Relations between convergences (1)

Examples of relations for functions on $[0, 1]$:

- $f_n(x) = x^n$
  $\hookrightarrow$ converges almost everywhere, not pointwise
- $g_n(x) = n\mathbf{1}_{[0,1/n]}(x)$
  $\hookrightarrow$ converges almost everywhere, not in L^1

Example 1 $\quad f_n(x) = x^n \quad , x \in [0,1]$

Then

(i) If $x \in [0,1)$, then

$$f_n(x) = x^n \longrightarrow 0$$

(ii) If $x = 1$, then

$$f_n(x) = f_n(1) = 1^n = 1 \longrightarrow 1$$

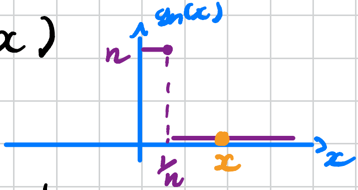
Thus $\quad f_n \longrightarrow 0 \quad$ a.e $\quad (\lambda([0,1)) = 1)$

$\quad\quad f_n \not\to 0 \quad$ pointwise

Example 2

$g_n : [0,1] \to \mathbb{R}$ given by

$$g_n(x) = n \, \mathbb{1}_{[0,1/n]}(x)$$



We have

(i) If $x \in (0,1]$, then $\exists n_0$ s.t. $\forall n \geq n_0$ we have $x > 1/n$

$$\Rightarrow \mathbb{1}_{[0,\frac{1}{n}]}(x) = 0 \quad \forall n \geq n_0$$

$$\Rightarrow g_n(x) = 0 \quad \forall n \geq n_0$$

$$\Rightarrow \lim_{n\to\infty} g_n(x) = 0 \Rightarrow g_n \to 0 \text{ a.e}$$

$g_n = n\,\mathbb{1}_{[0,1/n]}$

(ii) The sequence g_n does not converge to 0 in L^1:

$$\|g_n - 0\|_{L^1} = \|g_n\|_{L^1}$$
$$= \int_0^1 |g_n(x)|\,dx = n\int_0^1 \mathbb{1}_{[0,\frac{1}{n}]}(x)\,dx$$
$$= n\int_0^{1/n} dx$$
$$= 1 \not\to 0$$

Rmk Generally, this type of problem occurs when either (i) some mass escapes to ∞ or (ii) Big bump

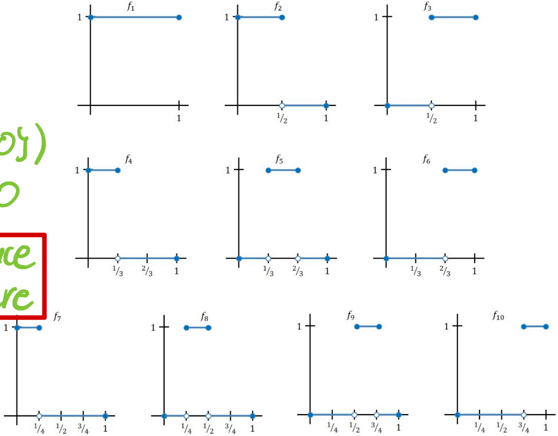
Relations between convergences (2)

Another example of relation for functions on $[0,1]$:

- $h_n = \mathbf{1}_{[0,1]}, \mathbf{1}_{[0,1/2]}, \mathbf{1}_{[1/2,1]}, \mathbf{1}_{[0,1/3]}, \mathbf{1}_{[1/3,2/3]}, \ldots$
  $\hookrightarrow$ converges in measure, not almost everywhere

$\lambda(\{\omega; h_n(\omega) \neq 0\}) \le \frac{1}{\sqrt{n}} \xrightarrow{n\to\infty} 0$

$\Rightarrow$ $\boxed{\text{convergence in measure}}$



Claim : The sequence h_n does not converge to 0 a.e.

In fact for every $x \in [0,1]$ and $n_0 \geq 1$, there exists $n > n_0$ s.t.

$$h_n(x) = 1$$

$$\Rightarrow \boxed{h_n(x) \text{ does not converge}}$$

Outline

Outline

Almost sure convergence

($\simeq$ a.e convergence for functions)

Definition 10.

Let

- $\{X_n;\, n \geq 1\}$ sequence of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$
- Another random variable X defined on $(\Omega, \mathcal{F}, \mathbf{P})$

Pointwise : $\forall \omega$, $X_n(\omega) \to X(\omega)$

We assume

$$\mathbf{P}\left(\left\{\omega \in \Omega;\ \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Then we say that

$$X_n \longrightarrow X \text{ almost surely}$$

Convergence in L^p

Definition 11.

Let

- $\{X_n;\ n \geq 1\}$ sequence of r.v in $L^r(\Omega)$
- Another random variable $X \in L^r(\Omega)$

We assume

$$\lim_{n\to\infty} \mathbf{E}\left[|X_n - X|^r\right] = 0.$$

Then we say that

$$X_n \longrightarrow X \text{ in } L^r(\Omega) \text{ (or in } r\text{-th mean)}$$

Convergence in probability $\simeq$ Convergence in measure for functions

Definition 12.

Let

- $\{X_n;\, n \geq 1\}$ sequence of random variables on $(\Omega, \mathcal{F}, \mathbf{P})$
- Another random variable X defined on $(\Omega, \mathcal{F}, \mathbf{P})$

We assume that for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbf{P}\left(|X_n - X| > \varepsilon\right) = 0.$$

Then we say that

$$X_n \xrightarrow{\mathcal{P}} X \text{ in probability}$$

Convergence in distribution

F_{X_n} = cdf of X_n
F_X = cdf of X

Definition 13.

Let

- $\{X_n;\ n \geq 1\}$ sequence of random variables on $(\Omega_n, \mathcal{F}_n, \mathbf{P}_n)$
- Another random variable X defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$

We assume that for all points $x \in \mathbb{R}$ such that F_X is continuous,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x).$$

Then we say that

$$X_n \xrightarrow{(d)} X \text{ in distribution}$$

Rmk $X_n \xrightarrow{d} X$ if the measure on $\mathbb{R}$ induced by X_n, given by

$$\mu_{X_n}((-\infty, x]) = F_{X_n}(x),$$

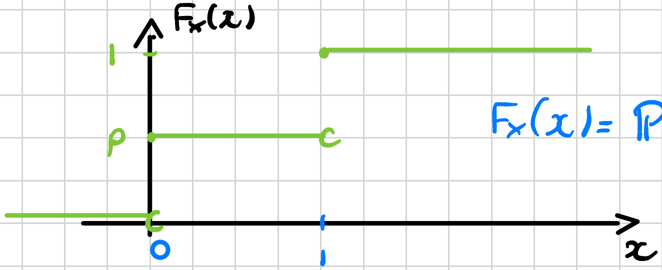
is such that "$\mu_{X_n} \to \mu_X$"

Rmk 2 The cdf of a r.v is not necessarily continuous. If F_X has a jump at $x \in \mathbb{R}$, it means that

$$\mathbb{P}(X = x) = a > 0$$

$\hookrightarrow$ size of the jump

Example If $X \sim \mathcal{B}(p)$. Then



$F_X(x) = \mathbb{P}(X \leq x)$

In general, F is right continuous with limits on the left (rcll)

Standard notation: càdlàg

Limit on (d): only occur at points of continuity for F_X

Remarks about convergence in distribution

1. The central limit theorem
   $\hookrightarrow$ is a convergence in distribution
2. Ergodic theorems for Markov chains
   $\hookrightarrow$ are convergences in distributions
3. Convergence in distribution
   $\hookrightarrow$ does not refer to a specific $(\Omega, \mathcal{F}, \mathbf{P})$

A Bernoulli example

A Bernoulli sequence: We consider

- $X \sim \mathcal{B}(1/2)$
- $X_n = X$ for all $n \geq 1$
- $Y = 1 - X$

Convergences:

1. We have

$$X_n \xrightarrow{(d)} X$$

2. X_n does not converge to X in any other mode

Situation: We take $X_n = X$ with $X \sim \mathcal{B}(\frac{1}{2})$

Set $Y = 1 - X$

Then $\mathbb{P}(Y=0) = \mathbb{P}(X=1) = \frac{1}{2}$
$\mathbb{P}(Y=1) = \mathbb{P}(X=0) = \frac{1}{2}$

$\Rightarrow Y \sim \mathcal{B}(\frac{1}{2})$

$\Rightarrow \boxed{X_n \xrightarrow{(d)} Y}$

However $X - \overset{=1-X}{\overbrace{Y}} = 2X - 1 = \begin{cases} 1 & \text{if } X=1 \\ -1 & \text{if } X=0 \end{cases}$

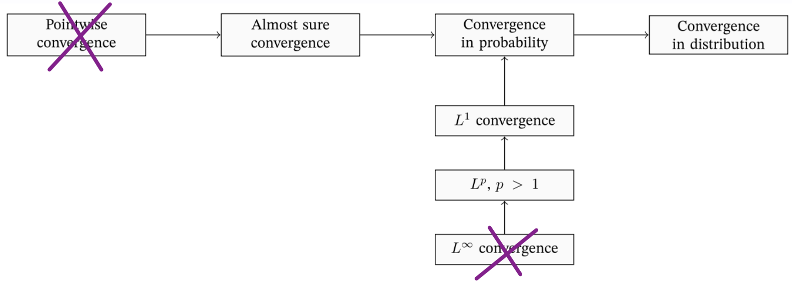
$\Rightarrow \mathbb{P}(|X_n - Y| = 1) = 1$

<u>Conclusion</u> : If $|X_n - Y| = 1$ a.s,

- $\mathbb{P}(|X_n - Y| > \varepsilon) \not\to 0$ as $n \to \infty$
- $\mathbb{E}[|X_n - Y|^r] = 1 \not\to 0$ as $n \to \infty$
- $|X_n - Y| \not\to 0$ a.s

Relations between modes of convergence

Theorem 14.



Outline

Convergence in probability and in distribution

Proposition 15.

Let

- X_n sequence of random variables
- Assume $X_n \xrightarrow{\mathrm{P}} X$

Then

$$X_n \xrightarrow{(\mathrm{d})} X$$

Notation We set

$$F_n(x) = \mathbb{P}(X_n \leq x)$$

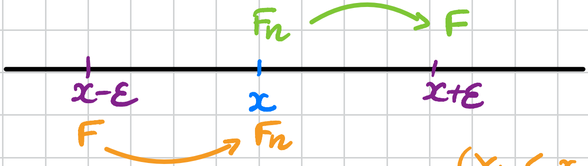
$$F(x) = \mathbb{P}(X \leq x)$$

We assume $X_n \xrightarrow{P} X$. We wish to show

$$F_n(x) \longrightarrow F(x)$$

at any point of continuity of F

Hyp: $\mathbb{P}(|X_n - X| \geq \varepsilon) \to 0 \quad \forall \varepsilon > 0$

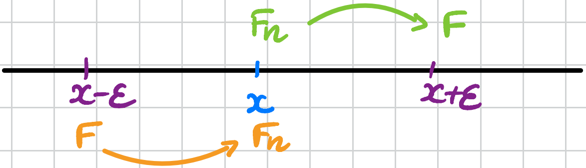


$(X_n \leq x, X > x+\varepsilon) \Rightarrow |X - X_n| > \varepsilon$

① $F_n(x) = \mathbb{P}(X_n \leq x)$

$$= \mathbb{P}(X_n \leq x,\ X \leq x+\varepsilon) + \mathbb{P}(X_n \leq x,\ X > x+\varepsilon)$$

$$\leq \mathbb{P}(X \leq x+\varepsilon) + \underbrace{\mathbb{P}(|X - X_n| > \varepsilon)}_{\equiv\, a_n(\varepsilon)}$$



② $$F(x-\varepsilon) = \mathbb{P}(X \le x-\varepsilon)$$

$$= \mathbb{P}(X \le x-\varepsilon,\ X_n \le x) + \mathbb{P}(X \le x-\varepsilon,\ X_n > x)$$

$$\le \mathbb{P}(X_n \le x) + \overbrace{\mathbb{P}(|X_n - X| > \varepsilon)}^{a_n(\varepsilon)}$$

$$\Rightarrow \quad \boxed{\begin{aligned} F(x-\varepsilon) &\le F_n(x) + a_n(\varepsilon) \\ F_n(x) &\le F(x) + a_n(\varepsilon) \end{aligned}}$$