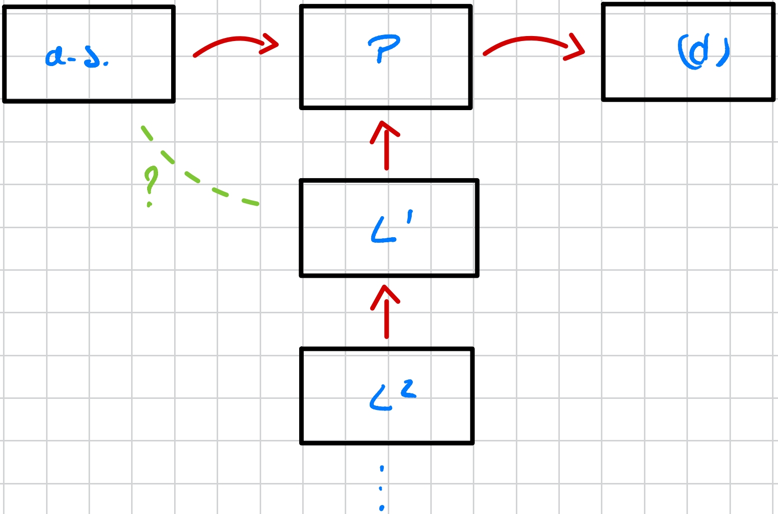
Summary



Non comparison between a.s and L^1-convergence

Proposition 23.

One can find

1. $\{X_n;\, n \geq 1\}$ sequence of random variables such that

$$X_n \xrightarrow{\text{a.s}} X\,, \quad \text{but} \quad X_n \overset{L^1}{\not\to} X$$

2. $\{Y_n;\, n \geq 1\}$ sequence of random variables such that

$$Y_n \xrightarrow{L^1} Y\,, \quad \text{but} \quad Y_n \overset{\text{a.s}}{\not\to} Y$$

Counter-example 1 $(X_n)_{n\geq 1}$ of $\perp\!\!\!\perp$ r.v. with

$$\mathbb{P}(X_n = n^3) = \frac{1}{n^2}, \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n^2}$$

Then

(i) $\boxed{X_n \xrightarrow{a.s.} 0}$. To prove this, consider

$$A_n(\varepsilon) = \{|X - X_n| > \varepsilon\}$$

It is sufficient to prove that $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n(\varepsilon)) < \infty$$

$\mathbb{P}(X_n = n^3) = \frac{1}{n^2}$, $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^2}$

Here, with $X = 0$, and $\varepsilon > 0$,

$$\mathbb{P}(A_n(\varepsilon)) = \mathbb{P}(|X - X_n| > \varepsilon)$$

$$= \mathbb{P}(X_n > \varepsilon) = \mathbb{P}(X_n = n^3) = \frac{1}{n^2}$$

Thus

$$\sum_{n \geq 1} \mathbb{P}(A_n(\varepsilon)) = \sum \frac{1}{n^2} < \infty$$

We get $\boxed{X_n \xrightarrow{a.s.} 0}$

$\mathbb{P}(X_n = n^3) = \frac{1}{n^2}$, $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^2}$

(ii) $\boxed{X_n \not\xrightarrow{L^1} 0}$. Indeed

$$\mathbb{E}[|X_n - 0|] = \mathbb{E}[X_n]$$

$$= 0 \times \left(1 - \frac{1}{n^2}\right) + n^3 \times \frac{1}{n^2}$$

$$= n \not\to 0$$

Counter example 2 Consider $(X_n)_{n\geq 1}$ sequence of $\perp\!\!\!\perp$ r.v with

$$X_n \sim \mathcal{B}\left(\frac{1}{n}\right)$$

Then

(i) We have seen that $\boxed{X_n \overset{a.s.}{\not\to} 0}$

(ii) $\boxed{X_n \xrightarrow{L^1} 0}$, since

$$\mathbb{E}\left[|X_n - 0|\right] = \mathbb{E}\left[X_n\right] = \frac{1}{n} \xrightarrow{n\to\infty} 0$$

Proof of counter-example for X_n (1)

Definition of a sequence (repeated):
We consider independent r.v with

$$\mathbf{P}\left(X_n = n^3\right) = \frac{1}{n^2}, \qquad \mathbf{P}\left(X_n = 0\right) = 1 - \frac{1}{n^2}$$

Convergence: We have

1. $X_n \xrightarrow{\text{a.s}} 0$
2. X_n does not converge in L^1

Proof of counter-example for X_n (2)

Some notation: For $\varepsilon > 0$ set:

$$A_k(\varepsilon) = \{|X_k - X| > \varepsilon\}$$

Almost sure convergence: We have

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{P}(A_n(\varepsilon)) &= \sum_{n=1}^{\infty} \mathbf{P}\left(X_n = n^3\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &< \infty \end{aligned}$$

Thus

$$X_n \xrightarrow{\text{a.s}} 0$$

Proof of counter-example for X_n (3)

Non convergence in $L^1(\Omega)$: We have already seen that

$$\mathbf{E}[|X_n|] = \mathbf{E}[X_n] = n$$

Thus

$$X_n \stackrel{L^1}{\not\to} 0$$

Outline

Case for which $\xrightarrow{(d)}$ yields $\xrightarrow{P}$

Proposition 24.

Consider

- $\{X_n;\ n \geq 1\}$ sequence of random variables

Assume

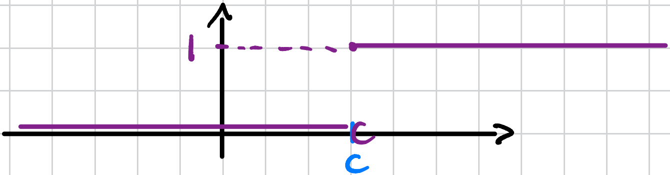
$$X_n \xrightarrow{(d)} c, \text{ where } c \text{ is a constant}$$

Then we have:

$$X_n \xrightarrow{P} c$$

Hyp $\overbrace{\mathbb{P}(X_n \le x)}^{F_n(x)} \longrightarrow \overbrace{\mathbb{P}(X \le x)}^{F(x)}$ for every point of continuity of F

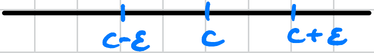
Hyp 2 $X = c \implies F(x) = \mathbb{1}_{[c,\infty)}(x)$



Aim: Prove $\mathbb{P}(|X_n - c| > \varepsilon) \xrightarrow{n\to\infty} 0$

$\forall\, \varepsilon > 0$

Comparation



$$\mathbb{P}(|X - X_n| > \varepsilon) = \mathbb{P}(|X_n - c| > \varepsilon)$$

$$= \mathbb{P}(X_n < c-\varepsilon) + \mathbb{P}(X_n > c+\varepsilon)$$

$$= \mathbb{P}(X_n < c-\varepsilon) + 1 - \mathbb{P}(X_n \leq c+\varepsilon)$$

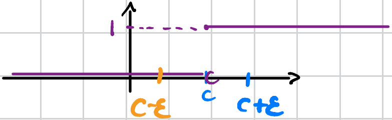
$$\leq \mathbb{P}(X_n \leq c-\varepsilon) \xrightarrow{n \to \infty} F(c-\varepsilon) = 0$$

$$\mathbb{P}(X_n \leq c+\varepsilon) \xrightarrow{n \to \infty} F(c+\varepsilon) = 1$$

Conclusion: As $n \to \infty$,

$$\mathbb{P}(|X - X_n| > \varepsilon) \longrightarrow 0$$

$$\Rightarrow \boxed{X_n \xrightarrow{P} c}$$



Proof of Proposition 24

Expression in terms of cdf: We have

$$\begin{aligned}\mathbf{P}(|X_n - c| > \varepsilon) &= \mathbf{P}(X_n < c - \varepsilon) + \mathbf{P}(X_n > c + \varepsilon) \\ &= \mathbf{P}(X_n < c - \varepsilon) + 1 - \mathbf{P}(X_n \leq c + \varepsilon)\end{aligned}$$

Convergence: Since $X_n \xrightarrow{(d)} X$, we get

$$\lim_{n\to\infty} \mathbf{P}(|X_n - c| > \varepsilon) = 0$$

Case for which $\xrightarrow{\mathrm{P}}$ yields $\xrightarrow{\mathrm{a.s}}$

Proposition 25.

Consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables

Assume

$$X_n \xrightarrow{\mathrm{P}} X$$

Then there exists a subsequence $\{n_k;\, k \geq 1\}$ such that:

$$X_{n_k} \xrightarrow{\mathrm{a.s}} X$$

Proof Consider event $(|X - X_n| > \varepsilon)$
We know $\mathbb{P}(|X - X_n| > \varepsilon) \to 0 \quad \forall \varepsilon$

Define a subsequence recursively: $n_1 = 1$
and

$$n_k = \inf\{n > n_{k-1} \,;\, \mathbb{P}(|X_n - X| > \tfrac{1}{k}) \leq \tfrac{1}{k^2}\}$$

Subsequence :

$$Y_k = X_{n_k}$$

Consider $A_k(\varepsilon) = \{|Y_k - X| > \varepsilon\}$

Aim: prove that $\sum_{k \geq 1/\varepsilon} \mathbb{P}(A_k(\varepsilon)) < \infty$

Recall: $\mathbb{P}(|Y_k - X| > \frac{1}{k}) \leq \frac{1}{k^2}$

Computation

$$\sum_{k \geq \frac{1}{\varepsilon}} \mathbb{P}(A_k(\varepsilon)) \leq \sum_{k \geq \frac{1}{\varepsilon}} \frac{1}{k^2} < \infty$$

Conclusion

$$Y_k \xrightarrow{a.s.} X$$

Proof of Proposition 25 (1)

Definition of n_k: Recursively we set

$$n_k = \inf\left\{n > n_{k-1};\ \mathbf{P}\left(|X_n - X| > \frac{1}{k}\right) \le \frac{1}{k^2}\right\}$$

Some notation: For $\varepsilon > 0$ define:

$$\begin{aligned} Y_k &= X_{n_k} \\ A_k(\varepsilon) &= \{|Y_k - X| > \varepsilon\} \end{aligned}$$

Proof of Proposition 25 (2)

Almost sure convergence: We have

$$\sum_{k=\varepsilon^{-1}}^{\infty} \mathbf{P}\left(A_n(\varepsilon)\right) = \sum_{k=\varepsilon^{-1}}^{\infty} \mathbf{P}\left(|X_{n_k} - X| > \frac{1}{k}\right) \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

Thus

$$Y_k \xrightarrow{\text{a.s}} X$$

Case for which $\xrightarrow{P}$ yields $\xrightarrow{L^r}$

Proposition 26.

Consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables

Assume (bounded convergence) $\xrightarrow{\text{better?}}$ dominated convergence

$X_n \xrightarrow{P} X$, and $|X_n| \leq k$ a.s for all $n \geq 1$ and a given $k > 0$

Then for all $r \geq 1$ we have:

$$X_n \xrightarrow{L^r} X$$

Proof - 1st step : we know that a.s., $|X_n| \leq k$ $\forall n$. We can prove that X is also bounded. To this aim:

$$B_\delta = (|X| \leq k+\delta) \Rightarrow \mathbb{P}(B_\delta) = 1$$

Then

$$\mathbb{P}(B_\delta^c) = \mathbb{P}(|X| > k+\delta) \quad \mathbb{P}(|X_n| \leq k) = 1$$

$$= \mathbb{P}(|X| > k+\delta , |X_n| \leq k)$$

$$\leq \mathbb{P}(|X - X_n| > \delta) \xrightarrow{n \to \infty} 0 \quad \forall \delta > 0$$

Taking $\delta \to 0$, we get $\boxed{\mathbb{P}(|X| \leq k) = 1}$

Step 2: Assume $X_n \xrightarrow{P} X$, $|X_n| \le k$. Then

$$\mathbb{E}[|X-X_n|^r] = \mathbb{E}[|X-X_n|^r \mathbb{1}_{(|X-X_n|>\varepsilon)}] + \mathbb{E}[|X-X_n|^r \mathbb{1}_{(|X-X_n|\le\varepsilon)}]$$

$$\le (2k)^r \, \mathbb{P}(|X-X_n|>\varepsilon) + \varepsilon^r$$

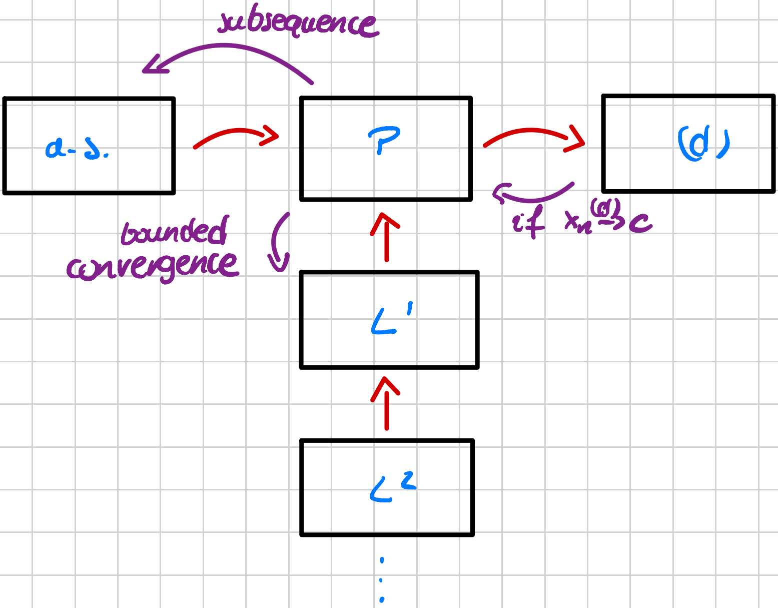
$\longrightarrow \varepsilon^r$ as $n \to \infty$, for all $\varepsilon > 0$

Conclusion

Def: $\mathbb{E}[|X-X_n|^r] \xrightarrow{n\to\infty} 0$

$$\boxed{X_n \xrightarrow{L^r} X}$$

New picture


subsequence
a.s.
P
(d)
bounded convergence
if $x_n \xrightarrow{(d)} c$
L^1
L^2
⋮

Proof of Proposition 26 (1)

Boundedness of X: For $\delta > 0$, set

$$B_\delta = (|X| \le k + \delta)$$

Then for all $n \ge 1$ we have

$$\begin{aligned}
\mathbf{P}\left(B_\delta\right) &\ge \mathbf{P}\left(|X - X_n| \le \delta, \, |X_n| \le k\right) \\
&\ge \mathbf{P}\left(|X_n| \le k\right) - \mathbf{P}\left(|X - X_n| > \delta\right) \\
&= 1 - \mathbf{P}\left(|X - X_n| > \delta\right)
\end{aligned}$$

Taking limits in n, δ we get

$$\mathbf{P}\left(|X| \le k\right) = 1$$

Proof of Proposition 26 (2)

Decomposition of $X_n - X$: For $\varepsilon > 0$ and $n \geq 1$ set

$$A_{n,\varepsilon} = \{|X_n - X| > \varepsilon\}$$

Then

$$|X_n - X|^r \leq \varepsilon^r \mathbf{1}_{A^c_{n,\varepsilon}} + (2k)^r \mathbf{1}_{A_{n,\varepsilon}}$$

Taking expectations: We obtain

$$\mathbf{E}\left[|X_n - X|^r\right] \leq \varepsilon^r \mathbf{1}_{A^c_{n,\varepsilon}} + (2k)^r \mathbf{P}\left(A_{n,\varepsilon}\right)$$

Taking limits: With $n \to \infty$ and $\varepsilon \to 0$ we end up with

$$\lim_{n\to\infty} \mathbf{E}\left[|X_n - X|^r\right] = 0$$

Outline

Right inverse (1)

Definition 27.

Let $F : \mathbb{R} \to [0,1]$ continuous cdf
We define the right inverse F^{-1} as

$$F^{-1} : (0,1) \to \mathbb{R}, \quad y \mapsto \inf \{a \in \mathbb{R};\ F(a) \geq y\}$$

Right inverse (2)

Remarks on right inverse:

(i) If F is strictly increasing, F^{-1} is the inverse of F
$\hookrightarrow$ i.e. $F \circ F^{-1} = F^{-1} \circ F = \mathrm{Id}$

(ii) Graphical method to construct F^{-1}:

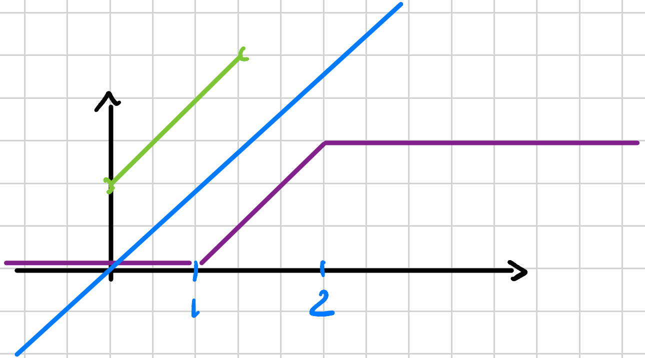
1. Symmetry wrt diagonal
2. Then erase vertical parts

Example: $F(x) = (x-1)\mathbf{1}_{[1,2)}(x) + \mathbf{1}_{[2,\infty)}(x)$
$\hookrightarrow$ $F^{-1}(y) = (1+y)\mathbf{1}_{(0,1)}(y)$

$F \leftarrow \mathcal{U}([1,2])$

Example of right inverse

$$F(x) = (x-1)\, \mathbb{1}_{[1,2)}(x) + \mathbb{1}_{[2,\infty)}(x)$$



$$\boxed{F^{-1}(y) = (y+1)\, \mathbb{1}_{(0,1)}(y)}$$

Rmk F is the cdf of $\mathcal{U}([1,2])$

$\Rightarrow$ $[1,2]$ is what is revelant for the r.v.

On $(1,2)$, F is strictly monotone
$\Rightarrow$ one can compute the inverse

$$F(x) = y \Leftrightarrow x-1 = y$$
$$\Leftrightarrow x = \boxed{y+1 = F^{-1}(y)}$$