{\pi}}$$

$\mathbb{P}(A_i) = \frac{2}{\pi}$

2nd step: Count the # hit. Set

$$X_i = \mathbb{1}_{A_i} = \begin{cases} 1 & \text{if } i\text{-th hits a line} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow X_i \sim \mathcal{B}\left(p = \frac{2}{\pi}\right)$$

$S_n =$ total number of hits $= \sum_{i=1}^{n} X_i$

$$\Rightarrow S_n \sim \text{Bin}\left(n, \frac{2}{\pi}\right)$$

According to de Moivre,

$$\boxed{\frac{S_n}{n} \longrightarrow \frac{2}{\pi}}$$

Proof of Proposition 5 (1)

Notation: We define

- $(X_i, Y_i) \equiv$ Coordinates of the center of the i-th needle
- $\Theta_i \equiv$ angle (i-th needle, x-axis)
- $Z_i = d\left((X_i, Y_i),\ \text{nearest line underneath}\right) = Y_i - \lfloor Y_i \rfloor$

Model: We assume

1. $Z_i \sim \mathcal{U}([0,1])$
2. $\Theta_i \sim \mathcal{U}([0,\pi])$
3. $Z_i \perp\!\!\!\perp \Theta_i$
4. $\{Z_i;\ i \geq 1\}$ i.i.d sequence
5. $\{\Theta_i;\ i \geq 1\}$ i.i.d sequence

Proof of Proposition 5 (2)

Expression for A_i: We have

$$A_i = A_i^- \cup A_i^+$$

with

$$\begin{aligned} A_i^- &= \left\{ Z_i \le \frac{1}{2}, \text{ and } Z_i < \frac{1}{2}\sin(\Theta_i) \right\} \\ A_i^+ &= \left\{ Z_i > \frac{1}{2}, \text{ and } 1 - Z_i < \frac{1}{2}\sin(\Theta_i) \right\} \end{aligned}$$

Proof of Proposition 5 (3)

Computing $\mathbf{P}(A_i)$: We write

$$\begin{aligned}\mathbf{P}(A_i) &= \mathbf{P}(A_i^-) + \mathbf{P}(A_i^+) \\ &= 2\mathbf{P}(A_i^-) \\ &= \frac{2}{\pi}\int_0^{\pi} d\theta \int_0^{\frac{1}{2}\sin(\theta)} dz\end{aligned}$$

Thus

$$\mathbf{P}(A_i) = \frac{2}{\pi}$$

Proof of Proposition 5 (4)

Some laws: We have

$$X_i \sim \mathcal{B}\left(\frac{2}{\pi}\right)$$
$$S_n \sim \text{Bin}\left(n, \frac{2}{\pi}\right)$$

Limit: By De Moivre,

$$\frac{S_n}{n} \longrightarrow \frac{2}{\pi}$$

Outline

Aim of this chapter

Problem with limit statement:

- For every $n \geq 1$, we have $S_n : \Omega \to \mathbb{R}$
- S_n is thus a function
- We don't know exactly what $\frac{S_n}{n} \longrightarrow \frac{2}{\pi}$ means!

Aim of this chapter:

- Explore different modes of convergence for random variables

Preliminary step:

- Explore different modes of convergence for functions

Setting for convergence of functions

Sequence of functions: We consider

- A sequence $\{f_n;\ n \geq 1\}$ with

$$f_n : [0,1] \longrightarrow \mathbb{R}$$

Aim of subsection: Review modes for

$$\lim_{n\to\infty} f_n$$

Pointwise convergence

Definition 6.

Let

- $\{f_n;\ n \geq 1\}$ sequence of measurable functions

We assume

$u_n \longrightarrow \ell$

$$\lim_{n\to\infty} f_n(x) = f(x)\,, \ \text{for all } x \in [0,1]$$

Then we say that

$$f_n \longrightarrow f \text{ pointwise}$$

Almost everywhere convergence

Definition 7.

Let

- $\{f_n;\ n \geq 1\}$ sequence of measurable functions

$([0,1], \mathcal{B}([0,1]), \lambda)$

We assume

$$\lim_{n\to\infty} f_n(x) = f(x)\,, \text{ for almost every } x \in [0,1]$$

Then we say that

$$f_n \longrightarrow f \text{ almost everywhere}$$

Question: what do we mean by

$\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in [0,1]$?

Answer: If $\lambda \equiv$ Lebesgue measure on $[0,1]$,

$\lambda(\{ x \in [0,1];$ $f_n(x)$ does not converge to $f(x) \}) = 0$

L^p convergence

Definition 8.

Let

- $\{f_n;\ n \geq 1\}$ sequence of measurable functions

We assume $\|f_n - f\|_{L^p} = \left(\int_0^1 |f_n(x) - f(x)|^p\, dx\right)^{1/p}$

$$\lim_{n\to\infty} \|f_n(\cancel{x}) - f(\cancel{x})\|_{L^p([0,1])} = 0$$

Then we say that

$$f_n \longrightarrow f \text{ in } L^p([0,1])$$

Convergence in measure

Definition 9.

Let

- $\{f_n;\ n \geq 1\}$ sequence of measurable functions

We assume that for all $\varepsilon > 0$

$f_n(u) - f(u)$

$$\lim_{n\to\infty} \lambda\left(\{u \in [0,1];\ |g(u) \cancel{-} h(u)| > \varepsilon\}\right) = 0$$

Then we say that

$$f_n \longrightarrow f \text{ in measure}$$