Solutions for

MA/STAT 538 Spring 2025
Probability Theory

Midterm

- You can use a calculator.
- A 2 pages long handwritten cheat sheet is allowed. It should only contain formulae and theorems (no example, no solved problem).
- You have 60 minutes.
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- GOOD LUCK!

Name:

Purdue ID:

Problem 1. We consider two sequences of random variables $\{X_n;\, n \geq 1\}$ and $\{Y_n;\, n \geq 1\}$, as well as another random variable Y. All random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We assume that

$$X_n \xrightarrow{(\mathrm{d})} 0, \quad \text{and} \quad Y_n \xrightarrow{\mathrm{P}} Y.$$

Let also $g : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function such that $y \mapsto g(x, y)$ is a continuous function of y for all x, and $(x, y) \mapsto g(x, y)$ is continuous at every point of the form $(0, y)$. The aim of the problem is to prove that $g(X_n, Y_n) \xrightarrow{\mathrm{P}} g(0, Y)$.

1.1. Prove that the set $\{\omega \in \Omega;\, g(X_n(\omega), Y_n(\omega)) \to g(0, Y(\omega))\}$ is an element of the σ-algebra $\mathcal{F}$.

Reformulation of the problem set

$Z_n = g(X_n, Y_n)$

$Z = g(0, Y)$

Since X_n, Y_n and Y are r.v. and g is meaxurable, we have that

Z_n, Z are r.v.

Next set

$$\boxed{C = \{\omega \in \Omega \;;\; \lim_{n \to \infty} Z_n(\omega) = Z(\omega)\}}$$

we have to prove that

$C \in \mathcal{F}$

Expression with lim sup

For $\varepsilon > 0$ set

$$A_n(\varepsilon) = \{ |Z_n - Z| > \varepsilon \}$$

we have seen in class (see proof of Prop 21 - convergence of r.v) that

$$C^c = \bigcup_{m \geq 1} \limsup_n A_n\left(\frac{1}{m}\right) \qquad (1)$$

Now $A_n(\frac{1}{m}) \in \mathcal{F}$, due to the fact that $Z_n - Z$ is a r.v. From (1) we thus get

$$\boxed{C \in \mathcal{F}}$$

1.2. Let $\delta, \hat{\varepsilon} > 0$ be given. Prove that one can find $M = M_{\delta,\hat{\varepsilon}}$ large enough such that for $n \geq M$ we have

$$\mathbf{P}(|X_n| > \delta) < \hat{\varepsilon}, \quad \mathbf{P}(|Y - Y_n| > \delta) < \hat{\varepsilon}, \quad \text{and} \quad \mathbf{P}(|Y| > M) < \hat{\varepsilon}.$$

(i) Since $X_n \xrightarrow{(d)} 0$ and 0 is a constant, we also have

$$X_n \xrightarrow{P} 0,$$

due to Prop 24 - Convergence of r.v.
Hence given $\delta > 0$ one can find n_1 s.t. for all $n \geq n_1$ we have

$$\mathbb{P}(|X_n| > \delta) < \hat{\varepsilon}$$

(ii) Along the same lines, since $Y_n \xrightarrow{P} Y$, one can find n_2 s.t. for all $n \geq n_2$ we have

$$\mathbb{P}(|Y - Y_n| > \delta) < \hat{\varepsilon}$$

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(iii) Let F be the cdf of Y.
Then

$$\mathbb{P}(|Y| > K) = 1 - F(K) + F(-K)$$

Moreover, any cdf verifies

$$\lim_{K \to \infty} (1 - F(K)) = 0$$

$$\lim_{K \to \infty} F(-K) = 0$$

Hence one can find K_0 large enough such that for $K \geq K_0$ we have

$$\mathbb{P}(|Y| > K) < \hat{\varepsilon}$$

(iv) Take $M = \sup\{n_1, n_2, K_0\}$.
We obtain that for $n \geq M$ we have

$$\boxed{\begin{aligned} &\mathbb{P}(|X_n| > \delta) < \hat{\varepsilon} \\ &\mathbb{P}(|Y - Y_n| > \delta) < \hat{\varepsilon} \\ &\mathbb{P}(|Y| > M) < \hat{\varepsilon} \end{aligned}}$$

1.3. Prove that g is uniformly continuous on the region of $\mathbb{R}^2$ defined by $D_M = \{0\} \times [-M, M]$, where M is given in Question 1.2. Deduce that for $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in [-M, M]$ we have

$$\Big(|\xi| \le \delta \ \text{ and } \ |\eta - y| \le \delta\Big) \quad \Longrightarrow \quad |g(\xi, \eta) - g(0, y)| < \varepsilon$$

(i) Our assumption is that g is continuous at every point $(x,y) \in D_M$.
Moreover D_M is compact in $\mathbb{R}^2$.
Hence

g is uniformly continuous on D_M

$\longrightarrow$

(ii) Here we consider two norms on $\mathbb{R}^2$:

$$\|(a,b)\|_2 = (a^2+b^2)^{\frac{1}{2}}$$

$$\|(a,b)\|_\infty = \sup\{|a|, |b|\}$$

Those 2 norms are equivalent on $\mathbb{R}^2$. The usual definition of absolute continuity for g on D_M would be that for all $\varepsilon>0$ there exists $\delta>0$ such that for all $y \in [-M,M]$ we have

$$\|(\xi,\eta)-(0,y)\|_2 < \delta$$

$$\Rightarrow \quad |g(\xi,\eta)-g(0,y)| < \varepsilon$$

Since $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, we also have, for all $y \in [-M,M]$,

$$\boxed{\begin{aligned} &\|(\xi,\eta)-(0,y)\|_\infty < \delta \\ &\Rightarrow \quad |g(\xi,\eta)-g(0,y)| < \varepsilon \end{aligned}}$$

1.4. Prove that

$$\mathbf{P}\left(|g(X_n, Y_n)| \geq \varepsilon\right) \leq \mathbf{P}\left(|X_n| > \delta\right) + \mathbf{P}\left(|Y_n - Y| > \delta\right) + \mathbf{P}\left(|Y| > M\right).$$

Reduction to a single probability we set

$$B^n_{\varepsilon,\delta} = (|X_n| \leq \delta) \cap (|Y_n - Y| \leq \delta) \cap (|Y| \leq M)$$

we evaluate

$$\mathbb{P}(|g(X_n, Y_n) - g(0,Y)| \geq \varepsilon)$$

$$= \mathbb{P}((|g(X_n, Y_n) - g(0,Y)| \geq \varepsilon) \cap B^n_{\varepsilon,\delta})$$

$$+ \mathbb{P}(|g(X_n, Y_n) - g(0,Y)| \geq \varepsilon \cap (B^n_{\varepsilon,\delta})^c)$$

Furthermore, Question 1.3 asserts that on $B^n_{\varepsilon,\delta}$ we have

$$|g(X_n, Y_n) - g(0,Y)| < \varepsilon$$

Hence

$$\mathbb{P}(|g(X_n, Y_n) - g(0,Y)| \geq \varepsilon) \leq \mathbb{P}((B^n_{\varepsilon,\delta})^c) \tag{2}$$

Evaluating $\mathbb{P}((B^n_{\varepsilon,\delta})^c)$ we have

$$\mathbb{P}((B^n_{\varepsilon,\delta})^c)$$

$$= \mathbb{P}((|X_n|>\delta) \cup (|Y_n-Y|>\delta) \cup (|Y|>M))$$

$$\leq \mathbb{P}(|X_n|>\delta) + \mathbb{P}(|Y_n-Y|>\delta) + \mathbb{P}(|Y|>M) \quad (3)$$

Conclusion Gathering (2) and (3) we get

$$\boxed{\begin{aligned} &\mathbb{P}(|g(X_n,Y_n)-g(0,Y)| \geq \varepsilon) \\ &\leq \mathbb{P}(|X_n|>\delta) + \mathbb{P}(|Y_n-Y|>\delta) \\ &\quad + \mathbb{P}(|Y|>M) \end{aligned}}$$

1.5. Prove that $g(X_n, Y_n) \xrightarrow{\text{P}} g(0, Y)$.

According to 1-2, for a given $\hat{\varepsilon} > 0$ one can find $M = M_{\delta,\hat{\varepsilon}}$ large enough so that for $n \geq M$ we have

$$\mathbb{P}\left((B^n_{\varepsilon,\delta})^c \right) < 3\hat{\varepsilon}$$

Thanks to 1-4 we thus obtain

$$\mathbb{P}\left(|g(X_n, Y_n) - g(0, Y)| \geq \varepsilon \right) \leq 3\hat{\varepsilon}.$$

Since ε and $\hat{\varepsilon}$ are arbitrarily small, we have proved

$$\boxed{g(X_n, Y_n) \xrightarrow{\mathbb{P}} g(0, Y)}$$

Problem 2. In this problem we consider a sequence of independent random variables $\{X_n; n \geq 1\}$, with common law $\mathcal{N}(0,1)$. All random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Our aim is to prove that one has to renormalize $|X_n|$ by $(2\ln(n))^{1/2}$ in order to get bounded fluctuations.

2.1. We set $f(x)$ for the density of X_1. Prove that f solves the differential equation $f' = -xf$.

Density

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Derivative we easily derive

$$f'(x) = -x \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\Rightarrow \quad \boxed{f'(x) = -x\, f(x)}$$

2.2. Let Φ be the cdf of the random variable X_1. Applying Question 2.1 and integration by parts, show that for $x \geq 0$ we have

$$1 - \Phi(x) = \frac{f(x)}{x} - \int_x^{\infty} \frac{f(r)}{r^2}\, \mathrm{d}r\,.$$

Deduce that $1 - \Phi(x) \leq \frac{f(x)}{x}$.

Remark: For the sequel, we will in fact admit that $(1 - \Phi(x)) \sim \frac{f(x)}{x}$ as $x \to \infty$.

Integral representation we have

$$1 - \phi(x) = \int_x^{\infty} f(r)\, dr$$

$$\overset{2-1}{=} -\int_x^{\infty} \frac{f'(r)}{r}\, dr$$

Integration by parts This yields

$$1 - \phi(x) = -\frac{f(r)}{r}\Big|_x^{\infty} - \int_x^{\infty} \frac{f(r)}{r^2}\, dr$$

$$1 - \phi(x) = \frac{f(x)}{x} - \int_x^{\infty} \frac{f(r)}{r^2}\, dr \qquad (1)$$

Bound Since $f(r) \geq 0$ and $r^2 \geq 0$, we get

$$\boxed{1 - \phi(x) \leq \frac{f(x)}{x}}$$

2.3. Let $-\frac{1}{2} < \varepsilon < \frac{1}{2}$ be fixed. Prove that we have the following equivalent as $n \to \infty$:

$$\mathbf{P}\left(|X_1| \geq (2\ln(n))^{1/2}(1+\varepsilon)\right) \sim \frac{1}{2\left(\pi \ln(n)\right)^{1/2}(1+\varepsilon)\, n^{(1+\varepsilon)^2}}.$$

Applying 2-2 we have

$$\lim_{n\to\infty} (2\ln(n))^{\frac{1}{2}}(1+\varepsilon) = \infty$$

Hence one can apply 2-2 in order to get

$$\boxed{P(|X_1| \geq (2\ln(n))^{\frac{1}{2}}(1+\varepsilon))}$$

$$\sim \frac{\phi(2^{\frac{1}{2}}(1+\varepsilon)(\ln(n))^{\frac{1}{2}})}{2^{\frac{1}{2}}(1+\varepsilon)(\ln(n))^{\frac{1}{2}}}$$

$$= \frac{1}{2(\pi\ln(n))^{\frac{1}{2}}(1+\varepsilon)} \exp\left(-\frac{1}{2} \times (2^{\frac{1}{2}}(1+\varepsilon)\ln(n)^{\frac{1}{2}})^2\right)$$

$$= \frac{1}{2(\pi\ln(n))^{\frac{1}{2}}(1+\varepsilon)} \exp\left(-(1+\varepsilon)^2\ln(n)\right)$$

$$= \boxed{\frac{1}{2(\pi\ln(n))^{\frac{1}{2}}(1+\varepsilon)\; n^{(1+\varepsilon)^2}}}$$

2.4. For a given $\varepsilon > 0$, show that we have

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{(\ln(n))^{1/2}} \geq \sqrt{2}(1+\varepsilon)\right) = 0.$$

Preparing for Borel-Cantelli Let $A_n(\varepsilon)$ be the event defined by

$$A_n(\varepsilon) = \frac{|X_n|}{(\ln(n))^{\frac{1}{2}}} \geq \sqrt{2}\,(1+\varepsilon)$$

According to 2-3 we have

$$\mathbb{P}(A_n(\varepsilon)) \overset{n\to\infty}{\sim} \frac{1}{2(\pi \ln(n))^{\frac{1}{2}}(1+\varepsilon)\; n^{(1+\varepsilon)^2}}$$

Thus if $\varepsilon > 0$ we get

$$\sum_{n\geq 1} \mathbb{P}(A_n(\varepsilon)) < \infty \qquad (1)$$

Applying Borel-Cantelli Owing to (1), Borel - Cantelli asserts that

$$\mathbb{P}\left(\limsup_{n\to\infty} A_n(\varepsilon) \right) = 0 \qquad (2)$$

This can be transformed into a statement about limsup $|X_n|/(\ln(n))^{\frac{1}{2}}$. Namely consider

$$D(\omega) = \left\{ n \; ; \; \frac{|X_n|}{(\ln(n))^{\frac{1}{2}}} \geq \sqrt{2}\,(1+\varepsilon) \right\}$$

Then according to (2) we have

$$\mathbb{P}(|D| < \infty) = 1$$

This implies

$$\boxed{\mathbb{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{(\ln(n))^{\frac{1}{2}}} \geq \sqrt{2}\,(1+\varepsilon) \right) = 0}$$

2.5. Prove that we have

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{|X_n|}{(\ln(n))^{1/2}}=\sqrt{2}\right)=1.$$

Case $-\frac{1}{2}<\varepsilon\leq 0$ For $\varepsilon\leq 0$, along the same lines as for 2.4 we have

$$\sum_{n=1}^{\infty}\mathbb{P}(A_n(\varepsilon))=\infty$$

Since the X_n's are ⫫, one can apply the reversed Borel-Cantelli. This yields

$$\mathbb{P}(\limsup_{n\to\infty}A_n(\varepsilon))=1$$

For $\varepsilon=0$ we get that the set

$$D_0(\omega):=\left\{n\geq 1\,;\ \frac{|X_n|}{(\ln(n))^{\frac{1}{2}}}\geq\sqrt{2}\right\}$$

satisfies

$$\mathbb{P}(\,|D_0(\omega)|=\infty\,)=1$$

Thus

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{|X_n|}{(\ln(n))^{\frac{1}{2}}}\geq\sqrt{2}\right)=1\qquad(1)$$

Conclusion Gathering (1) and 2.4, we get that for all $m \geq 1$,

$$\mathbb{P}\left(\sqrt{2} \leq \limsup_{n\to\infty} \frac{|X_n|}{(\ln(n))^{\frac{1}{2}}} \leq \sqrt{2}\left(1+\frac{1}{m}\right)\right) = 1$$

Taking limits in $m \to \infty$, we end up with

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{(\ln(n))^{\frac{1}{2}}} = \sqrt{2}\right) = 1$$