Borel-Cantelli lemma

Theorem 20.

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We assume

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$$

Then we have

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 0$$

$\limsup A_n = \bigcap_{n=1}^{\infty} \boxed{\bigcup_{k\geq n} A_k} \equiv B_n$

Proof of B-C

We have that $n \mapsto B_n$ is $\searrow$

Recall: (i) If $n \mapsto C_n$ is $\nearrow$, then
$\mathbb{P}\left(\bigcup_{n=1}^{\infty} C_n\right) = \lim_{n\to\infty} \mathbb{P}(C_n)$

(ii) If $n \mapsto D_n$ is $\searrow$, then
$\mathbb{P}\left(\bigcap_{n=1}^{\infty} D_n\right) = \lim_{n\to\infty} \mathbb{P}(D_n)$

Here $\limsup A_n = \bigcap_{n=1}^{\infty} B_n$

$$\limsup A_n = \bigcap_{n=1}^{\infty} \boxed{\bigcup_{k\geq n} A_k} \equiv B_n$$

Computation

$$\mathbb{P}(\limsup A_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) \qquad n \mapsto B_n \searrow$$

$$= \lim_{n\to\infty} \mathbb{P}(B_n)$$

$$= \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k\geq n} A_k\right) \quad \text{rough bound!}$$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \underbrace{\mathbb{P}(A_k)}_{a_k} = 0$$

Recall: If $\sum a_k < \infty$, then

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = 0$$

Emile Borel

Emile Borel's life:

- Lifespan: 1872-1956, $\simeq$ Paris
- $\#$ 1 student in France
  $\hookrightarrow$ for his academic year
- Contributions in analysis and probability
- Active in politics
- Minister of Navy in 1924-25
- Resistance against nazi occupation
- Introduced the ∞ monkey theorem

Fact: "Only" 14 mathematical objects named after Borel ...

Proof of Theorem 20 (1)

A non-increasing sequence: For $N \geq 1$ define

$$B_N = \bigcup_{k=N}^{\infty} A_k$$

Then

1. $N \mapsto B_N$ is non-increasing
2. $\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} B_N$

Proof of Theorem 20 (2)

Computing the probability: We have

$$
\begin{aligned}
\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) &= \mathbf{P}\left(\bigcap_{N=1}^{\infty} B_N\right) \\
&= \lim_{N\to\infty} \mathbf{P}\left(B_N\right) \\
&= \lim_{N\to\infty} \mathbf{P}\left(\bigcup_{k=N}^{\infty} A_k\right) \\
&\leq \lim_{N\to\infty} \sum_{k=N}^{\infty} \mathbf{P}\left(A_k\right) \\
&= 0
\end{aligned}
$$

A.s convergence and limsup

Proposition 21.

Consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables
- For $\varepsilon > 0$ set $A_n(\varepsilon) = \{|X_n - X| > \varepsilon\}$

Then

1. We have

   $X \xrightarrow{P} X \iff P(A_n(\varepsilon)) \to 0 \quad \forall \varepsilon > 0$

   $$X_n \xrightarrow{\text{a.s}} X \iff \mathbf{P}\left(\limsup_{n\to\infty} A_n(\varepsilon)\right) = 0 \text{ for all } \varepsilon > 0$$

2. It holds:

   $$\sum_{n=1}^{\infty} \mathbf{P}\left(A_n(\varepsilon)\right) < \infty \text{ for all } \varepsilon > 0 \implies X_n \xrightarrow{\text{a.s}} X$$

Notation Consider

Q: Is $C \in \mathcal{F}$?

$$C = \{\omega \in \Omega ;\ \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$$

$$A_n(\varepsilon) = \{\omega \in \Omega ;\ |X(\omega) - X_n(\omega)| > \varepsilon\}$$

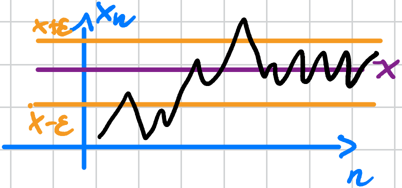
$$A(\varepsilon) = \limsup A_n(\varepsilon)$$

Claim

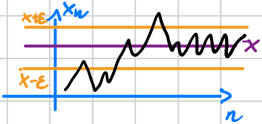
The difference $|X - X_n|$ will be $\leq \varepsilon$ for $n \geq n_0(\omega)$

$$C = \bigcap_{\varepsilon > 0} (A(\varepsilon))^c$$

$$= \bigcap_{m \geq 1} \left(A\left(\tfrac{1}{m}\right)\right)^c$$



$$C = \bigcap_{\varepsilon>0} (A(\varepsilon))^c = \bigcap_{m\geq 1} \left(A\left(\tfrac{1}{m}\right)\right)^c$$



Look now at C^c. Then

$$X_n \xrightarrow{a.s} X \iff \mathbb{P}(C^c) = 0$$

$$\iff \mathbb{P}\left(\bigcup_{m\geq 1} A\left(\tfrac{1}{m}\right)\right) = 0$$

Rmk $m \mapsto A_m$ is $\nearrow$

$$\iff \lim_{m\to\infty} \mathbb{P}\left(A\left(\tfrac{1}{m}\right)\right) = 0$$

Conclusion: $X_n \xrightarrow{a.s.} X \iff \lim_{m\to\infty} \mathbb{P}\left(A\left(\tfrac{1}{m}\right)\right) = 0$

Conclusion: $X_n \xrightarrow{a.s.} X \iff \lim_{m\to\infty} \mathbb{P}(A(\tfrac{1}{m})) = 0$

Rmk $a_m = \mathbb{P}(A(\tfrac{1}{m}))$ is $\nearrow$

If $a_m \nearrow$, $a_m \geq 0$ and $\lim a_m = 0$, we get

$$0 \leq a_m \leq \lim a_m = 0$$

$\Rightarrow$ $a_m = 0$ for all m's

Conclusion 2: $\boxed{X_n \xrightarrow{a.s.} X \iff \mathbb{P}(A(\tfrac{1}{m})) = 0 \quad \forall\, m \geq 1}$

2nd item of the prop: Assume

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n(\varepsilon)) < \infty \quad \forall \varepsilon$$

$$\overset{\text{B-C}}{\Longrightarrow} \quad \mathbb{P}(\limsup A_n(\varepsilon)) = 0$$

$$\overset{\text{item1}}{\Longrightarrow} \quad \boxed{X_n \to X \quad \text{a.s.}}$$

Proof of Proposition 21 (1)

Claim: Let

$$
\begin{aligned}
C &= \left\{\omega \in \Omega;\ \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} \\
A(\varepsilon) &= \limsup_{n\to\infty} A_n(\varepsilon)
\end{aligned}
$$

Then we have

$$
C = \bigcap_{\varepsilon>0} (A(\varepsilon))^c = \bigcap_{m\geq 1} \left(A\left(\frac{1}{m}\right)\right)^c
$$

Proof of Proposition 21 (2)

Application for almost sure convergence: We have

$$\begin{aligned}
\mathbf{P}\left(C^c\right) = 0 \iff & \mathbf{P}\left(\bigcup_{m \geq 1} A\left(\frac{1}{m}\right)\right) = 0 \\
\iff & \lim_{n \to \infty} \mathbf{P}\left(A\left(\frac{1}{m}\right)\right) = 0 \\
\iff & \mathbf{P}\left(A\left(\frac{1}{m}\right)\right) = 0, \quad \text{for all } m \geq 1
\end{aligned}$$

This proves item 1

Proof of Proposition 21 (3)

Proof of item 2: We write

$$\sum_{n=1}^{\infty} \mathbf{P}\left(A_n(\varepsilon)\right) < \infty \text{ for all } \varepsilon > 0$$

$$\Longrightarrow \mathbf{P}\left(\limsup_{n\to\infty} A_n(\varepsilon)\right) = 0 \text{ for all } \varepsilon > 0$$

$$\Longrightarrow \mathbf{P}\left(\lim_{n\to\infty} X_n = X\right) = 1$$

A.s convergence and convergence in probability

Proposition 22.

Consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables

Then we have:

$$X_n \xrightarrow{\text{a.s}} X \quad \Longrightarrow \quad X_n \xrightarrow{\text{P}} X$$

Recall : If $A_n(\varepsilon) = \{|X - X_n| > \varepsilon\}$ then

$$X_n \xrightarrow{P} X \quad \text{iff} \quad \lim_{n\to\infty} \mathbb{P}(A_n(\varepsilon)) = 0 \quad \forall \varepsilon$$

$$X_n \xrightarrow{a.s.} X \quad \text{iff} \quad \mathbb{P}(\limsup A_n(\varepsilon)) = 0$$

Also recall that $\limsup A_n(\varepsilon)$

$$= \bigcap_{n=1}^{\infty} B_n(\varepsilon)$$

with $B_n(\varepsilon) = \bigcup_{k\geq n} A_k(\varepsilon)$. If $X_n \xrightarrow{a.s.} X$,

we have seen that $\lim_{n\to\infty} \mathbb{P}(B_n(\varepsilon)) = 0$

Now $\mathbb{P}(B_n(\varepsilon)) = \mathbb{P}(\bigcup_{k\geq n} A_k(\varepsilon)) \geq \mathbb{P}(A_n(\varepsilon))$

Conclusion : If $\mathbb{P}(B_n(\varepsilon)) \to 0$

we have $\mathbb{P}(A_n(\varepsilon)) \to 0 \quad \forall \varepsilon$

$$\Rightarrow \quad \boxed{X_n \xrightarrow{P} X}$$

Counter example. Consider $(X_n)_{n\geq 1}$ with X_i's $\perp\!\!\!\perp$, such that

$$\boxed{X_n \sim \mathcal{B}\left(p = \frac{1}{n}\right)}$$

$\Rightarrow$ $\mathbb{P}(X_n = 1) = \frac{1}{n}$, $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}$

For $\varepsilon > 0$,

$$\mathbb{P}(A_n(\varepsilon)) = \mathbb{P}(X_n \geq \varepsilon)$$

$$= \mathbb{P}(X_n = 1) = \frac{1}{n} \xrightarrow{n \to \infty} 0$$

Thus $\boxed{X_n \xrightarrow{P} 0}$

A.S. convergence: If we want a.s. convergence, let

$$B_n(\varepsilon) = \bigcup_{k \geq n} A_k(\varepsilon)$$

We should prove that $\boxed{\lim \mathbb{P}(B_n(\varepsilon)) = 0}$

Here $\boxed{\mathbb{P}(B_n(\varepsilon))} = 1 - \mathbb{P}(B_n(\varepsilon)^c)$

$$= 1 - \mathbb{P}\Big(\big(\bigcup_{k \geq n} A_k(\varepsilon)\big)^c\Big)$$

$$= 1 - \mathbb{P}\Big(\bigcap_{k \geq n} A_k(\varepsilon)^c\Big)$$

$$= 1 - \mathbb{P}\Big(\bigcap_{k \geq n} (X_k = 0)\Big) \overset{\perp\!\!\!\perp}{=} \boxed{1 - \prod_{k=n}^{\infty} \mathbb{P}(X_k = 0)}$$

Summary we have

$$\boxed{\mathbb{P}(B_n(\varepsilon))} = 1 - \prod_{k=n}^{\infty} \mathbb{P}(X_k = 0)$$

$$= 1 - \prod_{k=n}^{\infty} \left(1 - \frac{1}{k}\right)$$

$\ln(1-u) \leq -u$
if $u \in (0, \frac{1}{2})$

$$\boxed{= 1}$$

Brief analysis for the $\prod$:

$$\boxed{\prod_{k=n}^{\infty} \left(1 - \frac{1}{k}\right)} = \exp\left(\ln \prod_{k=n}^{\infty} \left(1 - \frac{1}{k}\right)\right)$$

$$= \exp\left(\sum_{k=n}^{\infty} \ln\left(1 - \frac{1}{k}\right)\right)$$

$$\leq \exp\left(-\sum_{k=n}^{\infty} \frac{1}{k}\right) \boxed{= 0}$$

Conclusion since

$\mathbb{P}(B_n(\varepsilon)) \to 1$,

we don't have $X_n \xrightarrow{a.s.} X$