Laws of large numbers

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Probability Theory 1 - MA 538

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by Grimmett-Stirzaker

Outline

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Generalized Markov's inequality

Proposition 1.

Let

- $h : \mathbb{R} \to [0, \infty)$ non-negative function
- X random variable with $h(X) \in L^1(\Omega)$

Then for all $a > 0$ we have

$$\mathbf{P}\left(h(X) \geq a\right) \leq \frac{\mathbf{E}\left[h(X)\right]}{a}$$

Proof of Proposition 1

Deterministic inequality: Set

$$A = \{h(X) \geq a\}$$

Then we have

$$h(X) \geq a\, \mathbf{1}_A$$

Expectations: Taking expectations above, we get

$$\mathbf{E}\left[h(X)\right] \geq a\, \mathbf{P}(A)$$

Particular cases of Proposition 1:

Case $h(X) = |X|$: We get Markov's inequality,

$$\mathbf{P}\left(|X| \geq a\right) \leq \frac{\mathbf{E}\left[|X|\right]}{a}$$

Case $h(X) = X^2$: We get Chebyshev's inequality,

$$\mathbf{P}\left(|X| \geq a\right) \leq \frac{\mathbf{E}\left[|X|^2\right]}{a^2}$$

Reversed Markov type inequality

Proposition 2.

Let

- $h : \mathbb{R} \to [0, M)$ non-negative bounded function
- X random variable

Then for all $0 < a < M$ we have

$$\mathbf{P}\left(h(X) \geq a\right) \geq \frac{\mathbf{E}\left[h(X)\right] - a}{M - a}$$

Proof of Proposition 2

Deterministic inequality: Set

$$A = \{h(X) \geq a\}$$

Then we have

$$h(X) \leq M\,\mathbf{1}_A + a\,\mathbf{1}_{A^c}$$

Expectations: Taking expectations above, we get

$$\mathbf{E}\,[h(X)] \leq M\,\mathbf{P}(A) + a\,(1 - \mathbf{P}(A))$$

$$\Longrightarrow \mathbf{P}\,(A) \leq \frac{\mathbf{E}\,[h(X)] - a}{M - a}$$

Hölder's inequality

Proposition 3.

Let

- X, Y random variables
- $p, q > 1$ such that $p^{-1} + q^{-1} = 1$

Then we have

$$\|X\,Y\|_{L^1} \leq \|X\|_{L^p} \|Y\|_{L^q}$$

Minkowski's inequality

Proposition 4.

Let

- X, Y random variables
- $p \geq 1$

Then we have

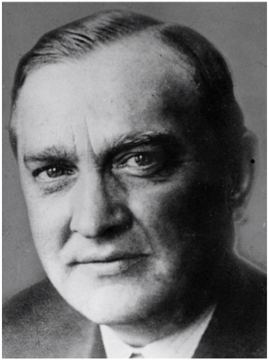
$$\|X+Y\|_{L^p} \leq \|X\|_{L^p} + \|Y\|_{L^p}$$

Remark:
$(L^p(\Omega), \|\cdot\|_{L^p})$ is a Banach space

Stefan Banach

Some facts about Banach:

- Lifespan: 1892-1945, in Krakow and Lviv
- Among greatest 20-th century mathematicians
- Founder of a new field
  $\hookrightarrow$ Functional Analysis
- Survived 2 world wars in tough conditions
- Then dies in 1945 from lung cancer



Limits of sums

Proposition 5.

Let

- X, Y random variables
- X_n, Y_n sequences of random variables

Then we have

1. $X_n \xrightarrow{a.s} X$ and $Y_n \xrightarrow{a.s} Y \implies X_n + Y_n \xrightarrow{a.s} X + Y$
2. $X_n \xrightarrow{L^p} X$ and $Y_n \xrightarrow{L^p} Y \implies X_n + Y_n \xrightarrow{L^p} X + Y$
3. $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y \implies X_n + Y_n \xrightarrow{P} X + Y$
4. $X_n \xrightarrow{(d)} X$ and $Y_n \xrightarrow{(d)} Y \not\Rightarrow X_n + Y_n \xrightarrow{(d)} X + Y$

Counter-example for Proposition 5 - item 4

Example of sequence: Consider

- $X_n = X \sim \mathcal{B}(\frac{1}{2})$
- $Y_n = Y = 1 - X$

Convergences: We have

$$X_n \xrightarrow{(d)} X, \qquad Y_n \xrightarrow{(d)} X\,.$$

However

$$X_n + Y_n \xrightarrow{(d)} 1$$

Outline

Limsup of sets

Definition 6.

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We define

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

Interpretation: We also have

$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega;\, \omega \text{ belongs to an infinity of } A_n\text{'s}\}$$

Borel-Cantelli lemma

Theorem 7.

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We assume

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) < \infty$$

Then we have

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 0$$

Reversed Borel-Cantelli lemma

Theorem 8.

Let

- $\{A_n;\, n \geq 1\}$ sequence in $\mathcal{F}$

We assume

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty, \quad \text{and} \quad A_n\text{'s independent}$$

Then we have

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) = 1$$

Proof of Theorem 8 (1)

Notation: We set

$$A = \limsup_{n\to\infty} A_n$$

Complement of A: We have

$$A^c = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k^c$$

Monotone convergence: We will use

$$\mathbf{P}\left(A^c\right) = \lim_{n\to\infty} \mathbf{P}\left(\cap_{k=n}^{\infty} A_k^c\right) \tag{1}$$

Proof of Theorem 8 (2)

Computation: We have

$$
\begin{aligned}
\mathbf{P}\left(\cap_{k=n}^{\infty} A_k^c\right) &= \lim_{r\to\infty} \mathbf{P}\left(\cap_{k=n}^{r} A_k^c\right) \\
&= \prod_{k=n}^{\infty} \mathbf{P}\left(A_k^c\right) \\
&= \prod_{k=n}^{\infty} \left[1 - \mathbf{P}\left(A_k\right)\right] \\
&\leq \prod_{k=n}^{\infty} \exp\left(-\mathbf{P}\left(A_k\right)\right) \\
&\leq \exp\left(-\sum_{k=n}^{\infty} \mathbf{P}\left(A_k\right)\right) \\
&= 0
\end{aligned}
$$

Proof of Theorem 8 (3)

Conclusion: Taking limits in (1) we get

$$\mathbf{P}(A) = 0$$

Remarks about Borel-Cantelli (1)

Recovering a result on Markov chains: Assume the following,

- X_n Markov chain with $X_0 = i$
- $A_n = \{X_n = i\}$
- $\sum_{n=1}^{\infty} p_n(i,i) < \infty$

Then by Borel-Cantelli,

$$\mathbf{P}(A_n \text{ occurs i.o}) = 0$$

However, one cannot apply reversed Borel-Cantelli

Remarks about Borel-Cantelli (2)

First case of 0-1 law: If the A_n's independent, we have obtain

$$\mathbf{P}\left(\limsup_{n\to\infty} A_n\right) \in \{0, 1\}$$

We will see generalizations of this kind of statement

Tail σ-field

Definition 9.

We consider

- $\{X_n;\ n \geq 1\}$ sequence of random variables
- $\mathcal{F}'_n = \sigma\left(X_k;\ k \geq n\right)$

We set

$$\mathcal{T} = \bigcap_{n \geq 1} \mathcal{F}'_n$$

The σ-field $\mathcal{T}$ is called Tail σ-field

Interpretation: We have

$A \in \mathcal{T}$ if changing a finite number of X_n's does not change the occurence of A.

Examples of events in $\mathcal{T}$

General setting: We consider

- $\{X_n;\ n \geq 1\}$ sequence of random variables
- $S_n = \sum_{k=1}^n X_i$

Then we have

1. $(X_n > 0 \text{ i.o}) \in \mathcal{T}$
2. $(\lim_{n\to\infty} S_n \text{ exists}) \in \mathcal{T}$
3. $(\limsup_{n\to\infty} X_n > 0) \in \mathcal{T}$
4. $(\limsup_{n\to\infty} S_n > 0) \notin \mathcal{T}$
5. $(\limsup_{n\to\infty} \frac{1}{a_n} S_n > 0) \in \mathcal{T}$ if $\lim_{n\to\infty} a_n = \infty$

Kolmogorov's 0-1 law

Theorem 10.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of independent random variables
- The tail σ-field $\mathcal{T}$

Then $\mathcal{T}$ is trivial, that is:

1. If $A \in \mathcal{T}$ we have

$$\mathbf{P}(A) \in \{0, 1\}$$

2. If $Y \in \mathcal{T}$, there exists $k \in [-\infty, \infty]$ such that

$$\mathbf{P}(Y = k) = 1$$

Recalling π-systems and λ-systems

π-system: Let $\mathcal{P}$ family of subsets of Ω. $\mathcal{P}$ is a π-system if:

$$A, B \in \mathcal{P} \quad \Longrightarrow \quad A \cap B \in \mathcal{P}$$

λ-system: Let $\mathcal{L}$ family of subsets of Ω. $\mathcal{L}$ is a λ-system if:

1. $\Omega \in \mathcal{L}$
2. If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$
3. If for $j \geq 1$ we have:
   - $A_j \in \mathcal{L}$
   - $A_j \cap A_i = \varnothing$ if $j \neq i$

   Then $\cup_{j \geq 1} A_j \in \mathcal{L}$

Recalling Dynkin's π-λ lemma

Proposition 11.

Let $\mathcal{P}$ et $\mathcal{L}$ such that:

- $\mathcal{P}$ is a π-system
- $\mathcal{L}$ is a λ-system
- $\mathcal{P} \subset \mathcal{L}$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$

Proof of Theorem 10 (1)

Strategy: For $A \in \mathcal{T}$,

1. We will prove $A \perp\!\!\!\perp A$
2. If $A \perp\!\!\!\perp A$, then

$$\mathbf{P}(A)^2 = \mathbf{P}(A), \quad \text{thus} \quad \mathbf{P}(A) \in \{0, 1\}$$

Proof of Theorem 10 (2)

Step 1: We will prove that

$$A \in \sigma(X_1, \ldots, X_k),\; B \in \sigma(X_{k+1}, \ldots) \quad \Longrightarrow \quad A \perp\!\!\!\perp B$$

Proof of Theorem 10 (3)

Proof of Step 1: We have

- Let $\mathcal{K}_{k,j} = \sigma(X_{k+1}, \ldots, X_{k+j})$. Then $\cup_{j\geq 0}\mathcal{K}_{k,j}$ is a π-system
- Let $A \in \sigma(X_1, \ldots, X_k)$ and

$$\mathcal{L} = \{B;\ \mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)\}$$

  Then $\mathcal{L}$ is a λ-system such that $\mathcal{L} \supset (\cup_{j\geq 0}\mathcal{K}_{k,j})$

Thus

$$\mathcal{L} \supset \sigma\left(\cup_{j\geq 0}\mathcal{K}_{k,j}\right) = \sigma(X_{k+1}, \ldots)$$

Proof of Theorem 10 (4)

Step 2: We will prove that

$$B \in \sigma(X_1, \ldots), \quad \text{and} \quad A \in \mathcal{T} \quad \Longrightarrow \quad A \perp\!\!\!\perp B$$

Conclusion: If $A \in \mathcal{T}$ we have

$$A \in \sigma(X_1, \ldots), \quad \text{and} \quad A \in \mathcal{T}. \quad \text{Thus} \quad A \perp\!\!\!\perp A$$

Proof of Theorem 10 (5)

Proof of Step 2: We have

- Let $\mathcal{F}_k = \sigma(X_1, \ldots, X_k)$. Then $\cup_{k\geq 1}\mathcal{F}_k$ is a π-system
- Let $A \in \mathcal{T}$ and

$$\mathcal{L} = \{B;\, \mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)\}$$

Then $\mathcal{L}$ is a λ-system such that $\mathcal{L} \supset (\cup_{k\geq 1}\mathcal{F}_k)$

Thus

$$\mathcal{L} \supset \sigma\left(\cup_{j\geq 0}\mathcal{K}_j\right) = \sigma(X_1, \ldots)$$

Proof that $\mathcal{L} \supset (\cup_{k\geq 1}\mathcal{F}_k)$: If $B \in \mathcal{F}_k$ and $A \in \mathcal{T}$, then

$$A \in \mathcal{K}_{k+1}, \quad \text{and thus} \quad A \perp\!\!\!\perp B$$

Application to law of large numbers

Theorem 12.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of independent random variables
- $S_n = \sum_{i=1}^{n} X_i$
- $Z_1 = \liminf_{n\to\infty} \frac{1}{n} S_n$, and $Z_2 = \limsup_{n\to\infty} \frac{1}{n} S_n$

Then the following holds true:

1. There exists $k_1, k_2 \in [-\infty, \infty]$ such that

$$Z_1 = k_1, \quad \text{and} \quad Z_2 = k_2 \quad \text{a.s}$$

2. If $A \equiv (\lim_{n\to\infty} \frac{1}{n} S_n \text{ exists})$, we have

$$\mathbf{P}(A) \in \{0, 1\}$$

Outline

Statement of the problem

General problem: We consider

- $\{X_n;\ n \geq 1\}$ sequence of random variables
- $S_n = \sum_{i=1}^{n} X_i$

Then we wish to investigate a convergence of the form

$$\frac{S_n}{n} - a_n \longrightarrow S$$

To be specified:

1. Constants a_n, b_n
2. Random variable S
3. Mode of convergence

Reviewing old results

Proposition 13.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- $\mathbf{E}[X_1] = \mu$ and $\mathbf{Var}(X_1) = \sigma^2$
- $S_n = \sum_{i=1}^{n} X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{(d)} \mu, \quad \text{and} \quad \sqrt{n}\, \frac{(\bar{X}_n - \mu)}{\sigma} \xrightarrow{(d)} \mathcal{N}(0,1)$$

Proof of Proposition 13 (1)

Characteristic functions: For $t, u \in \mathbb{R}$ set

$$\phi(u) = \mathbf{E}\left[\exp\left(\imath u X_1\right)\right], \quad \text{and} \quad \phi_n(t) = \mathbf{E}\left[\exp\left(\imath t \bar{X}_n\right)\right],$$

Then we have

$$\phi_n(t) = \left[\phi\left(\frac{t}{n}\right)\right]^n$$

Expansion for ϕ_n: We get

$$\phi_n(t) = \left(1 + \imath\frac{\mu\, t}{n} + o\left(\frac{1}{n}\right)\right)^n$$

Proof of Proposition 13 (2)

Limit for ϕ_n: By Taylor expansions arguments, for all $t \in \mathbb{R}$ we have

$$\lim_{n \to \infty} \phi_n(t) = \exp(\imath \mu t)$$

Conclusion: By characteristic function method,

$$\bar{X}_n \xrightarrow{(d)} \mu$$

Method for CLT part:
$\hookrightarrow$ Expansions of order 2 for characteristic functions

A first improvement: weak LLN

Proposition 14.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- Hyp: $X_1 \in L^1(\Omega)$ and $\mathbf{E}[X_1] = \mu$
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\mathrm{P}} \mu,$$

Proof of Proposition 14

Quick proof: The result stems from

- $\bar{X}_n \xrightarrow{(d)} \mu$
- μ is a constant

Strong LLN under L^2 conditions

Proposition 15.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- Hyp: $X_1 \in L^2(\Omega)$ and $\mathbf{E}[X_1] = \mu$, $\mathbf{Var}(X_1) = \sigma^2$
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu, \quad \text{and} \quad \bar{X}_n \xrightarrow{L^2} \mu$$

Proof of Proposition 15 (1)

L^2 convergence: We compute

$$\begin{aligned}
\mathbf{E}\left[\left(\bar{X}_n - \mu\right)^2\right] &= \frac{1}{n^2}\,\mathbf{E}\left[\left(\sum_{i=1}^{n}(X_i - \mu)\right)^2\right] \\
&= \frac{1}{n^2}\,\mathbf{Var}\left(\sum_{i=1}^{n}(X_i - \mu)\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbf{Var}\left(X_i\right) \\
&= \frac{1}{n}\,\mathbf{Var}\left(X_1\right)
\end{aligned}$$

Conclusion:

$$\lim_{n\to\infty}\mathbf{E}\left[\left(\bar{X}_n - \mu\right)^2\right] = 0$$

Proof of Proposition 15 (2)

General result for a subsequence: Since $\bar{X}_n \xrightarrow{\text{P}} \mu$, we have:

There exists a subsequence $\{n_k;\, k \geq 1\}$ such that $\bar{X}_{n_k} \xrightarrow{\text{a.s}} \mu$

Proof of Proposition 15 (3)

A more concrete subsequence: Set $n_k = k^2$ and

$$A_k(\varepsilon) = \left\{ |\bar{X}_{n_k} - \mu| > \varepsilon \right\}$$

Then by Chebyshev,

$$\mathbf{P}\left(A_k(\varepsilon)\right) \le \frac{\mathbf{E}\left[\left(\bar{X}_{k^2} - \mu\right)^2\right]}{\varepsilon^2} \le \frac{\mathbf{Var}(X_1)}{k^2 \varepsilon^2}$$

Almost sure convergence: We have

$$\sum_{k=1}^{\infty} \mathbf{P}\left(A_k(\varepsilon)\right) < \infty \text{ for all } \varepsilon > 0, \quad \text{and thus} \quad \bar{X}_{k^2} \xrightarrow{\text{a.s}} \mu$$

Proof of Proposition 15 (4)

Case of a positive sequence: If $X_n \geq 0$, then if $k^2 \leq n \leq (k+1)^2$

$$\begin{aligned} S_{k^2} &\leq S_n \leq & S_{(k+1)^2} \\ \frac{S_{k^2}}{(k+1)^2} &\leq \frac{S_n}{n} \leq & \frac{S_{(k+1)^2}}{k^2} \end{aligned}$$

Taking $n \to \infty$ we get

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu$$

Proof of Proposition 15 (5)

Signed sequence case: For a general X_n we argue as follows:

1. Write $X_n = X_n^+ - X_n^-$
2. Apply positive sequence case to both X_n^+ and X_n^-
3. This is allowed since X_n^+ i.i.d with $\mathbf{Var}(X_1^+) < \infty$

Conclusion: We still have

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu$$

Outline

The strong law

Theorem 16.

We consider

- $\{X_n;\ n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu, \quad \Longleftrightarrow \quad X_1 \in L^1(\Omega)$$

Nsc for weak convergence

Theorem 17.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\bar{X}_n \xrightarrow{\mathbf{P}} \mu \quad \Longleftrightarrow \quad \text{Condition (2) or (3) holds,}$$

with

$$\lim_{n\to\infty} n\,\mathbf{P}\left(|X_1| > n\right) = 0, \text{ and } \lim_{n\to\infty} \mathbf{E}\left[X_1\,\mathbf{1}_{(|X_1|\leq n)}\right] = \mu \qquad (2)$$

$$\phi \text{ differentiable at } 0, \quad \text{and} \quad \phi'(0) = \imath\,\mu \qquad (3)$$

Example of WLLN without SLLN

Proposition 18.

We consider

- $\{X_n;\ n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^{n} X_i$ and $\bar{X}_n = \frac{1}{n} S_n$
- X_1 symmetric random variable
- Common cdf satisfies $1 - F(x) \sim \frac{1}{x \ln(x)}$ as $x \to \infty$

Then

$$\bar{X}_n \xrightarrow{\mathrm{P}} 0, \quad \text{but} \quad \bar{X}_n \text{ does not converge a.s}$$

Cauchy random variable (1)

Notation:

$$\text{Cauchy}(\alpha), \text{ with } \alpha \in \mathbb{R}$$

State space:

$$\mathbb{R}$$

Density:

$$f(x) = \frac{1}{\pi} \frac{1}{1 + (x - \alpha)^2}$$

Expected value and variance:

Not defined (divergent integrals)!

Cauchy random variable (2)

Use 1: Trigonometric function of a uniform r.v
Namely if

- $X \sim \mathcal{U}([-\frac{\pi}{2}, \frac{\pi}{2}])$
- $Y = \tan(X)$

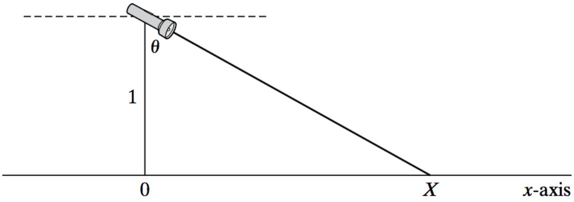
Then $Y \sim \text{Cauchy} \equiv \text{Cauchy}(0)$

Use 2:
Typical example of r.v with no mean

Example: beam (1)

Experiment:

- Narrow-beam flashlight spun around its center
- Center located a unit distance from the x-axis
- $X =$ point at which the beam intersects the x-axis when the flashlight has stopped spinning



Example: beam (2)

Model:

- We assume $\theta \sim \mathcal{U}([-\frac{\pi}{2}, \frac{\pi}{2}])$
- We have $X \sim \tan(\theta)$

Conclusion:

$$X \sim \text{Cauchy}$$

Example with no WLLN

Proposition 19.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$
- $X_1 \sim$ Cauchy

Then

$$\bar{X}_n \xrightarrow{(d)} \text{Cauchy}, \quad \text{but} \quad \bar{X}_n \text{ does not converge in P}$$

Proof of Theorem 16 (1)

Particular case: We assume

$$X_1 \geq 0 \text{ a.s}, \qquad \mathbf{E}[|X_1|] = \mathbf{E}[X_1] = \mu < \infty$$

Truncation: For $n \geq 1$ we set

$$Y_n = X_n \mathbf{1}_{(X_n < n)}$$

Claim about the truncation: Define

$$A_n = (X_n \neq Y_n)$$

Then

$$\mathbf{P}(A_n \text{ occurs i.o}) = 0 \tag{4}$$

Proof of Theorem 16 (2)

Proof of claim (4): We have

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbf{P}(A_n) &= \sum_{n=1}^{\infty} \mathbf{P}(X_n \geq n) \\ &\leq \mathbf{E}[X_1] < \infty\end{aligned}$$

Thus (4) holds thanks to Borel-Cantelli

Proof of Theorem 16 (3)

Reduction of the proof: According to (4), we have

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - Y_k) \xrightarrow{\text{a.s}} 0$$

Hence we just need to show

$$\bar{Y}_n \xrightarrow{\text{a.s}} \mu$$

Proof of Theorem 16 (4)

Elementary relation: Let $\alpha > 1$ and $\beta_k = \lfloor \alpha^k \rfloor$.
Then there exists $A > 0$ such that

$$\sum_{k=m}^{\infty} \frac{1}{\beta_k^2} \leq \frac{A}{\beta_m^2} \tag{5}$$

Brief proof of (5): Stems from

$$\beta_k \asymp \alpha^k, \quad \text{for large } k\text{'s}$$

Proof of Theorem 16 (5)

Claim 2 about the truncation: Write $S'_n = \sum_{k=1}^{n} Y_k$. Then

$$\frac{1}{\beta_n}\left(S'_{\beta_n} - \mathbf{E}\left[S'_{\beta_n}\right]\right) \xrightarrow{\text{a.s}} 0 \tag{6}$$

Proof of Theorem 16 (6)

Proof of (6): For $\varepsilon > 0$, set

$$B_n(\varepsilon) = \left(\frac{1}{\beta_n} \left| S'_n - \mathbf{E}\left[S'_n\right] \right| > \varepsilon \right)$$

Then the following yields (6) by Borel-Cantelli:

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbf{P}\left(B_{\beta_n}(\varepsilon)\right) &\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \mathbf{Var}\left(S'_{\beta_n}\right) \\
&\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \sum_{k=1}^{\beta_n} \mathbf{Var}\left(Y_k\right) \\
&\leq \frac{A}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{E}\left[Y_k^2\right] \overset{\text{Claim 3}}{<} \infty
\end{aligned}$$

Proof of Theorem 16 (7)

Proof of Claim 3: This is where we use the truncation,

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{E}\left[Y_k^2\right] &= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} \mathbf{E}\left[Y_k^2 \, \mathbf{1}_{B_{kj}}\right] \\
&\leq \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 \mathbf{P}\left(B_{kj}\right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 \mathbf{P}\left(B_{1j}\right) \\
&= \sum_{j=1}^{\infty} j^2 \mathbf{P}\left(B_{1j}\right) \sum_{k=j}^{\infty} \frac{1}{k^2} \\
&\lesssim \sum_{j=1}^{\infty} j \mathbf{P}\left(B_{1j}\right) \lesssim 1 + \sum_{j=1}^{\infty} (j-1) \mathbf{P}\left(B_{1j}\right) \\
&\lesssim 1 + \mathbf{E}[X_1] < \infty
\end{aligned}
$$

Proof of Theorem 16 (8)

From (6) to the theorem: The missing steps are

1. We have $\mathbf{E}[Y_n] \to \mu$
   $\hookrightarrow$ by monotone convergence
2. Fill the gaps between β_n's
   $\hookrightarrow$ Similar to Proposition 15
3. Signed sequence, also like in Proposition 15:
   1. Write $X_n = X_n^+ - X_n^-$
   2. Apply positive sequence case to both X_n^+ and X_n^-
   3. This is allowed since $X_n^{\pm}$ i.i.d with $\mathbf{E}[X_1^{\pm}] < \infty$

Conclusion: We have

$$X_1 \in L^1 \quad \Longrightarrow \quad \bar{X}_n \xrightarrow{\text{a.s}} \mu$$

Proof of Theorem 16 (9)

Converse result: We have

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu \quad \overset{\text{results on series}}{\Longrightarrow} \quad \frac{X_n}{n} \xrightarrow{\text{a.s}} 0$$

$$\overset{\text{reversed Borel-C}}{\Longrightarrow} \quad \sum_{n=1}^{\infty} \mathbf{P}\left(|X_n| \geq n\right) < \infty$$

$$\overset{\text{i.i.d Hyp}}{\Longrightarrow} \quad \sum_{n=1}^{\infty} \mathbf{P}\left(|X_1| \geq n\right) < \infty$$

Hence

$$\mathbf{E}[|X_1|] \overset{\text{Problem 4.14.3}}{\leq} 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(|X_1| \geq n\right) < \infty$$

Outline

The law of iterated logarithm

Theorem 20.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- Hyp: $X_1 \in L^2(\Omega)$ and $\mathbf{E}[X_1] = 0$, $\mathbf{Var}(X_1) = 1$
- $S_n = \sum_{i=1}^n X_i$

Then

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{S_n}{(2n \ln\ln(n))^{1/2}} = 1\right) = 1$$

$$\mathbf{P}\left(\liminf_{n\to\infty} \frac{S_n}{(2n \ln\ln(n))^{1/2}} = -1\right) = 1$$

LIL – second version

Theorem 21.

We consider

- $\{X_n;\, n \geq 1\}$ sequence of i.i.d random variables
- Hyp: $X_1 \in L^2(\Omega)$ and $\mathbf{E}[X_1] = \mu$, $\mathbf{Var}(X_1) = \sigma^2$
- $S_n = \sum_{i=1}^n X_i$ and $\bar{X}_n = \frac{1}{n} S_n$

Then

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\left(2\ln\ln(n)\right)^{1/2}\sigma} = 1 \right) = 1$$

$$\mathbf{P}\left(\liminf_{n\to\infty} \frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\left(2\ln\ln(n)\right)^{1/2}\sigma} = -1 \right) = 1$$

Interpretation of LIL

Heuristics: We have

1. LLN states that

$$\bar{X}_n \longrightarrow \mu$$

2. CLT states that

$$\bar{X}_n \simeq \mu + \frac{\sigma}{\sqrt{n}} \mathcal{N}(0,1)$$

3. LIL states that

$$\bar{X}_n = \mu + \text{ rare fluctuations of order } \frac{(2 \ln \ln(n))^{1/2} \sigma}{\sqrt{n}}$$

Hints about the proof of Theorem 20 (1)

0-1 law: Asserts that if

$$U \equiv \limsup_{n\to\infty} \frac{S_n}{(2n \ln\ln(n))^{1/2}},$$

then there exists $k \in [-\infty, \infty]$ such that

$$\mathbf{P}(U = k) = 1$$

Hints about the proof of Theorem 20 (2)

Global strategy: For $\alpha > 0$ set

$$A_n(\alpha) = \left(S_n \geq \alpha \left(2n \ln \ln(n)\right)^{1/2}\right)$$

Then with help of Borel-Cantelli we prove

$$\begin{aligned} \mathbf{P}\left(A_n(\alpha) \text{ occurs i.o}\right) &= 1\,, \quad \text{if } \alpha < 1 \\ \mathbf{P}\left(A_n(\alpha) \text{ occurs i.o}\right) &= 0\,, \quad \text{if } \alpha > 1 \end{aligned}$$