Prop 15

$$(X_n \xrightarrow{P} X) \Rightarrow (X_n \xrightarrow{(d)} X)$$

Proof. Assume $X_n \xrightarrow{P} X$. Let x be a point of continuity of F_X. We wish to prove that

$$\boxed{\lim_{n \to \infty} F_n(x) = F(x)}$$

Two inequalities

(i) $F_n(x) \leq F(x+\varepsilon) + \overbrace{\mathbb{P}(|X - X_n| > \varepsilon)}^{a_n(\varepsilon)}$

(ii) $F(x-\varepsilon) \leq F_n(x) + a_n(\varepsilon)$

$\Rightarrow \quad F_n(x) \geq F(x-\varepsilon) - a_n(\varepsilon)$

We thus get

$$\boxed{F(x-\varepsilon) - a_n(\varepsilon) \leq F_n(x) \leq F(x+\varepsilon) + a_n(\varepsilon)}$$

Recall: $\lim_{n \to \infty} a_n(\varepsilon) = 0$, due to $X_n \xrightarrow{P} X$

We have

$$\boxed{F(x-\varepsilon)-a_n(\varepsilon) \leq F_n(x) \leq F(x+\varepsilon)+a_n(\varepsilon)}$$

We wish to take limits in n on all sides of the inequalities

Problem: Even though $F_n(x) \in [0,1]$, we don't know if it is is convergent

Solution: Take lim sup, lim inf

Def For a sequence u_n

$$\limsup_n u_n = \lim_{n\to\infty} \overbrace{\sup_{k\geq n} u_k}^{\equiv M_n}$$

Note : $n \mapsto M_n$ is $\searrow$

If $|u_n|$ bounded

$\lim_{n\to\infty} M_n$ exists and is finite

Def $$\liminf_n u_n = \lim_{n\to\infty} \overbrace{\inf_{k\geq n} u_k}^{m_n}$$

If $|u_n|$ bounded, then

$\lim m_n$ exists and is finite

Recall

$$F(x-\varepsilon) - a_n(\varepsilon) \leq F_n(x) \leq F(x+\varepsilon) + a_n(\varepsilon)$$

We can take lim sup and lim inf
we get

$$F(x-\varepsilon) \leq \overbrace{\liminf_{n\to\infty} F_n(x)}^{\ell}$$

$$\leq \underbrace{\limsup_{n\to\infty} F_n(x)}_{L} \leq F(x+\varepsilon)$$

Summary : For all $\varepsilon > 0$, we have

$$F(x-\varepsilon) \leq \ell \leq L \leq F(x+\varepsilon)$$

We now take $\varepsilon \to 0$. If x is a point of continuity for F, we have

$$\lim_{\varepsilon \to 0} F(x-\varepsilon) = \lim_{\varepsilon \to 0} F(x+\varepsilon) = F(x)$$

Thus $\boxed{\ell = L = F(x)}$

Def : $X_n \xrightarrow{(d)} X$ is $\lim F_n(x) = F(x)$ $\forall x$ point of continuity of F

Conclusion we have obtained, if x point of continuity for F,

$$\liminf_n F_n(x) = \limsup_n F_n(x) = F(x)$$

Thus

$$\lim_n F_n(x) = F(x)$$

and $\boxed{X_n \xrightarrow{(d)} X}$

Proof of Proposition 15 (1)

Notation: Set

$$F_n(x) = \mathbf{P}(X_n \leq x), \qquad F(x) = \mathbf{P}(X \leq x)$$

Aim: Prove that

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ if } F \text{ is continuous at } x$$

Proof of Proposition 15 (2)

1st decomposition: We have

$$\begin{aligned} F_n(x) &= \mathbf{P}\left(X_n \le x,\, X \le x+\varepsilon\right) + \mathbf{P}\left(X_n \le x,\, X > x+\varepsilon\right) \\ &\le F(x+\varepsilon) + \mathbf{P}\left(|X_n - X| > \varepsilon\right) \end{aligned}$$

2nd decomposition: We have

$$\begin{aligned} F(x-\varepsilon) &= \mathbf{P}\left(X \le x-\varepsilon,\, X_n \le x\right) + \mathbf{P}\left(X \le x-\varepsilon,\, X_n > x\right) \\ &\le F_n(x) + \mathbf{P}\left(|X_n - X| > \varepsilon\right) \end{aligned}$$

Summary:

$$F(x-\varepsilon) - \mathbf{P}\left(|X_n - X| > \varepsilon\right) \le F_n(x) \le F(x+\varepsilon) + \mathbf{P}\left(|X_n - X| > \varepsilon\right)$$

Proof of Proposition 15 (3)

Limits as $n \to \infty$: Since $X_n \xrightarrow{(P)} X$, we have

$$F(x-\varepsilon) \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F(x+\varepsilon)$$

Limits as $\varepsilon \to 0$: If F is continuous at x, we get

$$F(x) = \liminf_{n\to\infty} F_n(x) = \limsup_{n\to\infty} F_n(x) = F(x)$$

Convergence in $L^p(\Omega)$

Proposition 16.

Let

- X_n sequence of random variables
- Assume $X_n \xrightarrow{L^s} X$ for $s > r$

Then

$$X_n \xrightarrow{L^r} X$$

Def $X_n \xrightarrow{L^s} X$ if

$$\lim_{n\to\infty} \underbrace{\left(\mathbb{E}\left[|X_n - X|^s\right]\right)^{1/s}}_{= \|X_n - X\|_{L^s(\Omega)} \equiv \|X_n - X\|_s} = 0$$

Proof of Prop 16

If $X_n \xrightarrow{L^s} X$, then

$$0 \le \|X_n - X\|_{L^r} \overset{s>r}{\le} \|X_n - X\|_{L^s}$$

Thus

$$X_n \xrightarrow{L^r} X$$

<u>Example</u> of sequence such that if $s > r$ we have

$$\boxed{X_n \xrightarrow{L^r} X \quad \text{but} \quad X_n \not\xrightarrow{L^s} X}$$

Take $(X_n)_{n \geq 1}$ all $\perp\!\!\!\perp$ with

- $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^{(r+s)/2}}$
- $\mathbb{P}(X_n = n) = \frac{1}{n^{(r+s)/2}}$

Recall

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^{(r+s)/2}}$$

$$\mathbb{P}(X_n = n) = \frac{1}{n^{(r+s)/2}}$$

Claim 1: $\boxed{X_n \to 0 \text{ in } L^r}$. Indeed

$$\mathbb{E}[|X_n - 0|^r] = \mathbb{E}[(X_n)^r]$$

$$= 0 \times \mathbb{P}(X_n = 0) + n^r \, \mathbb{P}(X_n = n)$$

$$= n^r \times \frac{1}{n^{(r+s)/2}}$$

$$= \frac{1}{n^{(s-r)/2}} \xrightarrow{n \to \infty} 0$$

Claim 2: $\boxed{X_n \not\to 0 \text{ in } L^s}$. Indeed

$$\mathbb{E}[|X_n - 0|^s] = \mathbb{E}[(X_n)^s]$$

$$= 0 \times \mathbb{P}(X_n = 0) + n^s \, \mathbb{P}(X_n = n)$$

$$= n^s \times \frac{1}{n^{(r+s)/2}}$$

$$= n^{\frac{(s-r)}{2}} \xrightarrow{n \to \infty} \infty$$

Note Here $(X_n)_{n \geq 0}$ has an "escape to ∞" problem.

Proof of Proposition 16

Inequality on norms: We have

$$\|X_n - X\|_r \leq \|X_n - X\|_s$$

Counter-example

Definition of a sequence: We consider independent r.v with

$$\mathbf{P}(X_n = n) = \frac{1}{n^{\frac{1}{2}(r+s)}}, \qquad \mathbf{P}(X_n = 0) = 1 - \frac{1}{n^{\frac{1}{2}(r+s)}}$$

Convergence: If $r < s$ we have

1. $X_n \xrightarrow{L^r} 0$
2. X_n does not converge in L^s

Markov's inequality

Proposition 17.

Let

$$X \text{ random variable with } X \in L^1(\Omega)$$

Then for all $a > 0$ we have

$$\mathbf{P}\left(|X| \geq a\right) \leq \frac{\mathbf{E}\left[|X|\right]}{a}$$

$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$

Proof On the set

$A = \{ \omega ; |X(\omega)| > a \}$,

we have $|X(\omega)| > a$. Therefore

$$|X| \geq |X| \mathbb{1}_A \geq a \mathbb{1}_A$$

Since $X \in L^1(\Omega)$, one can take $\mathbb{E}$:

$$\mathbb{E}[|X|] \geq \mathbb{E}[a \mathbb{1}_A] = a \mathbb{P}(A)$$

$$\Rightarrow \boxed{\mathbb{P}(A) \leq \frac{\mathbb{E}[|X|]}{a}}$$

Proof of Proposition 17

Deterministic inequality: Set

$$A = \{|X| \geq a\}$$

Then we have

$$|X| \geq a \, \mathbf{1}_A \,, \quad \text{almost surely}$$

Expectations: Taking expectations above, we get

$$\mathbf{E}\left[|X|\right] \geq a \, \mathbf{P}(A)$$

Convergence in $L^p(\Omega)$ and in probablity

Proposition 18.

Let

- X_n sequence of random variables
- Assume $X_n \xrightarrow{L^1} X$

Then

$$X_n \xrightarrow{P} X$$

Proof We assume $X_n \xrightarrow{L^1} X$, that is

$$\mathbb{E}[\,|X_n - X|\,] \longrightarrow 0$$

We wish to prove $\boxed{X_n \xrightarrow{P} X.}$ This means that for all ε

$$\mathbb{P}(\,|X_n - X| > \varepsilon\,) \xrightarrow{n\to\infty} 0$$

However

$$\mathbb{P}(\,|X_n - X| \geq \varepsilon\,) \overset{\text{Markov}}{\leq} \frac{\mathbb{E}[\,|X_n - X|\,]}{\varepsilon} \xrightarrow[n\to\infty]{L^1\text{ convergence}} 0$$

Example of $X_n \xrightarrow{P} 0$ but $X_n \not\xrightarrow{L^1} 0$.

Take X_n all ⫫ such that

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^2} \qquad \mathbb{P}(X_n = n^3) = \frac{1}{n^2}$$

Then for all $0 < \varepsilon < 1$

$$\mathbb{P}(|X_n - 0| > \varepsilon) = \mathbb{P}(X_n = n^3)$$

$$= \frac{1}{n^2} \xrightarrow{n \to \infty} 0$$

Thus $\boxed{X_n \xrightarrow{P} 0}$

Recall

$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^2}$ $\qquad$ $\mathbb{P}(X_n = n^3) = \frac{1}{n^2}$

Claim: $\boxed{X_n \overset{L^1}{\not\to} 0}$. Indeed

$$\mathbb{E}[\,|X_n - 0|\,] = \mathbb{E}[X_n]$$

$$= 0 \times \mathbb{P}(X_n = 0) + n^3\, \mathbb{P}(X_n = n^3)$$

$$= \frac{n^3}{n^2} \xrightarrow{n \to \infty} \infty$$

Proof of Proposition 18

Applying Markov's inequality: For $\varepsilon > 0$, we have

$$\mathbf{P}\left(|X_n - X| > \varepsilon\right) \leq \frac{\mathbf{E}[|X_n - X|]}{\varepsilon}$$

Then take $n \to \infty$

Counter-example

Definition of a sequence: We consider independent r.v with

$$\mathbf{P}\left(X_n = n^3\right) = \frac{1}{n^2}, \qquad \mathbf{P}\left(X_n = 0\right) = 1 - \frac{1}{n^2}$$

Convergence: We have

1. $X_n \xrightarrow{P} 0$
2. X_n does not converge in L^1

Proof of counter-example for X_n (1)

Some notation: For $\varepsilon > 0$ and $X = 0$ set:

$$A_k(\varepsilon) = \{|X_k - X| > \varepsilon\}$$

Convergence in probability: We have

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{P}\left(A_n(\varepsilon)\right) &= \lim_{n\to\infty} \mathbf{P}\left(X_n = n^3\right) \\
&= \lim_{n\to\infty} \frac{1}{n^2} \\
&= 0
\end{aligned}$$

Thus

$$X_n \xrightarrow{\mathrm{P}} 0$$

Proof of counter-example for X_n (2)

Non convergence in L^1: We have

$$\mathbf{E}[|X_n|] = \mathbf{E}[X_n] = n$$

Thus

$$X_n \overset{L^1}{\not\to} 0$$

Outline

Limsup of sets

$\omega \in \limsup A_n$ iff $\forall n_0 \geq 1, \exists k \geq n_0$ s.t. $\omega \in A_k$
iff $\omega \in$ an infinity of A_n

Definition 19.

Let

- $\{A_n;\ n \geq 1\}$ sequence in $\mathcal{F}$

We define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \in \mathcal{F}$$

Interpretation: We also have

$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega;\ \omega \text{ belongs to an infinity of } A_n\text{'s}\}$$

Borel-Cantelli lemma

Theorem 20.

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We assume

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$$

Then we have

ω "never" belongs to an infinity of A_n's

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 0$$