

Two inequalities an (E) (i) $F_n(x) \leq F(x+\varepsilon) + \Re(x-x_n) > \varepsilon$) $(\alpha) F(x-\varepsilon) \leq F_n(x) + a_n(\varepsilon)$ \Rightarrow $F_n(x) \ge F(x-\varepsilon) - a_n(\varepsilon)$ we thus get $F(x-\varepsilon) - \alpha_n(\varepsilon) \leq F_n(x) \leq F(x+\varepsilon) + \alpha_n(\varepsilon)$ Recall: lim an (E) = 0, clue to n-200 Xn - 20 Xn - 20 Xn

we have

 $F(x-\varepsilon) - a_n(\varepsilon) \leq F_n(x) \leq F(x+\varepsilon) + a_n(\varepsilon)$

we wish to take limits in n on all

sides of the inequalities

Problem: Even though Fn(x) E TO, 13, we don't know if it is is convergent

Solution: Take limsup, liminf

Def for a sequence un = Mn lim sup Un = lim sup Uk Note: n ~ Mn is s If |Un| baunded limn-- Mn exists and is finite Def lim inf Un = lim inf Un n 200 k3h If Un bounded, then tim mn exists and is finite



 $F(x-\varepsilon) - a_n(\varepsilon) \leq F_n(x) \leq F(x+\varepsilon) + a_n(\varepsilon)$

We can take limsup and lim inf we get $F(x-\varepsilon) \in \lim inf F_n(x)$ $\leq \lim_{n \to \infty} \sup F_n(x)$ < F(X+E)

Jummary: For all E>O, we have

$F(x-\varepsilon) \leq l \leq L \leq F(x+\varepsilon)$

We now have $\varepsilon \rightarrow 0$. If x is a point of continuity for F, we have $\lim_{\varepsilon \to 0} F(z - \varepsilon) = \lim_{\varepsilon \to 0} F(z + \varepsilon) = F(z)$ Thus l = L = F(x)

Conclusion we have obtained, if x point of continuity for F.

 $\liminf_{x \to \infty} F_n(x) = \lim_{x \to \infty} F_n(x) = F(x)$

Thus

 $\lim F_n(x) = F(x)$

and $X_n \xrightarrow{(d)} X$

Proof of Proposition 15 (1)

Notation: Set

$$F_n(x) = \mathbf{P}(X_n \le x), \qquad F(x) = \mathbf{P}(X \le x)$$

Aim: Prove that

 $\lim_{n\to\infty} F_n(x) = F(x)$ if F is continuous at x

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Proof of Proposition 15 (2)

1st decomposition: We have

$$F_n(x) = \mathbf{P}(X_n \le x, X \le x + \varepsilon) + \mathbf{P}(X_n \le x, X > x + \varepsilon)$$

$$\le F(x + \varepsilon) + \mathbf{P}(|X_n - X| > \varepsilon)$$

2nd decomposition: We have

$$\begin{array}{rcl} F(x-\varepsilon) &=& \mathbf{P}\left(X \leq x-\varepsilon, \ X_n \leq x\right) + \mathbf{P}\left(X \leq x-\varepsilon, \ X_n > x\right) \\ &\leq& F_n(x) + \mathbf{P}\left(|X_n - X| > \varepsilon\right) \end{array}$$

Summary:

 $F(x-\varepsilon) - \mathbf{P}(|X_n - X| > \varepsilon) \le F_n(x) \le F(x+\varepsilon) + \mathbf{P}(|X_n - X| > \varepsilon)$

Proof of Proposition 15 (3)

Limits as $n \to \infty$: Since $X_n \xrightarrow{(P)} X$, we have

$$F(x-\varepsilon) \leq \liminf_{n\to\infty} F_n(x) \leq \limsup_{n\to\infty} F_n(x) \leq F(x+\varepsilon)$$

Limits as $\varepsilon \to 0$: If F is continuous at x, we get

$$F(x) = \liminf_{n \to \infty} F_n(x) = \limsup_{n \to \infty} F_n(x) = F(x)$$

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Image: A matrix

Convergence in $L^{p}(\Omega)$

Proposition 16.

Let

• X_n sequence of random variables

• Assume
$$X_n \xrightarrow{L^s} X$$
 for $s > r$

Then

$$X_n \xrightarrow{L^r} X$$

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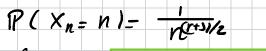
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Xn - L3 × if Pel $\lim_{x \to \infty} (E [[X_n - X]^3])^{\frac{1}{3}} = 0$ $= || \times_n - \times ||_{L^{S}(\mathcal{R})} = || \times_n - \times ||_{S}$ Proof of Prop 16 Il Xn SX rhen s>r $\leq \|X_n - X\|_{L^r} \leq$ $\|X_n - X\|_{L^s}$ 0 Thus Xn L' X

Example of sequence such that if s>r we have Xn Ex but Xn # X Take (Xn Inzi all I with $P(X_n = 0) = 1 - \frac{1}{n^{(C+3)/2}}$ • $P(X_{n} = n) = \frac{1}{\sqrt{(1+3)/2}}$

Recal

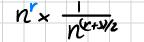
 $\mathbb{P}(X_n=0) = 1 - \frac{1}{n^{(c+s)/2}}$



Claim 1: Xn -> O in L' . Indeed

EIIXn-OITJ = E[(Xn/T]

 $O \times \mathbb{P}(X_n = O) + n' \mathbb{P}(X_n = n)$

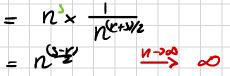




Claim 2: Xn ×0 in L°. Indeed

 $E \int [X_n - O]^3 J = E [(X_n)^3 J]$

 $O \times \mathbb{P}(X_n = O) + n^{S} \mathbb{P}(X_n = n)$



Note Here (Xn)nzo has an "escape to s" problem.

Proof of Proposition 16

Inequality on norms: We have

 $||X_n - X||_r \le ||X_n - X||_s$

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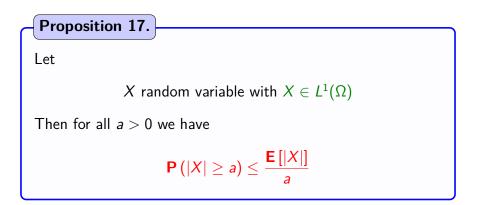
Definition of a sequence: We consider independent r.v with

$$\mathbf{P}(X_n = n) = \frac{1}{n^{\frac{1}{2}(r+s)}}, \qquad \mathbf{P}(X_n = 0) = 1 - \frac{1}{n^{\frac{1}{2}(r+s)}}$$

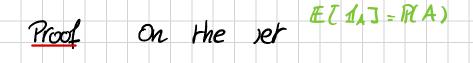
Convergence: If r < s we have

- $X_n \xrightarrow{L^r} 0$
- 2 X_n does not converge in L^s

Markov's inequality



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$A = \langle \omega; (X(\omega)) > a \rangle,$

we have $(X(\omega)) > \alpha$. Therefore

$|X| \ge |X| \mathbf{1}_A \ge a \mathbf{1}_A$

Since XEL'(SL), one can take E:

$E[IXI] \geq E[a \mathbf{1}_A] = a P(A)$

$\Rightarrow \mathbb{P}(A) \leq \frac{\mathbb{E}[X]}{a}$

Proof of Proposition 17

Deterministic inequality: Set

$$A = \{|X| \ge a\}$$

Then we have

 $|X| > a \mathbf{1}_A$, almost surely

Expectations: Taking expectations above, we get

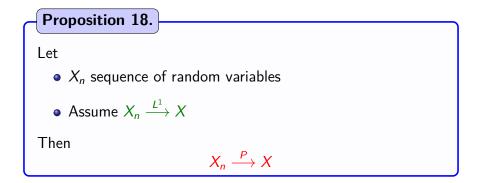
 $\mathbf{E}[|X|] > a \mathbf{P}(A)$

Image: A matrix

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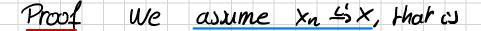
Convergence in $L^{p}(\Omega)$ and in probablity



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$E[I \times n - \times I] \longrightarrow O$

We wish to prove $X_n \xrightarrow{P} X$. This means that for all ε

 $\mathbb{P}(|X_n - X| > \varepsilon) \xrightarrow{n \to 0} O$

However $P(|X_n-X| \ge E) \le E$

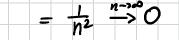
convergence

Example of Xn P>O but Xn #0. Take xn all 11 ruch that

$P(x_n=0)=1-\frac{1}{n^2}$ $P(x_n=n^3)=\frac{1}{n^2}$

Then for all OSE < 1

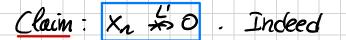
$\mathbb{P}(|X_n-0| \geq \varepsilon) = \mathbb{P}(|X_n=n^3|)$



Thus Xn -> 0



$P(x_n = 0) = 1 - \frac{1}{n^2}$ $P(x_n = n^3) = \frac{1}{n^2}$



$E[X_n-0] = E[X_n]$

$= O \times P(X_n = O) + n^3 P(X_n = n^3)$

$= \frac{n^3}{n^2} \xrightarrow{n \to \infty} \infty$

Proof of Proposition 18

Applying Markov's inequality: For $\varepsilon > 0$, we have

$$\mathsf{P}(|X_n - X| > \varepsilon) \le \frac{\mathsf{E}[|X_n - X|]}{\varepsilon}$$

Then take $n \to \infty$

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Definition of a sequence: We consider independent r.v with

$$\mathbf{P}(X_n = n^3) = \frac{1}{n^2}, \qquad \mathbf{P}(X_n = 0) = 1 - \frac{1}{n^2}$$

Convergence: We have

- $X_n \xrightarrow{P} 0$
- 2 X_n does not converge in L^1

Proof of counter-example for X_n (1)

Some notation: For $\varepsilon > 0$ and X = 0 set:

$$A_k(\varepsilon) = \{|X_k - X| > \varepsilon\}$$

Convergence in probability: We have

$$\lim_{n \to \infty} \mathbf{P} (A_n(\varepsilon)) = \lim_{n \to \infty} \mathbf{P} (X_n = n^3)$$
$$= \lim_{n \to \infty} \frac{1}{n^2}$$
$$= 0$$

 $X_n \xrightarrow{\mathsf{P}} 0$

Thus

gence of r.v	Probability Theory
gence of i.v	Frobability Theory

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Proof of counter-example for X_n (2)

Non convergence in L^1 : We have

$$\mathsf{E}[|X_n|] = \mathsf{E}[X_n] = n$$

Thus

 $X_n \not\rightarrow^{L^1} 0$

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Outline

Introduction

- 1.1 Basic probability structures
- 1.2 Buffon's needle
- 1.3 Convergence of functions

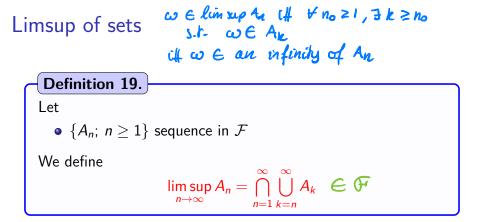
2 Modes of convergence

- 2.1 Reviewing the modes of convergence
- 2.2 Results for P and L^p convergences

2.3 Results for almost sure convergence

- 2.4 Cases of inverse relations for modes of convergence
- 2.5 Inverse method for simulation
- 2.6 Results for convergence in distribution

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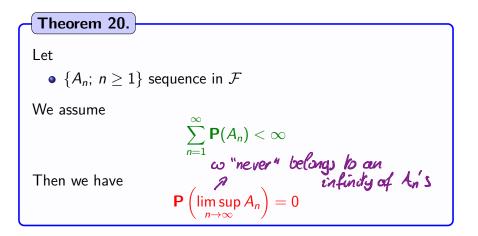
Interpretation: We also have

 $\limsup_{n\to\infty} A_n = \{\omega \in \Omega; \ \omega \text{ belongs to an infinity of } A_n \text{'s} \}$

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Borel-Cantelli lemma



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