Question For a general $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, how do you define $\mathbb{E}[X]$?

① If X discrete, $X : \Omega \to E = \{x_i ; i \geq 1\}$. Then

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} x_i \, \mathbb{P}(X = x_i)$$

② If X admits a density f. Then

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \, f(x) \, dx$$

③ X is $\mathbb{R}$-valued with cdf F. Then

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \, dF(x)$$

$\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$

④ General case: follow 544. Start with $X \geq 0$. Then

(i) One can approximate X by a sequence of X_n which are "simple":

$$X_n = \sum_{i=1}^{N} \alpha_i^{(n)} \mathbb{1}_{A_i^{(n)}}$$

and $X_n \nearrow X$

we have $\mathbb{E}[X_n] = \sum_{i=1}^{N} \alpha_i^{(n)} \mathbb{P}(A_i^{(n)})$

(ii) Since $X_n \nearrow$, $\mathbb{E}[X_n]$ is also increasing. Then

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n]$$

We say that $X \in L^1(\Omega)$ if $\mathbb{E}[X] < \infty$

Case X signed. Just write

$$X = X^+ - X^- \quad (\text{Rmk } X \in L^1 \Leftrightarrow |X| \in L^1)$$

Then $X \in L^1(\Omega)$ if $X^+ \in L^1(\Omega)$ and $X^- \in L^1(\Omega)$
and

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-] = \int_\Omega X(\omega)\, dP(\omega)$$