MA 538 - Review session 1 for the Final

Problem 56. Let $X_1, X_2, \ldots, X_n$, n be some real valued integrable random variables, independent and equally distributed. We set $m = \mathbf{E}[X_1]$ and $S_n = \sum_{i=1}^n X_i$.

56.1. Compute $\mathbf{E}[S_n | X_i]$ for all $i, 1 \leq i \leq n$.

$$\mathbb{E}[S_n \mid X_i] = \mathbb{E}\Big[\sum_{j=1}^n X_j \mid X_i\Big]$$

$$= \sum_{j=1}^n \mathbb{E}[X_j \mid X_i]$$

$$= \mathbb{E}[X_i \mid X_i] + \sum_{j \neq i} \overbrace{\mathbb{E}[X_j \mid X_i]}^{= \mathbb{E}[X_j] = m}$$

Hence

$$\boxed{\mathbb{E}[S_n \mid X_i] = X_i + (n-1)m}$$

56.2. Compute $\mathbf{E}[X_i|S_n]$ for all $i, 1 \leq i \leq n$.

we use two facts:

(i) $\mathcal{L}(X_1, \dots, X_i, \dots, X_j, \dots, X_n)$

$= \mathcal{L}(X_1, \dots, X_j, \dots, X_i, \dots, X_n)$

and

$S_n(X_1, \dots, X_i, \dots, X_j, \dots, X_n)$

$= S_n(X_1, \dots, X_j, \dots, X_i, \dots, X_n)$

Hence

$$\mathbb{E}[X_i | S_n] = \mathbb{E}[X_j | S_n]$$

(ii) $\sum_{i=1}^{n} \mathbb{E}[X_i | S_n] = \mathbb{E}[S_n | S_n] = S_n$

Thus

$$\boxed{\mathbb{E}[X_i | S_n] = \frac{S_n}{n}}$$

56.3. We assume now that $n = 2$ and that the random variables X_i have a common density f. Compute the conditional density of X_i given S_2. Give a specific expression whenever the law of each X_i is an exponential law.

Density for the couple (X_1, S_2) Let $\varphi \in C_b(\mathbb{R}^2)$. We compute

$$\mathbb{E}[\varphi(X_1, S_2)] = \mathbb{E}[\varphi(X_1, X_1 + X_2)]$$

$$= \int_{\mathbb{R}^2} \varphi(x_1, x_1 + x_2) f(x_1) f(x_2)\, dx_1\, dx_2$$

CV: $x = x_1$, $y = x_1 + x_2$

$\Rightarrow$ $x_1 = x$, $x_2 = y - x$, $|J| = 1$

Thus

$\mathbb{E}[\varphi(X_1, S_2)]$

$$= \int_{\mathbb{R}^2} \varphi(x, y) \overbrace{f(x) f(y-x)}^{\text{density } f_{X_1,S_2} \text{ of couple } (X_1, S_2)}\, dx\, dy \quad (1)$$

Case $X_1 \sim \mathcal{E}(\lambda)$ Then the density of the couple (X_1, S_2) is, according to (1),

$$f_{X_1,S_2}(x, y) = \lambda e^{-\lambda x} \times \lambda e^{-\lambda(y-x)} \mathbb{1}_{(y > x > 0)}$$

$$\Rightarrow \boxed{f_{X_1,S_2}(x, y) = \lambda^2 e^{-\lambda y} \mathbb{1}_{(y > x > 0)}}$$

Conditional density According to our formula in class, we have

$$\mathbb{E}[\varphi(X_1) \mid S_2] = h(S_2),$$

where

$$h(y) = \frac{\int \varphi(x) f_{X_1,S_2}(x,y)\,dx}{\int f_{X_1,S_2}(x)\,dx}$$

$$= \frac{\int \varphi(x) f(x) f(y-x)\,dx}{\int f(z) f(y-z)\,dz}$$

$$= \int \varphi(x) f_{X|Y}(x|y)\,dx,$$

where the conditional density is

$$f_{X|Y}(x|y) = \frac{f(x) f(y-x)}{\int f(z) f(y-z)\,dz}$$

Case $X_i \sim \mathcal{E}(\lambda)$ we have

$$f_{X|Y}(x|y) = \frac{f(x)\, f(y-x)}{\int f(z)\, f(y-z)\, dz}$$

Moreover

- $f(x)\, f(y-x) = \lambda^2 e^{-\lambda y}\, \mathbb{1}_{(0<x<y)}$

- $\int f(z)\, f(y-z)\, dz$
  $= \int \lambda^2 e^{-\lambda y}\, \mathbb{1}_{(y>x>0)}\, dx$
  $= \lambda^2 e^{-\lambda y}\, y$

Hence

$$\boxed{f_{X|Y}(x|y) = \frac{1}{y}\, \mathbb{1}_{[0,y]}(x)}$$

$\longrightarrow$

CRL we have obtained, in the $\mathcal{E}(\lambda)$ case, that

$$\mathcal{L}(X_1 \mid S_2) = \mathcal{U}([0, S_2]) \equiv \mu(\omega, \cdot)$$

This is a CRL. That is

(i) For $\varphi \in C_b(\mathbb{R})$,

$$\mu(\omega, \varphi) = \mathbb{E}[\varphi(X_1) \mid S_2]$$

$$= \frac{1}{S_2(\omega)} \int_0^{S_2(\omega)} \varphi(x) dx$$

Since $\omega \mapsto S_2(\omega)$ is measurable, we also have

$$\omega \mapsto \mu(\omega, \varphi) \text{ measurable}$$

(ii) For every $s > 0$, $\mathcal{U}([0, s])$ is a probability. Hence ω-a.s, $\mathcal{U}([0, S_2(\omega)])$ is a probability

Conclusion:

$$\boxed{\mathcal{U}([0, S_2]) \text{ is a CRL}}$$

Problem 58. Let X and Y be two real valued random variables, such that Y follows an exponential law. We assume that given Y, X is distributed according to a Poisson law with parameter Y (given as a conditional regular law).

58.1. Compute the law of the couple (X, Y), the law of X, and the law of Y given X as a conditional regular law.

Conditioning For $\varphi : \mathbb{N} \times \mathbb{R} \longrightarrow \mathbb{R}$ we have

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}\{\mathbb{E}[\varphi(X,Y) \mid Y]\}$$

$$\overset{\text{crl}}{=} \mathbb{E}\left\{ \sum_{k=0}^{\infty} \varphi(k,Y)\, e^{-Y} \frac{Y^k}{k!} \right\}.$$

Hence

$$\boxed{\mathbb{E}[\varphi(X,Y)] = \sum_{k=0}^{\infty} \int_0^{\infty} \varphi(k,y)\, e^{-2y} \frac{y^k}{k!}\, dy}$$

Expression for the law we can write

$$\mathcal{L}(X,Y) = \sum_{k=0}^{\infty} \delta_k \otimes \mu_k ,$$

where μ_k admits the density

$$f_k(y) = e^{-2y} \frac{y^k}{k!} \mathbb{1}_{\mathbb{R}_+}(y) \qquad (1)$$

computing $E[g(X)]$ From (1) we have

$$E[g(X)] = \sum_{k=0}^{\infty} \int_0^{\infty} g(k)\, e^{-2y} \frac{y^k}{k!}\, dy$$

$$= \sum_{k=0}^{\infty} \frac{g(k)}{k!} \underbrace{\int_0^{\infty} y^k e^{-2y}\, dy}_{= I_k}$$

The quantity I_k can be computed thanks to a change of variable, i.e setting $2y = z$ we get

$$I_k = \int_0^{\infty} \left(\frac{z}{2}\right)^k e^{-z} \frac{dz}{2}$$

$$= \frac{1}{2^{k+1}} \int_0^{\infty} z^k e^{-z}\, dz$$

$$= \frac{1}{2^{k+1}} \Gamma(k+1) = \frac{k!}{2^{k+1}}$$

Thus

$$\boxed{E[g(X)] = \sum_{k=0}^{\infty} \frac{g(k)}{2^{k+1}}}$$

Law of X Computing $\mathbb{E}[g(X)]$ for $g(k) = \mathbb{1}_{(k=n)}$, we get the pdf of X:

$$\boxed{\mathbb{P}(X = n) = \frac{1}{2^{n+1}}, \quad n \geq 0}$$

Note that $X \stackrel{(d)}{=} G - 1$, where $G \sim \mathcal{G}(\frac{1}{2})$.

Computing $\mathbb{E}[h(Y)\,|\,X]$ Since X is a discrete random variable, we have

$$\mathbb{E}[h(Y)\,|\,X] = \ell(X),$$

where

$$\ell(n) = \frac{\mathbb{E}[h(Y)\, \mathbb{1}_{(X=n)}]}{\mathbb{P}(X=n)}, \quad \text{for } n \geq 0$$

We thus compute, thanks to Q 1.1,

$$\ell(n) = 2^{n+1} \sum_{k=0}^{\infty} \int_0^\infty h(y)\, \mathbb{1}_{(k=n)}\, e^{-2y} \frac{y^k}{k!}\, dy$$

$$= \int_0^\infty h(y) \underbrace{2^{n+1} e^{-2y} \frac{y^n}{n!}}_{f_{Y|X}(y|n)}\, dy \qquad (1)$$

Problem 64. Let $\{Y_n; n \geq 1\}$ be a iid sequence of positive random variables such that $\mathbf{E}[Y_j] = 1$, $\mathbf{P}(Y_j = 1) < 1$ and $\mathbf{P}(Y_j = 0) = 0$. We set

$$X_n = \Pi_{j \leq n} Y_j.$$

64.1. Show that X is a martingale.

Filtration we take

$$\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$$

Integrability we have seen

$$Z, W \in L^1(\Omega), \quad Z \perp\!\!\!\perp W \implies ZW \in L^1(\Omega)$$

Here since $Y_j \in L^1(\Omega)$ and the Y_j's are independent, we get $X_n \in L^1(\Omega)$

Martingale property we have

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_n Y_{n+1} | \mathcal{F}_n]$$

$$= X_n \mathbb{E}[Y_{n+1} | \mathcal{F}_n] \qquad (X_n \in \mathcal{F}_n)$$

$$= X_n \mathbb{E}[Y_{n+1}] \qquad (Y_{n+1} \perp\!\!\!\perp \mathcal{F}_n)$$

$$= X_n$$

Hence $\boxed{(X_n) \text{ is a martingale}}$

64.2. Show that $\lim_{n\to\infty} X_n = 0$ a.s

Convergence we have

X_n martingale, $X_n \geq 0$

$\Rightarrow$ $X_n \to X_\infty$ a.s., with $X_\infty \in L^1(\Omega)$

we still need to show that $X_\infty = 0$.

Log-sequence set

$$Z_n = \ln(X_n)$$

Then

$$Z_n = \sum_{j=1}^{n} \ln(Y_j)$$

Easy case we have

$$\mathbb{E}[\ln(Y_j)] \le \ln(\mathbb{E}[Y_j]) \overset{Hyp}{=} 0$$

Assume

$$\ln(Y_j) \in L^1, \quad -m \equiv \mathbb{E}[\ln(Y_j)] < 0$$

Then by LLN we have

$$\lim_{n\to\infty} \frac{1}{n} z_n = -m \quad a.s.$$

$$\Rightarrow \quad \lim_{n\to\infty} z_n = -\infty \quad a.s.$$

$$\Rightarrow \quad \lim x_n = 0 \quad a.s.$$

General strategy we have seen that $x_n \to x_\infty$ a.s., with $x_\infty \in [0,\infty)$ a.s. For $z_n = \ln(x_n)$ this means

$$\lim_{n\to\infty} z_n = \begin{cases} -\infty & \text{if } x_\infty = 0 \\ z_\infty \in (-\infty,\infty) & \text{if } x_\infty > 0 \end{cases}$$

If we wish to prove $x_\infty = 0$, we just have to show that

$$z_\infty = \sum \ln(Y_j) \quad \text{divergent}$$

Proof that $\Sigma \ln(Y_j)$ divergent It is sufficient to prove that

$\ln(Y_j)$ does not converge to 0

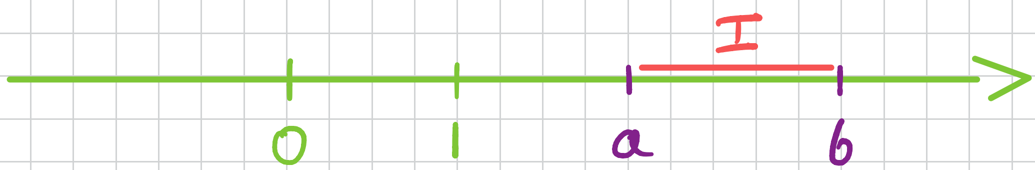
This is due to the fact that "Y_j is away from 1 i.o".

Bounding Y_j Since

$$\mathbb{P}(Y_j = 1) < 0 \quad , \quad E[Y_j] = 1$$

there exists $I = [a,b]$ with $a > 1$ s.t

$$\mathbb{P}(Y_j \in I) \equiv p > 0$$



Thus $\sum_{j=1}^{n} \mathbb{P}(Y_j \in I) = \infty$

Inverse B-C

$$\Longrightarrow \quad \mathbb{P}(Y_j \in I \text{ i.o}) = 1$$

$$\Longrightarrow \quad \boxed{Z_\infty = 0}$$

Problem 69. Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables with a common law given by $\mathbf{P}(Y_n = 1) = p = 1 - \mathbf{P}(Y_n = -1) = 1 - q$. We define $(S_n)_{n\in\mathbb{N}}$ by $S_0 = 0$ and $S_n = \sum_{k=1}^{n} Y_k$.

69.1. We assume that $p = q = \frac{1}{2}$. We set $T_a = \inf\{n \geq 0,\ S_n = a\}$ $(a \in \mathbb{Z}^*)$. Show that $\mathbf{E}(T_a) = +\infty$.

Setting of Pb 68 S_n is a martingale satisfying the conditions of Problem 68, i.e

$$|X_n - X_{n-1}| \leq 1 \equiv M$$

If $\nu \equiv T_a$ also fulfills the conditions of Pb 68, that is

$$T_a \in L^1(\Omega),$$

then we would get

$$E[S_{T_a}] = E[S_0] = 0 \qquad (1)$$

Contradiction Relation (1) is impossible: $S_{T_a} = a$ a.s, hence $E[S_{T_a}] = a$

Thus

$$\boxed{T_a \notin L^1(\Omega)}$$

69.2. Let $T = T_{a,b} = \inf\{n \geq 0,\ S_n = -a \text{ or } S_n = b\}$ $(a, b \in \mathbb{N})$. Using the value of $\mathbf{E}(S_T)$, compute the probability of the event $(S_T = -a)$.

T is a.s. finite If $x \in (-a, b)$ we have

$$\mathbb{P}(x + S_{a+b} \notin (-a,b)) \geq \frac{1}{2^{b+a}} > 0$$

Then write

$$\sum_{k=1}^{a+b} Y_{n(a+b)+k}$$

$$a_{n+1} \equiv \mathbb{P}(T > (n+1)(a+b))$$

$$= \mathbb{E}\{ \mathbb{1}_{(T > n(a+b))} \mathbb{1}_{(S_{n(a+b)} + \tilde{S}_{a+b} \in (-a,b))} \}$$

$$= \mathbb{E}\{ \mathbb{1}_{(T > n(a+b))} \mathbb{E}[\mathbb{1}_{(S_{n(a+b)} + \tilde{S}_{a+b} \in (-a,b))} \mid \mathcal{F}_{n(a+b)}] \}$$

$$= \mathbb{E}[\mathbb{1}_{(x + \tilde{S}_{a+b} \in (-a,b))}] \mid x = S_{n(a+b)}$$

$$\leq \sup_{x \in (-a,b)} \mathbb{P}(x + S_{a+b} \in (-a,b))$$

$$\leq 1 - \frac{1}{2^{b+a}} \equiv \rho$$

Hence

$$a_{n+1} \leq \rho\, \mathbb{E}[\mathbb{1}_{(T > n(a+b))}] = \rho\, a_n$$

we get $a_n \leq a_0 \rho^n$, hence $\lim_{n \to \infty} a_n = 0$ and $\boxed{T < \infty \text{ a.s.}}$

Decomposing S_T we have

$$S_T = -a \, \mathbb{1}_{(S_T=-a)} + b \, \mathbb{1}_{(S_T=b)}$$

Optional stopping we have that

$$\{ S_{n \wedge T} \, ; \, n \geq 1 \}$$

is a bounded martingale. Hence

$$\mathbb{E}[S_T] = 0$$

which can be written as

$$-a \, \underbrace{\mathbb{P}(S_T = -a)}_{x} + b \, \underbrace{\mathbb{P}(S_T = b)}_{y} = 0$$

System The couple (x,y) solves

$$\begin{cases} x + y = 1 \\ -ax + by = 0 \end{cases}$$

$$\Rightarrow \quad (a+b) \, y = a$$

$$\Rightarrow \quad y = \frac{a}{a+b} \qquad x = \frac{b}{a+b}$$

we get $\boxed{\mathbb{P}(S_T = -a) = \frac{b}{a+b}}$

69.3. Show that $Z_n = S_n^2 - n$ is a martingale, and from the value of $\mathbf{E}(Z_T)$ compute $\mathbf{E}(T)$.

<u>Z_n is a martingale</u> we have

(i) $Z_n = \varphi_n(S_n) \Rightarrow Z_n \in \mathcal{F}_n$

(ii) $|S_n| \leq n$, hence $Z_n \in L^\infty(\Omega)$

$\Rightarrow$ $Z_n \in L^1(\Omega)$

(iii) we have

$$Z_{n+1} = S_{n+1}^2 - (n+1)$$
$$= (S_n + Y_{n+1})^2 - (n+1)$$
$$= S_n^2 + 2 S_n Y_{n+1} + \overbrace{Y_{n+1}^2}^{=1} - (n+1)$$
$$= Z_n + 2 S_n Y_{n+1}$$

Hence

$$\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = Z_n + 2 S_n \mathbb{E}[Y_{n+1} | \mathcal{F}_n]$$
$$= Z_n + 2 S_n \mathbb{E}[Y_{n+1}] = Z_n$$

Hence

$$\boxed{(Z_n) \text{ is a martingale}}$$

Optional stopping problem Here T is not bounded and

$$\{ Z_{n \wedge T} \; ; n \geq 0 \} \text{ not bounded}$$

Application of the martingale prop For a fixed n we have

$$\mathbb{E}[Z_{n \wedge T}] = 0$$

$$\Rightarrow \quad \mathbb{E}[S^2_{n \wedge T}] = \mathbb{E}[T \wedge n] \quad (2)$$

Limit in lhs of (1) we have

$$S_{n \wedge T} \xrightarrow{a.s.} S_T \quad , \text{ since } T < \infty \text{ a.s.}$$

$$| S^2_{n \wedge T} | \leq \max (a^2, b^2)$$

By dominated convergence,

$$\lim_{n \to \infty} \mathbb{E}[S^2_{n \wedge T}] = \mathbb{E}[S_T]$$

Limit in lhs of (1) By monotone convergence

$$\lim_{n \to \infty} \mathbb{E}[T \wedge n] = \mathbb{E}[T]$$

Computing $\mathbb{E}[T]$ we have obtained

$$\mathbb{E}[T] = \mathbb{E}[S_T^2]$$

$$= a^2 \, \mathbb{P}(S_T = -a) + b^2 \, \mathbb{P}(S_T = b)$$

$$= \frac{a^2 b}{a+b} + \frac{a b^2}{a+b}$$

Hence

$$\mathbb{E}[T] = ab$$