# MA/STAT 538 Spring 2025
# Probability Theory

## Midterm

- You can use a calculator.
- A 2 pages long handwritten cheat sheet is allowed. It should only contain formulae and theorems (no example, no solved problem).
- You have 60 minutes.
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- GOOD LUCK!

**Name:**

**Purdue ID:**

**Problem 1.** We consider two sequences of random variables $\{X_n;\, n \geq 1\}$ and $\{Y_n;\, n \geq 1\}$, as well as another random variable $Y$. All random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We assume that

$$X_n \xrightarrow{(\mathrm{d})} 0, \quad \text{and} \quad Y_n \xrightarrow{\mathrm{P}} Y.$$

Let also $g : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function such that $y \mapsto g(x, y)$ is a continuous function of $y$ for all $x$, and $(x, y) \mapsto g(x, y)$ is continuous at every point of the form $(0, y)$. The aim of the problem is to prove that $g\left(X_n, Y_n\right) \xrightarrow{\mathrm{P}} g(0, Y)$.

**1.1.** Prove that the set $\{\omega \in \Omega;\, g\left(X_n(\omega), Y_n(\omega)\right) \to g(0, Y(\omega))\}$ is an element of the $\sigma$-algebra $\mathcal{F}$.

**Solution**:

**1.2.** Let $\delta, \hat{\varepsilon} > 0$ be given. Prove that one can find $M = M_{\delta,\hat{\varepsilon}}$ large enough such that for $n \geq M$ we have

$$\mathbf{P}\left(|X_n| > \delta\right) < \hat{\varepsilon}, \quad \mathbf{P}\left(|Y - Y_n| > \delta\right) < \hat{\varepsilon}, \quad \text{and} \quad \mathbf{P}\left(|Y| > M\right) < \hat{\varepsilon}.$$

**Solution**:

**1.3.** Prove that $g$ is uniformly continuous on the region of $\mathbb{R}^2$ defined by $D_M = \{0\} \times [-M, M]$, where $M$ is given in Question 1.2. Deduce that for a given small $\varepsilon > 0$, there exists $\delta > 0$ such that for every $y \in [-M, M]$ we have

$$\Big( |\xi| \leq \delta \ \text{ and } \ |\eta - y| \leq \delta \Big) \quad \Longrightarrow \quad |g(\xi, \eta) - g(0, y)| < \varepsilon$$

**Solution**:

**1.4.** With $M$ and $\varepsilon, \delta$ like in Question 1.3, prove that

$$\mathbf{P}\left(|g(X_n, Y_n) - g(0, Y)| \geq \varepsilon\right) \leq \mathbf{P}\left(|X_n| > \delta\right) + \mathbf{P}\left(|Y_n - Y| > \delta\right) + \mathbf{P}\left(|Y| > M\right).$$

**Solution**:

**1.5.** Prove that $g\left(X_n, Y_n\right) \xrightarrow{\mathrm{P}} g(0, Y)$.

**Solution**:

**Problem 2.** In this problem we consider a sequence of independent random variables $\{X_n;\, n \geq 1\}$, with common law $\mathcal{N}(0,1)$. All random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Our aim is to prove that one has to renormalize $|X_n|$ by $(2\ln(n))^{1/2}$ in order to get bounded fluctuations.

**2.1.** We set $f(x)$ for the density of $X_1$. Prove that $f$ solves the differential equation $f' = -xf$.

**Solution**:

**2.2.** Let $\Phi$ be the cdf of the random variable $X_1$. Applying Question 2.1 and integration by parts, show that for $x \geq 0$ we have

$$1 - \Phi(x) = \frac{f(x)}{x} - \int_x^\infty \frac{f(r)}{r^2}\,\mathrm{d}r\,.$$

Deduce that $1 - \Phi(x) \leq \frac{f(x)}{x}$.
*Remark:* For the sequel, we will in fact admit that $(1 - \Phi(x)) \sim \frac{f(x)}{x}$ as $x \to \infty$.

**Solution**:

**2.3.** Let $-\frac{1}{2} < \varepsilon < \frac{1}{2}$ be fixed. Prove that we have the following equivalent as $n \to \infty$:

$$\mathbf{P}\left(|X_1| \geq (2\ln(n))^{1/2}(1+\varepsilon)\right) \sim \frac{1}{2\left(\pi \ln(n)\right)^{1/2}(1+\varepsilon)\, n^{(1+\varepsilon)^2}}.$$

**Solution**:

**2.4.** For a given $0 < \varepsilon < \frac{1}{2}$, show that we have

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{(\ln(n))^{1/2}} \geq \sqrt{2}(1+\varepsilon)\right) = 0.$$

**Solution**:

**2.5.** Prove that we have

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{|X_n|}{(\ln(n))^{1/2}}=\sqrt{2}\right)=1.$$

**Solution**: