Conditional expectation

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Probability Theory 1 - MA 538

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- Baby conditional distributions: discrete case
- Baby conditional distributions: continuous case
- Definition with measure theory
- 2 Examples
- 3 Existence and uniqueness
- 4 Conditional expectation: properties
- 5 Conditional expectation as a projection
- 6 Conditional regular laws
 - Probability laws and expectations
 - Definition of the CRL



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General definition

Definition 1.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y
- y such that $p_Y(y) > 0$

Then the conditional pmf of X given Y = y is defined by

$$p_{X|Y}(x|y) = \mathbf{P}(X = x|Y = y) = \frac{p(x,y)}{p_Y(y)}$$

Example ctd: tossing 3 coins (1)

Experiment: Tossing a coin 3 times

Events: We consider

A = "At most one Head" B = "At least one Head and one Tail"

Random variables: Set

$$X_1 = \mathbf{1}_A, \qquad X_2 = \mathbf{1}_B, \qquad X = (X_1, X_2)$$

Example ctd: tossing 3 coins (2) Joint distribution of (X_1, X_2) :

$X_1 ackslash X_2$	0	1	Marg. X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg. X_2	1/4	3/4	1

Conditional probabilities given $X_1 = 0$:

$$p_{X_2|X_1}(0|0) = rac{1/8}{1/2} = rac{1}{4}, \quad p_{X_2|X_1}(1|0) = rac{3/8}{1/2} = rac{3}{4}$$

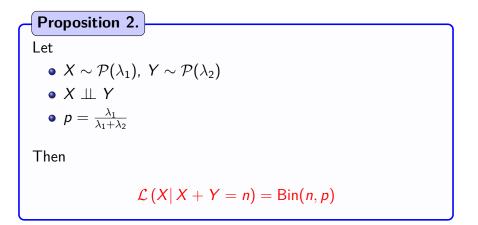
Conditional probabilities given $X_2 = 1$:

$$p_{X_1|X_2}(0|1) = rac{3/8}{3/4} = rac{1}{2}, \quad p_{X_1|X_2}(1|1) = rac{3/8}{3/4} = rac{1}{2}$$

Image: A matrix

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Conditioning Poisson random variables



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Proof (1)

Expression for the conditional probabilities: Let $0 \le k \le n$. Then invoking $X \perp Y$,

$$P(X = k | X + Y = n) = \frac{P(X = k) P(Y = n - k)}{P(X + Y = n)}$$

Law of X + Y: One can prove that

$$X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$$

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Proof (2)

Computation of the conditional probabilities:

$$\mathbf{P} \left(X = k | X + Y = n \right)$$

= $e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} \left[e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!} \right]^{-1}$
= $\binom{n}{k} p^k (1-p)^{n-k}$

Conclusion:

 $\mathcal{L}(X|X+Y=n)=\mathrm{Bin}(n,p)$

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Cond. expectation in the discrete case

Definition 3.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y, y such that $p_Y(y) > 0$
- $p_{X|Y}(x|y)$ conditional distribution

Then the conditional exp. of X given Y = y is defined by

$$\mathbf{E}[X|Y = y] = \sum_{x \in \mathcal{E}} x \, p_{X|Y}(x|y)$$

Binomial example (1)

Situation: Let

- $X, Y \sim Bin(n, p)$
- $X \parallel Y$
- Z = X + Y

Problem: We wish to compute

 $\mathbf{E}[X|Z=m]$

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Binomial example (2)

Distribution for *Z*:

$$Z = \sum_{i=1}^n X_i + \sum_{j=1}^n Y_j \sim \mathsf{Bin}(2n, p)$$

Computation for conditional pmf: For $k \leq \min(n, m)$ we have

$$\mathbf{P}(X = k | Z = m) = \frac{\mathbf{P}(X = k, X + Y = m)}{\mathbf{P}(Z = m)}$$
$$= \frac{\mathbf{P}(X = k, Y = m - k)}{\mathbf{P}(Z = m)}$$
$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

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Hypergeometric random variable (1)

Use: Consider the experiment

- Urn containing N balls
- m white balls, N m black balls
- Sample of size *n* is drawn without replacement
- Set X = # white balls drawn

Then

 $X \sim \mathsf{HypG}(n, N, m)$

Hypergeometric random variable (2) Notation:

 $X \sim \mathsf{HypG}(n, N, m)$, for $N \in \mathbb{N}^*$, $m, n \leq N$

State space:

 $\{0,\ldots,n\}$

Pmf:

$$\mathbf{P}(X=k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, \quad 0 \le k \le n$$

Expected value and variance: Set $p = \frac{m}{N}$. Then

$$\mathbf{E}[X] = np,$$
 $\mathbf{Var}(X) = np(1-p)\left(rac{N-n}{N-1}
ight)$

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Binomial example (3)

Conditional pmf: For $k \leq \min(n, m)$ we have seen

$$p_{X|Z}(k|m) = rac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

Recall: If $V \sim \text{HypG}(n, N, m)$ then

$$\mathbf{P}(X=k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

Identification of the conditional pmf: We have

 $p_{X|Z}(k|m) = \text{Pmf of HypG}(m, 2n, n)$

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Binomial example (4)

Conditional expectation: Let $V \sim HypG(m, 2n, n)$. Then

$$\mathbf{E}\left[X|Z=m\right]=\mathbf{E}[V]$$

Numerical value:

According to the values for hypergeometric distributions,

$$\mathbf{E}\left[X|Z=m\right]=m\times\frac{n}{2n}=\frac{m}{2}$$

Image: A matrix

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General definition

Definition 4.

Let

- (X, Y) couple of continuous random variables
- Joint density f
- Marginal densities f_X, f_Y
- y such that $f_Y(y) > 0$

Then the conditional density of X given Y = y is defined by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Justification of the definition

Heuristics: $f_{X|Y}(x|y)$ can be interpreted as

$$f_{X|Y}(x|y) dx = \frac{f(x,y) dx dy}{f_Y(y) dy}$$

$$\simeq \frac{\mathbf{P} (x \le X \le x + dx, y \le Y \le y + dy)}{\mathbf{P} (y \le Y \le y + dy)}$$

$$= \mathbf{P} (x \le X \le x + dx | y \le Y \le y + dy)$$

Use of the conditional probability: compute probabilities like

$$\mathbf{P}(X \in A | Y = y) = \int_A f_{X|Y}(x|y) \, dx$$

Rigorous definition: see next sections

Simple example of continuous conditioning (1)

Density: Let (X, Y) be a random vector with density

$$\frac{e^{-\frac{x}{y}}e^{-y}}{y}\mathbf{1}_{(0,\infty)}(x)\mathbf{1}_{(0,\infty)}(y)$$

Question: Compute

 $\mathbf{P}(X > 1 | Y = y)$

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Simple example of continuous conditioning (2)

Marginal distribution of Y: We have

$$f_{Y}(y) = \int_{0}^{\infty} f(x, y) dx$$

= $\frac{e^{-y}}{y} \left(\int_{0}^{\infty} e^{-\frac{x}{y}} dx \right) \mathbf{1}_{(0,\infty)}(y)$
= $e^{-y} \mathbf{1}_{(0,\infty)}(y)$

Conditional density: For y > 0 we have

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}}}{y} \mathbf{1}_{(0,\infty)}(x)$$

Namely $\mathcal{L}(X | Y = y) = \mathcal{E}(\frac{1}{y})$

Image: A matrix

Simple example of continuous conditioning (3)

Conditional probability:

$$P(X > 1 | Y = y) = \int_{1}^{\infty} f_{X|Y}(x|y) dx$$
$$= \int_{1}^{\infty} \frac{e^{-\frac{x}{y}}}{y} dx$$
$$= e^{-\frac{1}{y}}$$

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Cond. expectation in the continuous case

Definition 5.

Let

- (X, Y) couple of continuous random variables
- Joint density f
- Marginal densities f_X, f_Y, y such that $f_Y(y) > 0$
- $f_{X|Y}(x|y)$ conditional density

Then the conditional exp. of X given Y = y is defined by

$$\mathbf{E}[X|Y=y] = \int_{\mathbb{R}} x f_{X|Y}(x|y) dx$$

Example of continuous conditional expectation (1)

Density: Let (X, Y) be a random vector with density

$$\frac{e^{-\frac{x}{y}}e^{-y}}{y}\mathbf{1}_{(0,\infty)}(x)\mathbf{1}_{(0,\infty)}(y)$$

Question: Compute

$$\mathsf{E}\left[X \mid Y = y\right]$$

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Example of continuous conditional expectation (2)

Conditional density: For y > 0 we have seen that

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}}}{y} \mathbf{1}_{(0,\infty)}(x)$$

Namely
$$\mathcal{L}(X|Y=y) = \mathcal{E}(rac{1}{y})$$

Conditional expectation: We have

$$\mathbf{E}[X|Y = y] = \int_{\mathbb{R}} x f_{X|Y}(x|y) dx$$
$$= \int_{0}^{\infty} x \frac{e^{-\frac{x}{y}}}{y}$$
$$= y$$

Image: A matrix



Definition

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Formal definition

Definition 6.

We are given a probability space $(\Omega, \mathcal{F}_0, \mathbf{P})$ and

- A σ -algebra $\mathcal{F} \subset \mathcal{F}_0$.
- $X \in \mathcal{F}_0$ such that $\mathbf{E}[|X|] < \infty$.

Conditional expectation of X given \mathcal{F} :

- Denoted by $\mathbf{E}[X|\mathcal{F}]$
- Defined by: $\mathbf{E}[X|\mathcal{F}]$ is the $L^1(\Omega)$ r.v Y such that

(i) $Y \in \mathcal{F}$.

(ii) For all $A \in \mathcal{F}$, we have

 $\mathbf{E}[X\mathbf{1}_{A}] = \mathbf{E}[Y\mathbf{1}_{A}],$

or otherwise stated
$$\int_A X \, d\mathbf{P} = \int_A Y \, d\mathbf{P}$$
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Remarks

Notation: We use the notation $Y \in \mathcal{F}$ to say that a random variable Y is \mathcal{F} -measurable.

Interpretation: More intuitively

- \mathcal{F} represents a given information
- Y is the best prediction of X given the information in \mathcal{F} .

Existence and uniqueness:

To be seen after the examples.

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Easy examples (1)

Example 1: Assume

 $X \in \mathcal{F}$.

Then

 $\mathbf{E}[X|\mathcal{F}] = X$

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Independence of a r.v and a σ -field

Definition 7.

We say that $X \perp\!\!\!\perp \mathcal{F}$ if \hookrightarrow for all $A \in \mathcal{F}$ and $B \in \mathcal{B}(\mathbb{R})$, we have

 $\mathbf{P}((X \in B) \cap A) = \mathbf{P}(X \in B) \, \mathbf{P}(A),$

or otherwise stated:

 $X \perp\!\!\!\perp \mathbf{1}_A$

Easy examples (2)

Example 2: Assume

 $X \perp\!\!\!\perp \mathcal{F}.$

Then

 $\mathbf{E}[X|\mathcal{F}] = \mathbf{E}[X]$

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Proof: example 2

We have

(i) $\mathbf{E}[X] \in \mathcal{F}$ since $\mathbf{E}[X]$ is a constant. (ii) If $A \in \mathcal{F}$,

$$\mathbf{E}[X \mathbf{1}_A] = \mathbf{E}[X] \mathbf{E}[\mathbf{1}_A] = \mathbf{E}\Big[\mathbf{E}[X] \mathbf{1}_A\Big].$$

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Discrete conditional expectation

Example 3: We consider

{Ω_j; j ≥ 1} partition of Ω such that P(Ω_j) > 0 for all j ≥ 1.
F = σ(Ω_j; j ≥ 1).

Then

$$\mathbf{E}[X|\mathcal{F}] = \sum_{j \ge 1} \frac{\mathbf{E}[X \mathbf{1}_{\Omega j}]}{\mathbf{P}(\Omega_j)} \mathbf{1}_{\Omega j} \equiv Y.$$
(1)

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Proof: example 3

Strategy: Verify (i) and (ii) for the random variable Y.

(i) For all $j \ge 1$, we have $\mathbf{1}_{\Omega_j} \in \mathcal{F}$. Thus, for any sequence $(\alpha_j)_{j \ge 1}$, $\sum_{j \ge 1} \alpha_j \mathbf{1}_{\Omega_j} \in \mathcal{F}.$

(ii) It is enough to verify (1) for $A = \Omega_n$ and $n \ge 1$ fixed. However,

$$\mathsf{E}[Y\mathbf{1}_{\Omega_n}] = \mathsf{E}\left\{\frac{\mathsf{E}[X\mathbf{1}_{\Omega_n}]}{\mathsf{P}(\Omega_n)}\mathbf{1}_{\Omega_n}\right\} = \frac{\mathsf{E}[X\mathbf{1}_{\Omega_n}]}{\mathsf{P}(\Omega_n)}\mathsf{E}[\mathbf{1}_{\Omega_n}] = \mathsf{E}[X\mathbf{1}_{\Omega_n}].$$

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Undergrad conditional probability

Definition: For a generic measurable set $A \in \mathcal{F}_0$, we set

$$\mathsf{P}(A|\mathcal{F}) \equiv \mathsf{E}[\mathbf{1}_A|\mathcal{F}]$$

Discrete example setting: Let B, B^c be a partition of Ω , and $A \in \mathcal{F}_0$. Then **1** $\mathcal{F} = \sigma(B) = \{\Omega, \emptyset, B, B^c\}$ **2** We have $\mathbf{P}(A|\mathcal{F}) = \mathbf{P}(A|B) \mathbf{1}_B + \mathbf{P}(A|B^c) \mathbf{1}_{B^c}.$

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Dice throwing

Example: We consider

•
$$\Omega = \{1, 2, 3, 4, 5, 6\}, A = \{4\}, B = "even number".$$

Then
 $\mathbf{P}(A|T) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathbf{P}(A|\mathcal{F}) = \frac{1}{3}\mathbf{1}_B.$$

Image: A matrix

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Conditioning a r.v by another r.v

Definition 8.

Let

- X random variable such that $X \in L^1(\Omega)$
- Y random variable

We set

 $\mathbf{E}[X|Y] = \mathbf{E}[X|\sigma(Y)].$

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Characterizing $\sigma(Y)$

How to know if $A \in \sigma(Y)$: We have $A \in \sigma(Y)$ iff

$$A = \left\{ \omega; Y(\omega) \in B
ight\}, \quad ext{or} \quad \mathbf{1}_A = \mathbf{1}_B(Y)$$

How to know if $Z \in \sigma(Y)$: Let Z and Y be two random variables. Then

 $Z \in \sigma(Y)$ iff we can write Z = U(Y), with $U \in \mathcal{B}(\mathbb{R})$.

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Conditioning a r.v by a discrete r.v

Example 4: Whenever X and Y are discrete random variables \hookrightarrow Computation of $\mathbf{E}[X|Y]$ can be handled as in example 3.

More specifically:

- Assume $Y \in E$ with $E = \{y_i; i \ge 1\}$
- Hypothesis: $\mathbf{P}(Y = y_i) > 0$ for all $i \ge 1$.

Then $\mathbf{E}[X|Y] = h(Y)$ with $h: E \to \mathbb{R}$ defined by:

$$h(y) = \frac{\mathbf{E}[X \mathbf{1}_{(Y=y)}]}{\mathbf{P}(Y=y)}.$$

Conditioning a r.v by a continuous r.v

Example 5: Let (X, Y) couple of real random variables with measurable density $f : \mathbb{R}^2 \to \mathbb{R}_+$. We assume that

$$\int_{\mathbb{R}} f(x,y) dx > 0, \quad \text{for all } y \in \mathbb{R}.$$

Let $g : \mathbb{R} \to \mathbb{R}$ a measurable function such that $g(X) \in L^1(\Omega)$. Then $\mathbf{E}[g(X)|Y] = h(Y)$, with $h : \mathbb{R} \to \mathbb{R}$ defined by:

$$h(y) = \frac{\int_{\mathbb{R}} g(x) f(x, y) dx}{\int_{\mathbb{R}} f(x, y) dx}.$$

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Heuristic proof

Formally one can use a conditional density:

$$\mathbf{P}(X = x | Y = y)'' = "\frac{\mathbf{P}(X = x, Y = y)}{\mathbf{P}(Y = y)} = \frac{f(x, y)}{\int f(x, y) dx},$$

Integrating against this density we get:

$$\mathbf{E}[g(X)|Y = y] = \int g(x)\mathbf{P}(X = x|Y = y) dx$$
$$= \frac{\int g(x)f(x,y)dx}{\int f(x,y)dx}.$$

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Image: A matrix

Rigorous proof Strategy: Check (i) and (ii) in the definition for the r.v h(Y). (i) If $h \in \mathbb{B}(\mathbb{R})$, we have seen that $h(Y) \in \sigma(Y)$. (ii) Let $A \in \sigma(Y)$ Then

$$A = \left\{ \omega; Y(\omega) \in B \right\} \implies \mathbf{1}_A = \mathbf{1}_B(Y)$$

Thus

$$\begin{aligned} \mathbf{E}[h(Y)\mathbf{1}_{A}] &= \mathbf{E}[h(Y)\mathbf{1}_{B}(Y)] \\ &= \int_{B} \int_{\mathbb{R}} h(y)f(x,y)dxdy \\ &= \int_{B} dy \int_{\mathbb{R}} \left\{ \frac{\int g(z)f(z,y)dz}{\int f(z,y)dz} \right\} f(x,y)dx \\ &= \int_{B} dy \int g(z)f(z,y)dz = \mathbf{E}[g(X)\mathbf{1}_{B}(Y)]. \end{aligned}$$

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Weird example

Example 6: We take

•
$$\Omega = (0,1)$$
, $\mathcal{F}_0 = \mathcal{B}((0,1))$ and $\mathbf{P} = \lambda$.

We set $X(\omega) = \cos(\pi \omega)$, and

$$\mathcal{F} = \{A \subset (0,1); A ext{ or } A^c ext{countable}\}$$
 .

Then $\mathbf{E}[X|\mathcal{F}] = 0$.

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Proof

Strategy: Check (i) and (ii) in the definition for the r.v Y = 0. (i) Obviously $0 \in \mathcal{F}$.

(ii) Let $A \in \mathcal{F}$, such that A is countable. Then

$$\mathbf{E}[X\,\mathbf{1}_A] = \int_A \cos(\pi x) dx = 0.$$

Similarly, if $A \in \mathcal{F}$ is such that A^c is countable, we have

$$\mathbf{E}[X \mathbf{1}_A] = \int_0^1 \cos(\pi x) dx - \int_{A^c} \cos(\pi x) dx = 0,$$

which ends the proof.

Weird example: heuristics

Intuition: One could think that

- We know that $\{x\}$ occurred for all $x \in [0, 1]$
- $\ 2 \ \{x\} \in \mathcal{F}$
- **3** Thus $\mathbf{E}[X|\mathcal{F}] = X$.

Paradox: This is wrong because $X \notin \mathcal{F}$.

Correct intuition: If we know $\omega \in A_i$ for a finite number of $A_i \in \mathcal{F}$ then nothing is known about X.

Outline

Definition

- Baby conditional distributions: discrete case
- Baby conditional distributions: continuous case
- Definition with measure theory
- 2 Examples
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 - 4 Conditional expectation: properties
- 6 Conditional expectation as a projection
- 6 Conditional regular laws
 - Probability laws and expectations
 - Definition of the CRL

Conditional expectation: uniqueness

Proposition 9.

On the probability space $(\Omega, \mathcal{F}_0, \mathbf{P})$ consider

- A σ -algebra $\mathcal{F} \subset \mathcal{F}_0$.
- $X \in \mathcal{F}_0$ such that $\mathbf{E}[|X|] < \infty$.

Then if it exists, the random variable

$\mathbf{E}[X|\mathcal{F}]$

is uniquely defined.

Proof of uniqueness

Aim: Let Y, Y' satisfying (i) + (ii). \hookrightarrow Let us show Y = Y' a.s

General property: For all $A \in \mathcal{F}$, we have $\mathbf{E}[Y \mathbf{1}_A] = \mathbf{E}[Y' \mathbf{1}_A]$.

Particular case: Let $\epsilon > 0$, and set

$$A_{\epsilon} \equiv (Y - Y' \geqslant \epsilon).$$

Then $A_{\epsilon} \in \mathcal{F}$, and thus

$$0 = \mathsf{E}[(Y - Y') \mathbf{1}_{A_{\epsilon}}] \geq \epsilon \, \mathsf{E}[\mathbf{1}_{A_{\epsilon}}] = \epsilon \, \mathsf{P}(A_{\epsilon})$$

 $\Rightarrow \mathbf{P}(A_{\epsilon}) = 0.$

Proof of uniqueness (2)

Set A_+ : Let

$$A_+ \equiv (Y - Y' > 0) = \bigcup_{n \ge 1} A_{1/n}.$$

We have $n \mapsto A_{1/n}$ increasing, and thus

$$\mathbf{P}(A_+) = \mathbf{P}\left(\bigcup_{n \ge 1} A_{1/n}\right) = \lim_{n \to \infty} \mathbf{P}(A_{1/n}) = 0.$$

Set A_{-} : In the same way, if

$$A_{-} = \{Y - Y' < 0\}$$

we have $P(A_{-}) = 0$.

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Proof of uniqueness (3)

Conclusion: We obtain, setting

$$oldsymbol{A}_
eq\equiv \{Y
eq Y'\}=oldsymbol{A}_+\cupoldsymbol{A}_-,$$

that $\mathbf{P}(A_{\neq}) = 0$, and thus Y = Y' a.s.

Image: A matrix

Absolute continuity

Definition 10.

Let μ, ν two σ -finite measures on (Ω, \mathcal{F}) . We say that $\nu \ll \mu$ (μ is absolutely continuous w.r.t ν) if

$$\mu(A) = 0 \implies \nu(A) = 0 \text{ for all } A \in \mathcal{F}.$$

Radon-Nykodym theorem

Theorem 11.

Let

• $\mu, \nu \sigma$ -finite measures on (Ω, \mathcal{F}) , such that $\nu \ll \mu$. Then there exists $f \in \mathcal{F}$ such that for all $A \in \mathcal{F}$ we have

$$\nu(A) = \int_A f \, d\mu$$

The function f:

- $\bullet\,$ Is called Radon-Nykodym derivative of μ with respect to $\nu\,$
- Is denoted by $f \equiv \frac{d\nu}{d\mu}$.
- We have $f \ge 0$ μ -almost everywhere
- $f \in L^1(\mu)$.

Conditional expectation: existence

Proposition 12.

On the probability space $(\Omega, \mathcal{F}_0, \mathbf{P})$ consider

- A σ -algebra $\mathcal{F} \subset \mathcal{F}_0$.
- $X \in \mathcal{F}_0$ such that $\mathbf{E}[|X|] < \infty$.

Then the random variable

$\mathbf{E}[X|\mathcal{F}]$

exists and is uniquely defined.

Proof of existence

Hypothesis: We have

- A σ -algebra $\mathcal{F} \subset \mathcal{F}_0$.
- $X \in \mathcal{F}_0$ such that $\mathbf{E}[|X|] < \infty$.
- $X \ge 0$.

Defining two measures: we set

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$$\mu = P$$
, measure on (Ω, \mathcal{F}) .

Then ν is a measure (owing to Beppo-Levi).

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Proof of existence (2)

Absolute continuity: we have

$$\mathbf{P}(A) = 0 \Rightarrow \mathbf{1}_A = 0 \quad P\text{-a.s.}$$
$$\Rightarrow X \mathbf{1}_A = 0 \quad P\text{-a.s.}$$
$$\Rightarrow \nu(A) = 0$$

Thus $\nu \ll P$

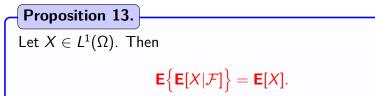
Conclusion: invoking Radon-Nykodym, there exists $f \in \mathcal{F}$ such that, for all $A \in \mathcal{F}$, we have $\nu(A) = \int_A f \, d\mathbf{P}$. \hookrightarrow We set $f = \mathbf{E}[X|\mathcal{F}]$.

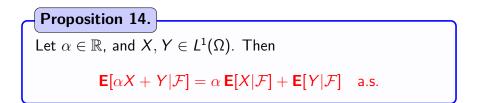
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Linearity, expectation





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Proof

Strategy: Check (i) and (ii) in the definition for the r.v

$$Z \equiv \alpha \, \mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}].$$

Verification: we have

(i) Z is a linear combination of $\mathbf{E}[X|\mathcal{F}]$ and $\mathbf{E}[Y|\mathcal{F}]$ $\hookrightarrow Z \in \mathcal{F}$.

(ii) For all $A \in \mathcal{F}$, we have

$$\mathbf{E}[Z \mathbf{1}_{A}] = \mathbf{E}\left\{ (\alpha \mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}]) \mathbf{1}_{A} \right\}$$

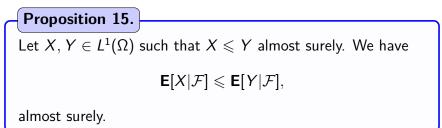
$$= \alpha \mathbf{E}\left\{ \mathbf{E}[X|\mathcal{F}] \mathbf{1}_{A} \right\} + \mathbf{E}\left\{ \mathbf{E}[Y|\mathcal{F}] \mathbf{1}_{A} \right\}$$

$$= \alpha \mathbf{E}[X \mathbf{1}_{A}] + \mathbf{E}[Y \mathbf{1}_{A}]$$

$$= \mathbf{E}[(\alpha X + Y) \mathbf{1}_{A}].$$

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Monotonicity



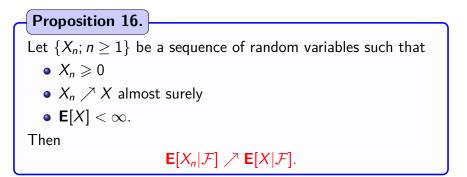
Proof: Along the same lines as proof of uniqueness for the conditional expectation. For instance if we set

$$A_{\varepsilon} = \{ \mathbf{E}[X|\mathcal{F}] - \mathbf{E}[Y|\mathcal{F}] \ge \varepsilon > 0 \},\$$

then it is readily checked that

$$\mathbf{P}(A_{\varepsilon}) = 0.$$

Monotone convergence



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Proof

Strategy: Set $Y_n \equiv X - X_n$. We are reduced to show $Z_n \equiv \mathbf{E}[Y_n | \mathcal{F}] \searrow 0$.

Existence of a limit: $n \mapsto Y_n$ is decreasing, and $Y_n \ge 0$ $\hookrightarrow Z_n$ is decreasing and $Z_n \ge 0$. $\hookrightarrow Z_n$ admits a limit a.s, denoted by Z_{∞} .

Aim: Show that $Z_{\infty} = 0$.

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Proof (2)

Expectation of Z_{∞} : we will show that $\mathbf{E}[Z_{\infty}] = 0$. Indeed

- X_n converges a.s. to X.
- $0 \leq X_n \leq X \in L^1(\Omega).$

Thus, by dominated convergence, $\mathbf{E}[X_n] \rightarrow \mathbf{E}[X]$.

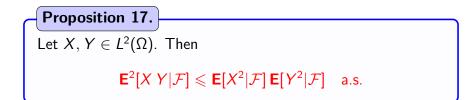
We deduce:

- $\mathbf{E}[Y_n] \to 0$
- Since $\mathbf{E}[Y_n] = \mathbf{E}[Z_n]$, we also have $\mathbf{E}[Z_n] \to 0$.

• By monotone convergence, we have $\mathbf{E}[Z_n] \to \mathbf{E}[Z_\infty]$ This yields $\mathbf{E}[Z_\infty] = 0$.

Conclusion: $Z_{\infty} \ge 0$ and $\mathbf{E}[Z_{\infty}] = 0$ $\hookrightarrow Z_{\infty} = 0$ almost surely.

Cauchy-Schwarz inequality



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Proof of Cauchy-Schwarz (1)

A family positive random variables: For all $\theta \in \mathbb{R}$, we have

$$\mathbf{E}[(X+\theta Y)^2|\mathcal{F}] \ge 0 \quad \text{a.s.}$$

Thus almost surely we have: for all $\theta \in \mathbb{Q}$,

 $\mathbf{E}[(X+\theta Y)^2|\mathcal{F}] \ge 0,$

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Proof of Cauchy-Schwarz (2)

Expansion: For all $\theta \in \mathbb{Q}$

$$\mathbf{E}[Y^2|\mathcal{F}]\theta^2 + 2\mathbf{E}[XY|\mathcal{F}]\theta + \mathbf{E}[X^2|\mathcal{F}] \ge 0.$$

Recall: If a polynomial satisfies $a\theta^2 + b\theta + c \ge 0$ for all $\theta \in \mathbb{Q}$ \hookrightarrow then we have $b^2 - 4ac \le 0$

Application: Almost surely, we have

 $E^{2}[XY|\mathcal{F}] - \mathbf{E}[X^{2}|\mathcal{F}]\mathbf{E}[Y^{2}|\mathcal{F}] \leqslant 0.$

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Jensen's inequality

Proposition 18. Let $X \in L^1(\Omega)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(X) \in L^1(\Omega)$ and φ convex. Then $\varphi(\mathbf{E}[X|\mathcal{F}]) \leq \mathbf{E}[\varphi(X)|\mathcal{F}]$ a.s.

Contraction in $L^p(\Omega)$

Proposition 19.

The conditional expectation is a

contraction in $L^p(\Omega)$ for all $p \ge 1$

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Proof of contraction in L^p

Application of Jensen's inequality: We have

$$X\in L^p(\Omega)\Rightarrow {\sf E}[X|{\mathcal F}]\in L^p(\Omega)$$

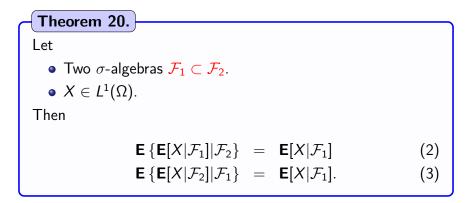
and

 $|\mathbf{E}[X|\mathcal{F}]|^{\rho} \leq \mathbf{E}[|X|^{\rho}|\mathcal{F}] \implies \mathbf{E}\{|\mathbf{E}[X|\mathcal{F}]|^{\rho}\} \leqslant \mathbf{E}[|X|^{\rho}]$

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Successive conditionings



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Proof

Proof of (2): We set $Z \equiv \mathbf{E}[X|\mathcal{F}_1]$. Then

$$Z \in \mathcal{F}_1 \subset \mathcal{F}_2.$$

According to Example 1, we have $\mathbf{E}[Z|\mathcal{F}_2] = Z$, i.e. (2).

Proof of (3): We set $U = \mathbf{E}[X|\mathcal{F}_2]$. \hookrightarrow We will show that $\mathbf{E}[U|\mathcal{F}_1] = Z$, via (i) and (ii) of Definition 6. (i) $Z \in \mathcal{F}_{1}$. (ii) If $A \in \mathcal{F}_1$, we have $A \in \mathcal{F}_1 \subset \mathcal{F}_2$, and thus $\mathbf{E}[Z\mathbf{1}_A] = \mathbf{E}[X\mathbf{1}_A] = \mathbf{E}[U\mathbf{1}_A].$

Conditional expectation for products

Theorem 21.
Let
$$X, Y \in L^2(\Omega)$$
, such that $X \in \mathcal{F}$. Then
 $\mathbf{E}[X Y | \mathcal{F}] = X \mathbf{E}[Y | \mathcal{F}].$

Proof: We use a 4 steps methodology

Image: A matrix

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Proof

Step 1: Assume $X = \mathbf{1}_B$, with $B \in \mathcal{F}$ We check (i) and (ii) of Definition 6. (i) We have $\mathbf{1}_{B}\mathbf{E}[Y|\mathcal{F}] \in \mathcal{F}$. (ii) For $A \in \mathcal{F}$, we have

$$\mathsf{E} \{ (\mathbf{1}_B \mathsf{E}[Y|\mathcal{F}]) \ \mathbf{1}_A \} = \mathsf{E} \{ \mathsf{E}[Y|\mathcal{F}] \ \mathbf{1}_{A \cap B} \}$$

$$= \mathsf{E}[Y \ \mathbf{1}_{A \cap B}]$$

$$= \mathsf{E}[(\mathbf{1}_B Y) \ \mathbf{1}_A],$$

and thus

$$\mathbf{1}_B \, \mathbf{\mathsf{E}}[Y|\mathcal{F}] = \mathbf{\mathsf{E}}[\mathbf{1}_B \, Y|\mathcal{F}].$$

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Proof (2) Step 2: If X is of the form

$$X=\sum_{i\leqslant n}\alpha_i\mathbf{1}_{B_i},$$

with $\alpha_i \in \mathbb{R}$ and $B_i \in \mathcal{F}$, then, by linearity we also get $\mathbf{E}[XY|\mathcal{F}] = X \mathbf{E}[Y|\mathcal{F}].$

Step 3: If $X, Y \ge 0$ \hookrightarrow There exists a sequence $\{X_n; n \ge 1\}$ of simple random variables such that

 $X_n \nearrow X$.

Then applying the monotone convergence we end up with:

$$\mathbf{E}[XY|\mathcal{F}] = X \, \mathbf{E}[Y|\mathcal{F}].$$

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Proof (3)

Step 4: General case $X \in L^2$

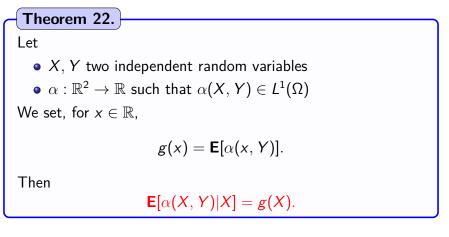
 \hookrightarrow Decompose $X = X^+ - X^-$ and $Y = Y^+ - Y^-$, which gives

 $\mathbf{E}[XY|\mathcal{F}] = X\mathbf{E}[Y|\mathcal{F}]$

by linearity.

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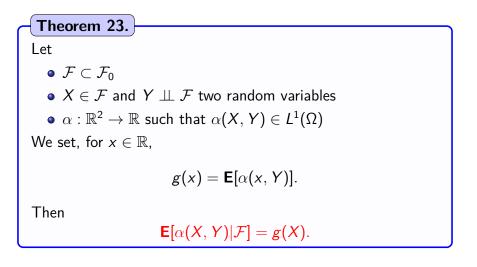
Conditional expectation and independence



Proof: with 4 steps method applied to α .

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Generalization of the previous theorem



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Orthogonal projection

Definition: Let

- *H* Hilbert space
 - \hookrightarrow complete vectorial space equipped with inner product.
- F closed subspace of H.
- Then, for all $x \in H$

• There exists a unique $y \in F$, denoted by $y = \pi_F(x)$ Satisfying one of the equivalent conditions (i) or (ii).

(i) For all $z \in F$, we have $\langle x - y, z \rangle = 0$.

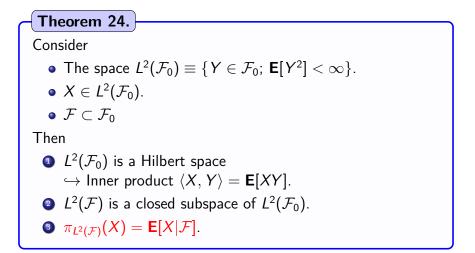
(ii) For all $z \in F$, we have $||x - y||_H \leq ||x - z||_H$.

 $\pi_F(x)$ is denoted orthogonal projection of x onto F.

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Conditional expectation and projection



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Proof

Proof of 2:

If $X_n \to X$ in $L^2 \Rightarrow$ There exists a subsequence $X_{n_k} \to X$ a.s. Thus, if $X_n \in \mathcal{F}$, we also have $X \in \mathcal{F}$.

Proof of 3: Let us check (i) in our definition of projection

Let
$$Z \in L^2(\mathcal{F})$$
.
 \hookrightarrow We have $\mathbf{E}[ZX|\mathcal{F}] = Z \mathbf{E}[X|\mathcal{F}]$, and thus
 $\mathbf{E} \{ Z \mathbf{E}[X|\mathcal{F}] \} = \mathbf{E} \{ \mathbf{E}[XZ|\mathcal{F}] \} = \mathbf{E} [XZ],$

which ensures (i) and $\mathbf{E}[X|\mathcal{F}] = \pi_{L^2(\mathcal{F})}(X)$.

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Application to Gaussian vectors

Example: Let

- (X, Y) centered Gaussian vector in \mathbb{R}^2
- Hypothesis: V(Y) > 0.

Then

$$\mathbf{E}[X|Y] = \alpha Y$$
, with $\alpha = \frac{\mathbf{E}[X|Y]}{V(Y)}$.

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Proof

Step 1: We look for α such that

$$Z = X - \alpha Y \quad \Longrightarrow \quad Z \perp\!\!\!\perp Y.$$

Recall: If (Z, Y) is a Gaussian vector $\hookrightarrow Z \perp \perp Y$ iff $\operatorname{cov}(Z, Y) = 0$

Application: $cov(Z, Y) = \mathbf{E}[Z Y]$. Thus

$$\operatorname{cov}(Z, Y) = \operatorname{\mathsf{E}}[(X - \alpha Y) Y] = \operatorname{\mathsf{E}}[X Y] - \alpha V(Y),$$

et

$$\operatorname{cov}(Z, Y) = 0$$
 iff $\alpha = \frac{\mathsf{E}[XY]}{V(Y)}$.

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Proof (2)

Step 2: We invoke (i) in the definition of π . \hookrightarrow Let $V \in L^2(\sigma(Y))$. Then

$$Y \perp\!\!\!\perp (X - \alpha Y) \implies V \perp\!\!\!\perp (X - \alpha Y)$$

and

$$\mathbf{E}[(X - \alpha Y) V] = \mathbf{E}[X - \alpha Y] \mathbf{E}[V] = 0.$$

Thus

$$\alpha Y = \pi_{\sigma(Y)}(X) = \mathbf{E}[X|Y].$$

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Outline

Definition

- Baby conditional distributions: discrete case
- Baby conditional distributions: continuous case
- Definition with measure theory
- 2 Examples
- 3 Existence and uniqueness
- 4 Conditional expectation: properties
- 5 Conditional expectation as a projection

6 Conditional regular laws

- Probability laws and expectations
- Definition of the CRL

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Aim of this section

Recall: We have seen that if

• $X \sim \mathcal{P}(\lambda_1), Y \sim \mathcal{P}(\lambda_2)$ • $X \perp Y$ • $p=rac{\lambda_1}{\lambda_1+\lambda_2}$,

then

$$\mathcal{L}(X|X+Y=n)=\mathrm{Bin}(n,p)$$

Question: How to translate this \hookrightarrow to the non-baby conditional expectation language?

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Outline

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Characterizing r.v by expected values

Notation:

 $C_b(\mathbb{R}) \equiv$ set of continuous and bounded functions on \mathbb{R} .

Theorem 25.

Let X be a r.v. We assume that

 $\mathbf{E}[\varphi(X)] = \int_{\mathbb{R}} \varphi(x) f(x) dx$, for all functions $\varphi \in C_b(\mathbb{R})$.

Then X is continuous, with density f.

Application: change of variable

Problem: Let

- X random variable with density f.
- Set Y = h(X) with $h : \mathbb{R} \to \mathbb{R}$.

We wish to find the density of Y.

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Application: change of variable (2)

Recipe: One proceeds as follows • For $\varphi \in C_b(\mathbb{R})$, write

$$\mathsf{E}[\varphi(Y)] = \mathsf{E}[\varphi(h(X))] = \int_{\mathbb{R}} \varphi(h(x)) f(x) \, dx.$$

Change variables y = h(x) in the integral. After some elementary computations we get

$$\mathsf{E}[\varphi(Y)] = \int_{\mathbb{R}} \varphi(y) \, g(y) \, dy.$$

This characterizes Y, which admits a density g

Example: normal r.v and linear transformations

Proposition 26.

Let

•
$$X \sim \mathcal{N}(0,1)$$

•
$$\mu \in \mathbb{R}$$
 and $\sigma > 0$

• Set
$$Y = \sigma X + \mu$$

Then

 $Y \sim \mathcal{N}(\mu, \sigma^2)$

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Proof

Recipe, item 1: for $\varphi \in C_b(\mathbb{R})$, write

$$\mathbf{E}[\varphi(Y)] = \mathbf{E}[\varphi(\sigma X + \mu)] = \int_{\mathbb{R}} \varphi(\sigma x + \mu) \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

Recipe, item 2: Change of variable: $y = \sigma x + \mu$:

$$\mathsf{E}[\varphi(Y)] = \int_{\mathbb{R}} \varphi(y) g(y) dx, \quad \text{with} \quad g(y) = \frac{e^{-(y-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}.$$

Recipe, item 3: *Y* is continuous with density *g*, therefore $Y \sim \mathcal{N}(\mu, \sigma^2)$.

Characterizing r.v by expected values (ctd)

Theorem 27.

Let $X : \Omega \to \mathbb{R}$ be a r.v. Then

$\{ \mathbf{E}[\varphi(X)]; \, \varphi \in C_b(\mathbb{R}) \}$ characterizes the law of X

Outline

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CRL

Definition 28.

Let

- (Ω, \mathcal{F}, P) a probability space
- (S,S) a measurable space of the form $\mathbb{R}^d, \mathbb{Z}^d$
- $X: (\Omega, \mathcal{F})
 ightarrow (S, \mathcal{S})$ a random variable in $L^1(\Omega)$
- \mathcal{G} a σ -algebra such that $\mathcal{G} \subset \mathcal{F}$.

We say that $\mu: \Omega \times S \to [0,1]$ is a Conditional regular law of X given G if

(i) For all $f \in C_b(S)$, the map $\omega \mapsto \mu(\omega, f)$ is a random variable, equal to $\mathbf{E}[f(X)|\mathcal{G}]$ a.s.

(ii) ω -a.s. $f \mapsto \mu(\omega, f)$ is a probability measure on (S, S).

Discrete example

Poisson law case: Let

• $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$ • $X \perp \!\!\!\perp Y$

We set S = X + Y.

Then

CRL of X given S is Bin(S, p), with $p = \frac{\lambda}{\lambda + \mu}$

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Proof for the discrete example

Proof: we have seen that for $n \leq m$

$$\mathbf{P}(X=n|S=m)=\binom{m}{n}p^n(1-p)^{m-n}$$
 with $p=\frac{\lambda}{\lambda+\mu}$.

Then we consider

• State space
$$\,=\,\mathbb{N}$$
, $\mathcal{G}=\sigma(\mathcal{S})$

and we verify that these conditional probabilities define a CRL.

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Continuous example

Exponential law case: Let

• $X \sim \mathcal{E}(1)$ and $Y \sim \mathcal{E}(1)$ • $X \perp Y$

We set S = X + Y.

Then

CRL of X given S is $\mathcal{U}([0, S])$.

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Continuous example

Proof: The joint density of (X, S) is given by

$$f(x,s)=e^{-s}\mathbf{1}_{\{0\leq x\leq s\}}.$$

Let then $\psi \in \mathcal{B}_b(\mathbb{R}_+)$. Thanks to Example 5, we have

 $\mathbf{E}[\psi(X)|S] = u(S),$

with

$$u(s) = \frac{\int_{\mathbb{R}_+} \psi(x) f(x,s) dx}{\int_{\mathbb{R}_+^2} f(x,s) dx} = \frac{1}{s} \int_0^s \psi(x) dx.$$

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Proof

In addition, $S \neq 0$ almost surely, and thus if $A \in \mathcal{B}(\mathbb{R})$ we have:

$$\mathsf{E}[\psi(X)|S] = \frac{\int_0^S \psi(x) dx}{S}$$

Considering the state space as $=\mathbb{R}_+$, $\mathcal{S}=\mathcal{B}(\mathbb{R}_+)$ and setting

$$\mu(\omega, f) = \frac{1}{S(\omega)} \int_0^{S(\omega)} \psi(x) dx,$$

one can verify that we have defined a conditional regular law.

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Existence of the CRL

Theorem 29.

Let

- X a random variable on $(\Omega, \mathcal{F}_0, P)$.
- Taking values in a space of the form $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.
- $\mathcal{G} \subset \mathcal{F}_0$ a σ -algebra.

Then the CRL of X given \mathcal{G} exists.

Proof: nontrivial and omitted.

Computation rules for CRL

(1) If $\mathcal{G} = \sigma(Y)$, with Y random variable with values in \mathbb{R}^m , we have

 $\mu(\omega, f) = \mu(Y(\omega), f),$

and one can define a CRL of X given Y as a family $\{\mu(y,.); y \in \mathbb{R}^m\}$ of probabilities on \mathbb{R}^n , such that for all $f \in C_b(\mathbb{R}^n)$ the function

$$y\mapsto \mu(y,f)$$

is measurable.

(2) If Y is a discrete r.v, this can be reduced to:

$$\mu(y,A) = \mathbf{P}(X \in A | Y = y) = \frac{\mathbf{P}(X \in A, Y = y)}{\mathbf{P}(Y = y)}.$$

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Computation rules for CRL (2)

(3) When one knows the CRL, quantities like the following (for $\phi \in \mathcal{B}(\mathbb{R}^n)$) can be computed:

$$\mathbf{E} \left[\phi(X) | \mathcal{G} \right] = \int_{\mathbb{R}^n} \phi(x) \, \mu(\omega, dx)$$

$$\mathbf{E} \left[\phi(X) | Y \right] = \int_{\mathbb{R}^n} \phi(x) \, \mu(Y, dx).$$

(4) The CRL is not unique.
 However if N₁, N₂ are 2 CRL of X given G
 → we have ω-almost surely:

$$N_1(\omega, f) = N_2(\omega, f)$$
 for all $f \in C_b(\mathbb{R}^n)$.