Brownian motion

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Outline

Outline

Stochastic processes

Definition 1.

Let:

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space.
- $I \subset \mathbb{R}_+$ interval.
- $\{X_t;\ t \in I\}$ family of random variables, $\mathbb{R}^n$-valued

Then:

1. If $\omega \mapsto X_t(\omega)$ measurable, X is a stochastic process
2. $t \mapsto X_t(\omega)$ is called a path
3. X is continuous if its paths are continuous a.s.

Modifications of processes

Definition 2.

Let X and Y be two processes on $(\Omega, \mathcal{F}, \mathbf{P})$.

1. X is a modification of Y if

$$\mathbf{P}\left(X_t = Y_t\right) = 1, \quad \text{for all} \quad t \in I$$

2. X and Y are non-distinguishable if

$$\mathbf{P}\left(X_t = Y_t \text{ for all } t \in I\right) = 1$$

Remarks:

(i) Relation (2) implicitly means that $(X_t = Y_t \text{ for all } t \in I) \in \mathcal{F}$

(ii) (2) is much stronger than (1)

(iii) If X and Y are continuous, (2) $\Longleftrightarrow$ (1)

Filtrations

Filtration: Increasing sequence of σ-algebras, i.e
$\hookrightarrow$ If $s < t$, then $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.

Interpretation: $\mathcal{F}_t$ summarizes an information obtained at time t

Negligible sets: $\mathcal{N} = \{F \in \mathcal{F};\ \mathbf{P}(F) = 0\}$

Complete filtration: Whenever $\mathcal{N} \subset \mathcal{F}_t$ for all $t \in I$

Stochastic basis: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbf{P})$ with a complete $(\mathcal{F}_t)_{t\in I}$

Remark: Filtration $(\mathcal{F}_t)_{t\in I}$ can always be thought of as complete
$\hookrightarrow$ One replaces $\mathcal{F}_t$ by $\hat{\mathcal{F}}_t = \sigma(\mathcal{F}_t, \mathcal{N})$

Adaptation

Definition 3.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbf{P})$ stochastic basis
- $\{X_t;\ t \in I\}$ stochastic process

We say that X is $\mathcal{F}_t$-adapted if for all $t \in I$:

$$X_t : (\Omega, \mathcal{F}_t) \longrightarrow (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)) \quad \text{is measurable}$$

Remarks:

(i) Let $\mathcal{F}_t^X = \sigma\{X_s;\ s \leq t\}$ the natural filtration of X.
$\hookrightarrow$ Process X is always $\mathcal{F}_t^X$-adapted.

(ii) A process X is $\mathcal{F}_t$-adapted iff $\mathcal{F}_t^X \subset \mathcal{F}_t$

Outline

Definition of the Wiener process

Notation: For a function f, $\delta f_{st} \equiv f_t - f_s$

Definition 4.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{W_t;\ t \geq 0\}$ stochastic process, $\mathbb{R}$-valued

We say that W is a Wiener process if:

1. $W_0 = 0$ almost surely
2. Let $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. The increments
$$\delta W_{t_0 t_1}, \delta W_{t_1 t_2}, \ldots, \delta W_{t_{n-1} t_n} \quad \text{are independent}$$
3. For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, t-s)$
4. W has continuous paths almost surely

Illustration: chaotic path

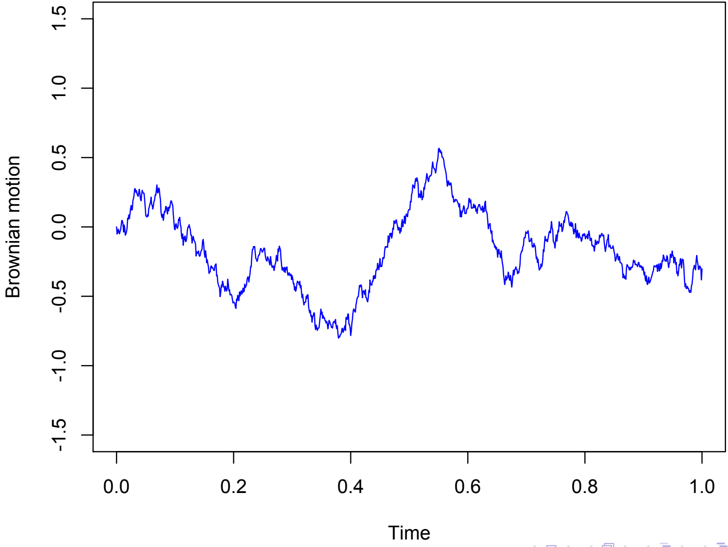
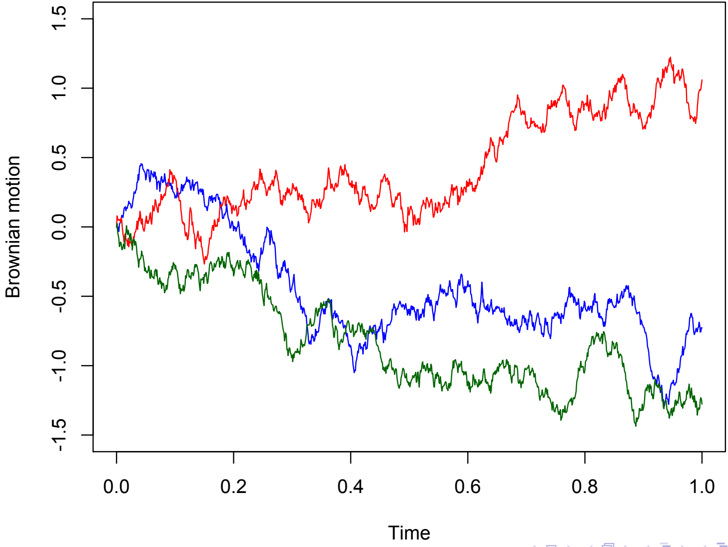


Illustration: random path



Existence of the Wiener process

Theorem 5.

There exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which one can construct a Wiener process.

Classical constructions:

- Kolmogorov's extension theorem
- Limit of a renormalized random walk
- Lévy-Ciesilski's construction

Haar functions

Definition 6.

We define a family of functions $\{h_k : [0,1] \to \mathbb{R};\ k \geq 0\}$:

$$\begin{aligned} h_0(t) &= \mathbf{1} \\ h_1(t) &= \mathbf{1}_{[0,1/2]}(t) - \mathbf{1}_{(1/2,1]}(t), \end{aligned}$$

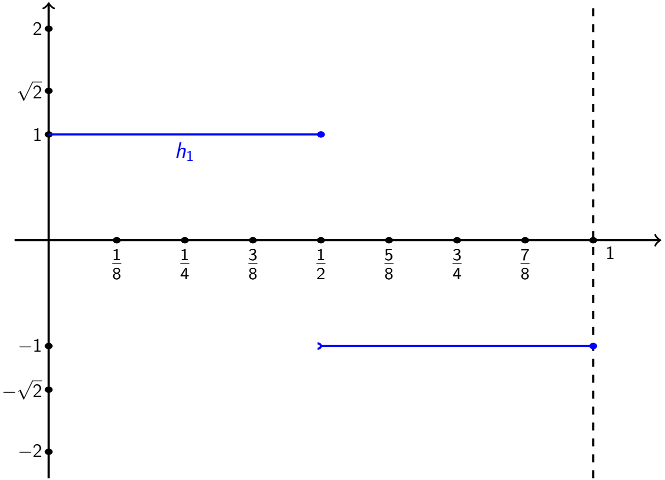
and for $n \geq 1$ and $2^n \leq k < 2^{n+1}$:

$$h_k(t) = 2^{n/2}\, \mathbf{1}_{\left[\frac{k-2^n}{2^n}, \frac{k-2^n+1/2}{2^n}\right]}(t) - 2^{n/2}\, \mathbf{1}_{\left(\frac{k-2^n+1/2}{2^n}, \frac{k-2^n+1}{2^n}\right]}(t)$$

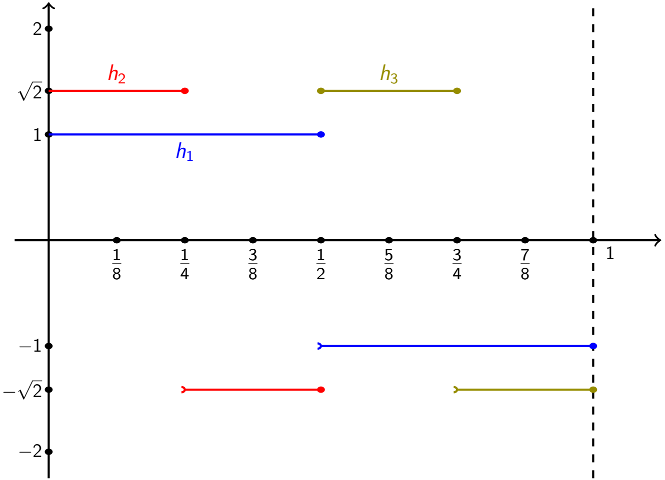
Lemma 7.

The functions $\{h_k : [0,1] \to \mathbb{R};\ k \geq 0\}$ form an orthonormal basis of $L^2([0,1])$.

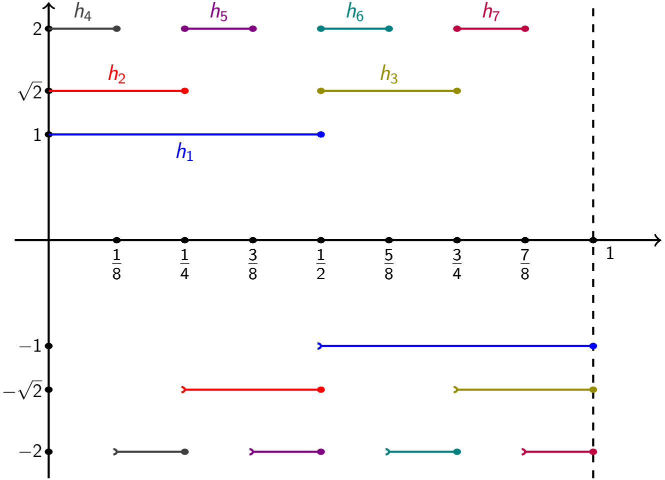
Haar functions: illustration



Haar functions: illustration



Haar functions: illustration



Proof

Norm: For $2^n \leq k < 2^{n+1}$, we have

$$\int_0^1 h_k^2(t)\,dt = 2^n \int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+1}{2^n}} dt = 1.$$

Orthogonality: If $k < l$, we have two situations:

(i) $\mathrm{Supp}(h_k) \cap \mathrm{Supp}(h_l) = \varnothing$.
Then trivially $\langle h_k,\, h_l\rangle_{L^2([0,1])} = 0$

(ii) $\mathrm{Supp}(h_l) \subset \mathrm{Supp}(h_k)$.
Then if $2^n \leq k < 2^{n+1}$ we have:

$$\langle h_k,\, h_l\rangle_{L^2([0,1])} = \pm 2^{n/2} \int_0^1 h_l(t)\,dt = 0.$$

Proof (2)

Complete system: Let $f \in L^2([0,1])$ s.t $\langle f,\, h_k \rangle = 0$ for all k.
$\hookrightarrow$ We will show that $f = 0$ almost everywhere.

Step 1: Analyzing the relations $\langle f,\, h_k \rangle = 0$
$\hookrightarrow$ We show that $\int_s^t f(u)\, du = 0$ for dyadic r, s.

Step 2: Since $\int_s^t f(u)\, du = 0$ for dyadic r, s, we have

$$f(t) = \partial_t \left(\int_0^t f(u)\, du \right) = 0, \quad \text{almost everywhere,}$$

according to Lebesgue's derivation theorem.

Schauder functions

Definition 8.

We define a family of functions $\{s_k : [0,1] \to \mathbb{R};\ k \geq 0\}$:

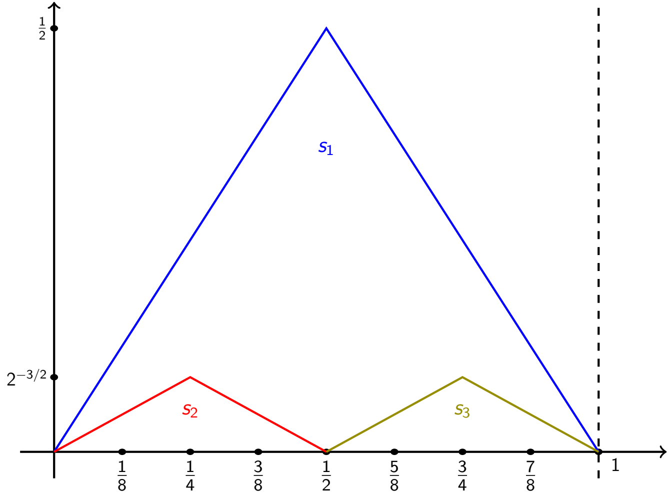
$$s_k(t) = \int_0^t h_k(u)\, du$$

Lemma 9.

Functions $\{s_k : [0,1] \to \mathbb{R};\ k \geq 0\}$ satisfy for $2^n \leq k < 2^{n+1}$:

1. $\mathrm{Supp}(s_k) = \mathrm{Supp}(h_k) = [\frac{k-2^n}{2^n}, \frac{k-2^n+1}{2^n}]$
2. $\|s_k\|_\infty = \frac{1}{2^{n/2+1}}$

Schauder functions: illustration



Gaussian supremum

Lemma 10.

Let $\{X_k;\, k \geq 1\}$ i.i.d sequence of $\mathcal{N}(0,1)$ r.v. We set:

$$M_n \equiv \sup\left\{|X_k|;\, 1 \leq k \leq n\right\}.$$

Then

$$M_n = O\left(\sqrt{\ln(n)}\right) \qquad \text{almost surely}$$

Proof

Gaussian tail: Let $x > 0$. We have:

$$\begin{aligned}
\mathbf{P}\left(|X_k| > x\right) &= \frac{2}{(2\pi)^{1/2}} \int_x^\infty e^{-\frac{z^2}{4}} e^{-\frac{z^2}{4}}\, dz \\
&\le c_1 e^{-\frac{x^2}{4}} \int_x^\infty e^{-\frac{z^2}{4}}\, dz \le c_2 e^{-\frac{x^2}{4}}.
\end{aligned}$$

Application of Borel-Cantelli: Let $A_k = (|X_k| \ge 4(\ln(k))^{1/2})$.
According to previous step we have:

$$\mathbf{P}\left(A_k\right) \le \frac{c}{k^4} \implies \sum_{k=1}^{\infty} \mathbf{P}\left(A_k\right) < \infty \implies \mathbf{P}\left(\limsup A_k\right) = 0$$

Conclusion: ω-a.s there exists $k_0 = k_0(\omega)$ such that
$\hookrightarrow |X_k(\omega)| \le 4[\ln(k)]^{1/2}$ for $k \ge k_0$.

Concrete construction on $[0, 1]$

Proposition 11.

Let

- $\{s_k;\, k \geq 0\}$ Schauder functions family
- $\{X_k;\, k \geq 0\}$ i.i.d sequence of $\mathcal{N}(0, 1)$ random variables.

We set:

$$W_t = \sum_{k \geq 0} X_k \, s_k(t).$$

Then W is a Wiener process on $[0, 1]$

$\hookrightarrow$ In the sense of Definition 4.

Proof: uniform convergence

Step 1: Show that $\sum_{k\geq 0} X_k\, s_k(t)$ converges

$\hookrightarrow$ Uniformly in $[0,1]$, almost surely.

$\hookrightarrow$ This also implies that W is continuous a.s

Problem reduction: See that for all $\varepsilon > 0$

$\hookrightarrow$ there exists $n_0 = n_0(\omega)$ such that for all $n_0 \leq m < n$ we have:

$$\left\| \sum_{k=2^m}^{2^n-1} X_k\, s_k \right\|_\infty \leq \varepsilon.$$

Proof: uniform convergence (2)

Useful bounds:

1. Let $\eta > 0$. We have (Lemma 10):

$$|X_k| \le c\, k^{\eta}, \quad \text{with} \quad c = c(\omega)$$

2. For $2^p \le k < 2^{p+1}$, functions s_k have disjoint support. Thus

$$\left\| \sum_{k=2^p}^{2^{p+1}-1} s_k \right\|_\infty \le \frac{1}{2^{\frac{p}{2}+1}}.$$

Proof: uniform convergence (3)

Uniform convergence: for all $t \in [0,1]$ we have:

$$\begin{aligned}
\left| \sum_{k=2^m}^{2^n} X_k \, s_k(t) \right| &\le \sum_{p \ge m} \sum_{k=2^p}^{2^{p+1}-1} |X_k| \, s_k(t) \\
&\le \sum_{p \ge m} \left(\sup_{2^p \le \ell \le 2^{p+1}-1} |X_\ell| \right) \left| \sum_{k=2^p}^{2^{p+1}-1} s_k(t) \right| \\
&\le c_1 \sum_{p \ge m} \frac{1}{2^{p\left(\frac{1}{2}-\eta\right)}} \\
&\le \frac{c_2}{2^{m\left(\frac{1}{2}-\eta\right)}},
\end{aligned}$$

which shows uniform convergence.

Proof: law of δW_{rt}

Step 2: Show that $\delta W_{rt} \sim \mathcal{N}(0, t-s)$ for $0 \le r < t$.

Problem reduction: See that for all $\lambda \in \mathbb{R}$,

$$\mathbf{E}\left[e^{\imath\lambda\,\delta W_{rt}}\right] = e^{-\frac{(t-r)\lambda^2}{2}}.$$

Recall: $\delta W_{rt} = \sum_{k\ge0} X_k(s_k(t) - s_k(r))$

Computation of a characteristic function:
Invoking independence of X_k's and dominated convergence,

$$\begin{aligned}\mathbf{E}\left[e^{\imath\lambda\,\delta W_{rt}}\right] &= \prod_{k\ge0}\mathbf{E}\left[e^{\imath\lambda\,X_k(s_k(t)-s_k(r))}\right] \\ &= \prod_{k\ge0} e^{-\frac{\lambda^2(s_k(t)-s_k(r))^2}{2}} = e^{-\frac{\lambda^2}{2}\sum_{k\ge0}(s_k(t)-s_k(r))^2}\end{aligned}$$

Proof: law of δW_{rt} (2)

Inner product computation: For $0 \le r < t$ we have

$$\sum_{k\ge 0} s_k(r)\, s_k(t) = \sum_{k\ge 0} \left\langle h_k,\, \mathbf{1}_{[0,r]} \right\rangle \left\langle h_k,\, \mathbf{1}_{[0,t]} \right\rangle = \left\langle \mathbf{1}_{[0,r]},\, \mathbf{1}_{[0,t]} \right\rangle = r.$$

Thus:

$$\sum_{k\ge 0} [s_k(t) - s_k(r)]^2 = t - r.$$

Computation of a characteristic function (2): We get

$$\mathbf{E}\left[e^{\imath\lambda\, \delta W_{rt}} \right] = e^{-\frac{\lambda^2}{2} \sum_{k\ge 0} (s_k(t) - s_k(r))^2} = e^{-\frac{(t-r)\lambda^2}{2}}.$$

Proof: increment independence

Simple case: For $0 \leq r < t$, we show that $W_r \perp\!\!\!\perp \delta W_{rt}$

Computation of a characteristic function: for $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\mathbf{E}\left[e^{\imath(\lambda_1 W_r + \lambda_2\, \delta W_{rt})}\right] = \prod_{k\geq 0} \mathbf{E}\left[e^{\imath\, X_k[\lambda_1 s_k(r) + \lambda_2(s_k(t) - s_k(r))]}\right]$$

$$= e^{-\frac{1}{2}\sum_{k\geq 0}[\lambda_1 s_k(r) + \lambda_2(s_k(t) - s_k(r))]^2} = e^{-\frac{1}{2}[\lambda_1^2 r + \lambda_2^2(t-r)]}$$

Conclusion: We have, for all $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\mathbf{E}\left[e^{\imath(\lambda_1 W_r + \lambda_2\, \delta W_{rt})}\right] = \mathbf{E}\left[e^{\imath\lambda_1 W_r}\right]\, \mathbf{E}\left[e^{\imath\lambda_2\, \delta W_{rt}}\right],$$

and thus $W_r \perp\!\!\!\perp \delta W_{rt}$.

Effective construction on $[0, \infty)$

Proposition 12.

Let:

- For $k \geq 1$, a space $(\Omega_k, \mathcal{F}_k, \mathbf{P}_k)$
  $\hookrightarrow$ On which a Wiener process W^k on $[0, 1]$ is defined
- $\bar{\Omega} = \prod_{k\geq 1} \Omega_k$, $\bar{\mathcal{F}} = \otimes_{k\geq 1} \mathcal{F}_k$, $\bar{\mathbf{P}} = \otimes_{k\geq 1} \mathbf{P}_k$

We set $W_0 = 0$ and recursively:

$$W_t = W_n + W^{n+1}_{t-n}, \quad \text{if} \quad t \in [n, n+1].$$

Then W is a Wiener process on $\mathbb{R}_+$
$\hookrightarrow$ Defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$.

Partial proof

Aim: See that $\delta W_{st} \sim \mathcal{N}(0, t-s)$
$\hookrightarrow$ with $m \le s < m+1 \le n \le t < n+1$

Decomposition of δW_{st}: We have

$$\delta W_{st} = \sum_{k=1}^{n} W_1^k + W_{t-n}^{n+1} - \left(\sum_{k=1}^{m} W_1^k + W_{s-m}^{m+1} \right) = Z_1 + Z_2 + Z_3,$$

with

$$Z_1 = \sum_{k=m+2}^{n} W_1^k, \quad Z_2 = W_1^{m+1} - W_{s-m}^{m+1}, \quad Z_3 = W_{t-n}^{n+1}.$$

Law of δW_{st}: The Z_j's are independent centered Gaussian.
Thus $\delta W_{st} \sim \mathcal{N}(0, \sigma^2)$, with:

$$\sigma^2 = n - (m+1) + 1 - (s-m) + t - n = t - s.$$

Wiener process in $\mathbb{R}^n$

Definition 13.

Let

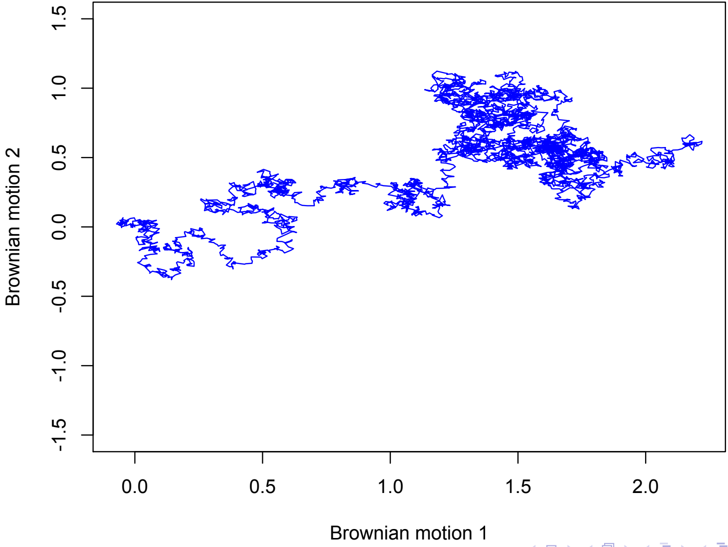
- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{W_t;\ t \geq 0\}$ $\mathbb{R}^n$-valued stochastic process

We say that W is a Wiener process if:

1. $W_0 = 0$ almost surely
2. Let $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. The increments
$$\delta W_{t_0 t_1}, \delta W_{t_1 t_2}, \ldots, \delta W_{t_{n-1} t_n} \quad \text{are independent}$$
3. For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, (t-s)\mathrm{Id}_{\mathbb{R}^n})$
4. W continuous paths, almost surely

Remark: One can construct W
$\hookrightarrow$ from n independent real valued Brownian motions.

Illustration: 2-d Brownian motion



Wiener process in a filtration

Definition 14.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ stochastic basis
- $\{W_t;\ t \geq 0\}$ stochastic process with values in $\mathbb{R}^n$

We say that W is a Wiener process
with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ if:

1. $W_0 = 0$ almost surely
2. Let $0 \leq s < t$. Then $\delta W_{st} \perp\!\!\!\perp \mathcal{F}_s$.
3. For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, (t-s)\mathrm{Id}_{\mathbb{R}^n})$
4. W has continuous paths almost surely

Remark: A Wiener process according to Definition 13 is a Wiener process in its natural filtration.

Brown

Some facts about Brown:

- Scottish, Lived 1773-1858
- Pioneer in the use of microscope
- Detailed description of cell nucleus
- Observed 2-d Brownian motion
  $\hookrightarrow$ Pollen particles in water

Outline

Gaussian property

Definition 15.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{X_t;\ t \geq 0\}$ stochastic process, with values in $\mathbb{R}$

We say that X is a Gaussian process if
for all $0 \leq t_1 < \cdots < t_n$ we have:

$$(X_{t_1}, \ldots, X_{t_n}) \quad \text{Gaussian vector.}$$

Proposition 16.

Let W be a real valued Brownian motion .
Then W is a Gaussian process.

Proof

Notation: For $0 = t_0 \leq t_1 < \cdots < t_n$ we set

- $X_n = (W_{t_1}, \ldots, W_{t_n})$
- $Y_n = (\delta W_{t_0 t_1}, \ldots, \delta W_{t_{n-1} t_n})$

Vector Y_n: Thanks to independence of increments of W
$\hookrightarrow$ Y_n is a Gaussian vector.

Vecteur X_n: There exists $M \in \mathbb{R}^{n,n}$ such that $X_n = M Y_n$
$\hookrightarrow$ X_n Gaussian vector

Covariance matrix: We have $\mathbf{E}[W_s W_t] = s \wedge t$. Thus

$$(W_{t_1}, \ldots, W_{t_n}) \sim \mathcal{N}(0, \Gamma_n), \quad \text{with} \quad \Gamma_n^{ij} = t_i \wedge t_j.$$

Consequence of Gaussian property

Characterization of a Gaussian process:
Let X Gaussian process. The law of X is characterized by:

$$\mu_t = \mathbf{E}[X_t], \quad \text{and} \quad \rho_{s,t} = \mathbf{Cov}(X_s, X_t).$$

Another characterization of Brownian motion:
Let W real-valued Gaussian process with

$$\mu_t = 0, \quad \text{and} \quad \rho_{s,t} = s \wedge t.$$

Then W is a Brownian motion.

Brownian scaling

Proposition 17.

Let

- W real-valued Brownian motion.
- A constant $a > 0$.

We define a process W^a by:

$$W_t^a = a^{-1/2}\, W_{at}, \quad \text{for} \quad t \geq 0.$$

Then W^a is a Brownian motion.

Proof:

Gaussian characterization of Brownian motion.

Canonical space

Proposition 18.

Let $E = \mathcal{C}([0,\infty);\mathbb{R}^n)$. We set:

$$d(f,g) = \sum_{k\geq 1} \frac{\|f-g\|_{\infty,k}}{2^k\left(1+\|f-g\|_{\infty,k}\right)}$$

where

$$\|f-g\|_{\infty,k} = sup\left\{|f_t - g_t|;\ t\in[0,k]\right\}.$$

Then E is a separable complete metric space.

Borel σ-algebra on E

Proposition 19.

Let $E = \mathcal{C}([0,\infty);\mathbb{R}^n)$. For $m \geq 1$ we consider:

- $0 \leq t_1 < \cdots < t_m$
- $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}^n)$

Let $\mathcal{A}$ be the σ-algebra generated by rectangles:

$$R_{t_1,\ldots,t_m}(A_1,\ldots,A_m) = \{x \in E;\, x_{t_1} \in A_1, \ldots, x_{t_m} \in A_m\}.$$

Then $\mathcal{A} = \mathcal{B}(E)$, Borel σ-algebra on E.

Wiener measure

Proposition 20.

Let

- W $\mathbb{R}^n$-valued Wiener process, defined on $(\Omega, \mathcal{F}, \mathbf{P})$.
- $T : (\Omega, \mathcal{F}) \to (E, \mathcal{A})$, such that $T(\omega) = \{W_t(\omega);\ t \geq 0\}$.

Then:

1. The application T is measurable
2. Let $\mathbf{P}_0 = \mathbf{P} \circ T^{-1}$, measure on $(E, \mathcal{A})$.
   $\mathbf{P}_0$ is called Wiener measure.
3. Under $\mathbf{P}_0$, the canonical process ω can be written as:

$$\omega_t = W_t, \quad \text{where} \quad W \text{ Brownian motion.}$$

Proof

Inverse image of rectangles: We have

$$T^{-1}\left(R_{t_1,\ldots,t_m}(A_1,\ldots,A_m)\right) = \left(W_{t_1} \in A_1,\ldots,W_{t_m} \in A_m\right) \in \mathcal{F}.$$

Conclusion: T measurable, since $\mathcal{A}$ generated by rectangles.

Einstein

Einstein and Brownian motion:

- In 1905, 4 revolutionary papers
  $\hookrightarrow$ While Einstein was employee
  at Patent Office in Zurich
- One of the 4 papers explains BM
  $\hookrightarrow$ Action of water molecules on pollen
- Lead to evidence for atoms
- First elements about
  transition kernel of Brownian motion

Outline

Martingale property

Definition 21.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ stochastic basis
- $\{X_t;\ t \geq 0\}$ stochastic process, with values in $\mathbb{R}^n$

We say that X is a $\mathcal{F}_t$-martingale if

1. $X_t \in L^1(\Omega)$ for all $t \geq 0$.
2. X is $\mathcal{F}_t$-adapted.
3. $\mathbf{E}[\delta X_{st} | \mathcal{F}_s] = 0$ for all $0 \leq s < t$.

Proposition 22.

Let W a $\mathcal{F}_t$-Brownian motion.
Then W is a $\mathcal{F}_t$-martingale.

Stopping time

Definition 23.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ stochastic basis.
- S random variable, with values in $[0, \infty]$.

We say that S is a stopping time if for all $t \geq 0$ we have:

$$(S \leq t) \in \mathcal{F}_t$$

Interpretation 1: If we know $X_{[0,t]}$
$\hookrightarrow$ One also knows if $S \leq t$ or $S > t$

Interpretation 2: $S \equiv$ instant for which one stops playing
$\hookrightarrow$ Only depends on information up to current time.

Typical examples of stopping time

Proposition 24.

Let:

- X process with values in $\mathbb{R}^d$, $\mathcal{F}_t$-adapted and continuous.
- G open set in $\mathbb{R}^d$.
- F closed set in $\mathbb{R}^d$.

We set:

$$T_G = \inf\{t \geq 0;\, X_t \in G\}, \quad T_F = \inf\{t \geq 0;\, X_t \in F\}.$$

Then:

- T_F is a stopping time.
- T_G is a stopping time when X is a Brownian motion.

Proof for T_G

First aim: prove that for $t > 0$ we have

$$(T_G < t) \in \mathcal{F}_t \qquad (1)$$

Problem reduction for (1): We show that

$$(T_G < t) = \bigcup_{s \in \mathbb{Q} \cap [0,t)} (X_s \in G) . \qquad (2)$$

Since $\cup_{s \in \mathbb{Q} \cap [0,t)} (X_s \in G) \in \mathcal{F}_t$, this proves our claim.

First inclusion for (2):

$$\bigcup_{s \in \mathbb{Q} \cap [0,t)} (X_s \in G) \subset (T_G < t) : \text{ trivial.}$$

Proof for T_G (2)

Second inclusion for (2): If $T_G < t$, then

- There exist $s < t$ such that $X_s \in G$. We set $X_s \equiv x$.
- Let $\varepsilon > 0$ such that $B(x, \varepsilon) \in G$

Then:

- There exists $\delta > 0$ such that $X_r \in B(x, \varepsilon)$ for all $r \in (s - \delta, s + \delta)$.
- In particular, there exists $q \in \mathbb{Q} \cap (s - \delta, s]$ such that $X_q \in G$.

Since

$$(\mathbb{Q} \cap (s - \delta, s]) \subset (\mathbb{Q} \cap [0, t)) \, ,$$

we have the second inclusion.

Proof for T_G (3)

Optional times: We say that $T : \Omega \to [0, \infty]$ is an optional time if

$$(T < t) \in \mathcal{F}_t.$$

Remark: Relation (1) proves that T_G is optional.

Optional times and stopping times:

- A stopping time is an optional time.
- An optional time satisfies $(T \leq t) \in \mathcal{F}_{t+} \equiv \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$.
- When X is a Brownian motion, $\mathcal{F}_{t+} = \mathcal{F}_t$ (by Markov prop.).
- When X is a Brownian motion, optional time = stopping time.

Conclusion: When X is a Brownian motion, T_G is a stopping time.

Simple properties of stopping times

Proposition 25.

Let S and T two stopping times. Then

1. $S \wedge T$
2. $S \vee T$

are stopping times.

Proposition 26.

If T is a deterministic time ($T = n$ almost surely)
$\hookrightarrow$ then T is a stopping time.

Information at time S

Definition 27.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ stochastic basis.
- S stopping time.

The σ-algebra $\mathcal{F}_S$ is defined by:

$$\mathcal{F}_S = \{A \in \mathcal{F};\ [A \cap (S \leq t)] \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Interpretation:

$\mathcal{F}_S \equiv$ Information up to time S.

Optional sampling theorem

Theorem 28.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ stochastic basis.
- S, T two stopping times, with $S \leq T$.
- X continuous martingale.

Hypothesis:

- $\{X_{t\wedge T};\ t \geq 0\}$ uniformly integrable martingale.

Then:

$$\mathbf{E}\left[X_T \,|\, \mathcal{F}_S\right] = X_S.$$

In particular:

$$\mathbf{E}\left[X_T\right] = \mathbf{E}\left[X_S\right] = \mathbf{E}\left[X_0\right].$$

Remarks

Strategy of proof:

- One starts from known discrete time result.
- X is approximated by a discrete time martingale

$$Y_m \equiv X_{t_m}, \quad \text{with} \quad t_m = \frac{m}{2^n}.$$

Checking the assumption: Set $Y_t = X_{t \wedge T}$.
$\{Y_t;\, t \geq 0\}$ uniformly integrable martingale in following cases:

- $|Y_t| \leq M$ with M deterministic constant independent of t.
- $\sup_{t\geq 0} \mathbf{E}[|Y_t|^2] \leq M$.
- $\sup_{t\geq 0} \mathbf{E}[|Y_t|^p] \leq M$ with $p > 1$.

Example of stopping time computation

Proposition 29.

Let:

- B standard Brownian motion, with $B_0 = 0$.
- $-a < 0 < b$
- $T_a = \inf\{t \geq 0 : \, B_t = -a\}$ and
  $T_b = \inf\{t \geq 0 : \, B_t = b\}$.
- $T = T_a \wedge T_b$.

Then:

$$\mathbf{P}\left(T_a < T_b\right) = \frac{b}{b+a}, \quad \text{and} \quad \mathbf{E}\left[T\right] = ab.$$

Proof

Optional sampling for $M_t = B_t$: yields

$$\mathbf{P}\left(T_a < T_b\right) = \frac{b}{b+a}.$$

Optional sampling for $M_t = B_t^2 - t$: yields, for a constant $\tau > 0$,

$$\mathbf{E}[B_{T\wedge\tau}^2] = \mathbf{E}\left[T \wedge \tau\right].$$

Limiting procedure: by dominated and monotone convergence,

$$\mathbf{E}[B_T^2] = \mathbf{E}\left[T\right].$$

Conclusion: we get

$$\mathbf{E}\left[T\right] = \mathbf{E}[B_T^2] = a^2\mathbf{P}\left(T_a < T_b\right) + b^2\mathbf{P}\left(T_b < T_a\right) = ab.$$

Bachelier

Bachelier and Brownian motion:

- Born 1870, son of a wine merchant
  $\hookrightarrow$ Used to business
- In 1900 (predates Einstein) PhD thesis
  $\hookrightarrow$ Theory of speculation
- Mathematical description of BM
  $\hookrightarrow$ Application to stock pricing
- Not well perceived by academic establishment

Outline

Wiener measure indexed by $\mathbb{R}^d$

Proposition 30.

Let:

- $x \in \mathbb{R}^d$.

There exists a probability measure $\mathbf{P}_x$ on $(E, \mathcal{A})$ such that:
$\hookrightarrow$ Under $\mathbf{P}_x$ the canonical process ω can be written as:

$$\omega_t = x + W_t, \quad \text{where} \quad W \text{ Brownian motion.}$$

Notations:

- We consider $\{\mathbf{P}_x;\, x \in \mathbb{R}^d\}$.
- Expected value under $\mathbf{P}_x$: $\mathbf{E}_x$.

Shift on paths

Definition 31.

Let

- $E = \mathcal{C}([0,\infty);\mathbb{R}^d)$, equipped with Borel $\mathcal{A} \equiv \sigma$-algebra.
- $t \geq 0$.

We set:

$$\theta_t : E \to E, \quad \{\omega_s;\, s \geq 0\} \mapsto \{\omega_{t+s};\, s \geq 0\}$$

Shift and future: Let $Y : E \to \mathbb{R}$ measurable.
$\hookrightarrow$ Then $Y \circ \theta_s$ depends on future after s.

Example: For $n \geq 1$, f measurable and $0 \leq t_1 < \cdots < t_n$,

$$Y(\omega) = f(\omega_{t_1}, \ldots, \omega_{t_n}) \quad \Longrightarrow \quad Y \circ \theta_s = f\left(W_{s+t_1}, \ldots, W_{s+t_n}\right).$$

Markov property

Theorem 32.

Let:

- W Wiener process, with values in $\mathbb{R}^d$.
- $Y : E \to \mathbb{R}$ bounded measurable.
- $s \geq 0$.

Then:

$$\mathbf{E}_x \left[Y \circ \theta_s \middle| \mathcal{F}_s \right] = \mathbf{E}_{W_s}[Y].$$

Interpretation:

Future after s can be predicted with value of W_s only.

Pseudo-proof

Very simple function: Consider $Y \equiv f(W_t)$, let $Y \circ \theta_s = f(W_{s+t})$.
For $0 \le s < t$, independence of increments for W gives

$$\begin{aligned} \mathbf{E}_x \left[Y \circ \theta_s \middle| \mathcal{F}_s \right] &= \mathbf{E}_x \left[f(W_{s+t}) \middle| \mathcal{F}_s \right] = p_t f(W_s) \\ &= \mathbf{E}_{W_s} \left[f(W_t) \right] = \mathbf{E}_{W_s} \left[Y \right], \end{aligned}$$

with

$$p_h : \mathcal{C}(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d), \quad p_h f(x) \equiv \int_{\mathbb{R}^d} f(y) \, \frac{\exp\left(-\frac{|y-x|^2}{2h}\right)}{(2\pi h)^{d/2}} \, dy.$$

Extension:

1. Random variable $Y = f(W_{t_1}, \ldots, W_{t_n})$.
2. General random variable: by π-λ-systems.

π-systems and λ-systems

π-system: Let $\mathcal{P}$ family of sets in Ω. $\mathcal{P}$ is a π-system if:

$$A, B \in \mathcal{P} \quad \Longrightarrow \quad A \cap B \in \mathcal{P}$$

λ-system: Let $\mathcal{L}$ family of sets in Ω. $\mathcal{L}$ is a λ-system if:

1. $\Omega \in \mathcal{L}$
2. If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$
3. If for $j \geq 1$ we have:
    - $A_j \in \mathcal{L}$
    - $A_j \cap A_i = \emptyset$ if $j \neq i$

   Then $\cup_{j \geq 1} A_j \in \mathcal{L}$

Dynkin's π-λ lemma

Lemma 33.

Let $\mathcal{P}$ and $\mathcal{L}$ such that:

- $\mathcal{P}$ is a π-system
- $\mathcal{L}$ is a λ-system
- $\mathcal{P} \subset \mathcal{L}$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$

Links with analysis

Heat semi-group: We have set $p_t f(x) = \mathbf{E}_x[f(W_t)]$. Then:

- The family $\{p_t;\ t \geq 0\}$ is a semi-group of operators.
- Generator of the semi-group: $\frac{\Delta}{2}$, with $\Delta \equiv$ Laplace operator.

Feynman-Kac formula: Let $f \in \mathcal{C}_b(\mathbb{R}^d)$ and PDE on $\mathbb{R}^d$:

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x), \qquad u(0,x) = f(x).$$

Then

$$u(t,x) = \mathbf{E}_x[f(W_t)] = p_t f(x)$$

Strong Markov property

Theorem 34.

Let:

- W Wiener process in $\mathbb{R}^d$.
- $Y : \mathbb{R}_+ \times E \to \mathbb{R}$ bounded measurable.
- S stopping time.

Then:

$$\mathbf{E}_x \left[Y_S \circ \theta_S \middle| \mathcal{F}_S \right] \mathbf{1}_{(S<\infty)} = \mathbf{E}_{W_S}[Y_S] \, \mathbf{1}_{(S<\infty)}.$$

Particular case: If S finite stopping time a.s. we have

$$\mathbf{E}_x \left[Y_S \circ \theta_S \middle| \mathcal{F}_S \right] = \mathbf{E}_{W_S}[Y_S]$$

Reflection principle

Theorem 35.

Let:

- W real-valued Brownian motion.
- $a > 0$.
- $T_a = \inf\{t \geq 0;\ W_t = a\}$.

Then:

$$\mathbf{P}_0\left(T_a < t\right) = 2\,\mathbf{P}_0\left(W_t > a\right).$$

Intuitive proof

Independence: If W reaches a for $s < t$
$\hookrightarrow W_t - W_{T_a} \perp\!\!\!\perp \mathcal{F}_{T_a}$.

Consequence:

$$\mathbf{P}_0\left(T_a < t,\ W_t > a\right) = \frac{1}{2}\,\mathbf{P}_0\left(T_a < t\right)$$

Furthermore:

$$\left(W_t > a\right) \subset \left(T_a < t\right) \quad \Longrightarrow \mathbf{P}_0\left(T_a < t,\ W_t > a\right) = \mathbf{P}_0\left(W_t > a\right).$$

Thus:

$$\mathbf{P}_0\left(T_a < t\right) = 2\,\mathbf{P}_0\left(W_t > a\right).$$

Rigorous proof

Reduction: We have to show

$$\mathbf{P}_0\left(T_a < t,\ W_t > a\right) = \frac{1}{2}\,\mathbf{P}_0\left(T_a < t\right)$$

Functional: We set (with $\inf \varnothing = \infty$)

$$S = \inf\left\{s < t;\ W_s = a\right\}, \qquad Y_s(\omega) = \mathbf{1}_{(s<t,\,\omega(t-s)>a)}.$$

Then:

1. $(S < \infty) = (T_a < t)$.
2. $Y_S \circ \theta_S = \mathbf{1}_{(S<t)}\,\mathbf{1}_{W_t>a} = \mathbf{1}_{(T_a<t)}\,\mathbf{1}_{W_t>a}$.

Rigorous proof (2)

Application of strong Markov:

$$\mathbf{E}_0 \left[Y_S \circ \theta_S \middle| \mathcal{F}_S \right] \mathbf{1}_{(S<\infty)} = \mathbf{E}_{W_S}[Y_S] \, \mathbf{1}_{(S<\infty)} = \varphi(W_S, S), \qquad (3)$$

with

$$\varphi(x, s) = \mathbf{E}_x \left[\mathbf{1}_{W_{t-s}>a} \right] \mathbf{1}_{(s<t)}.$$

Conclusion: Since $W_S = a$ if $S < \infty$ and $\mathbf{E}_a[\mathbf{1}_{W_{t-s}>a}] = \frac{1}{2}$,

$$\varphi(W_S, S) = \frac{1}{2} \mathbf{1}_{(S<t)}.$$

Taking expectations in (3) we end up with:

$$\mathbf{P}_0 \left(T_a < t, \, W_t > a \right) = \frac{1}{2} \mathbf{P}_0 \left(T_a < t \right).$$

Outline

Hölder continuity

Definition 36.

Let

$$f : [a, b] \longrightarrow \mathbb{R}^n.$$

We say that f is γ-Hölder if $\|f\|_\gamma < \infty$ with:

$$\|f\|_\gamma = \sup_{s,t\in[a,b],\, s\neq t} \frac{|\delta f_{st}|}{|t-s|^\gamma}.$$

Remark:

1. $\|\cdot\|_\gamma$ is a semi-norm.
2. Notation: $\mathcal{C}^\gamma$

Regularity of Brownian motion

Theorem 37.

Let

- $\tau > 0$
- W Wiener process on $[0, \tau]$
- $\gamma \in (0, 1/2)$

Then there exists a version $\hat{W}$ of W such that almost surely:
$\hookrightarrow$ The paths of $\hat{W}$ are γ-Hölder.

Remark: $\hat{W}$ and W are usually denoted in the same way.

Kolmogorov's criterion

Theorem 38.

Let $X = \{X_t;\ t \in [0,\tau]\}$ process defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that:

$$\mathbf{E}\left[|\delta X_{st}|^{\alpha}\right] \le c|t-s|^{1+\beta}, \quad \text{for} \quad s,t \in [0,\tau],\ c,\alpha,\beta > 0$$

Then there exists a modification $\hat{X}$ of X satisfying
$\hookrightarrow$ Almost surely $\hat{X} \in \mathcal{C}^{\gamma}$ for all $\gamma < \beta/\alpha$, i.e:

$$\mathbf{P}\left(\omega;\ \|\hat{X}(\omega)\|_{\gamma} < \infty\right) = 1.$$

Proof of Theorem 37

Law of δB_{st}: We have $\delta B_{st} \sim \mathcal{N}(0, t-s)$.

Moments of δB_{st}:
Using expression for the moments of $\mathcal{N}(0,1)$
$\hookrightarrow$ for $m \geq 1$, we have

$$\mathbf{E}\left[|\delta B_{st}|^{2m}\right] = c_m |t-s|^m \quad \text{i.e} \quad \mathbf{E}\left[|\delta B_{st}|^{2m}\right] = c_m |t-s|^{1+(m-1)}$$

Application of Kolmogorov's criterion:
B is γ-Hölder for $\gamma < \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}$
Taking limits $m \to \infty$, the proof is finished.

Levy's modulus of continuity

Theorem 39.

Let $\tau > 0$ and W Wiener process on $[0, \tau]$.
Then almost surely W satisfies:

$$\limsup_{\delta \to 0^+} \sup_{0 \le s < t \le \tau, |t-s| \le \delta} \frac{|\delta W_{st}|}{(2\delta \ln(1/\delta))^{1/2}} = 1.$$

Interpretation: W has Hölder-regularity $= \frac{1}{2}$ at each point
$\hookrightarrow$ up to a logarithmic factor.

Variations of a function

Interval partitions: Let $a < b$ two real numbers.

- We denote by π a set $\{t_0, \ldots, t_m\}$ with $a = t_0 < \ldots < t_m = b$
  We say that π is a partition of $[a, b]$.
- Write $\Pi_{a,b}$ for the set of partitions of $[a, b]$.

Definition 40.

Let $a < b$ and $f : [a, b] \to \mathbb{R}$. The variation of f on $[a, b]$ is:

$$V_{a,b}(f) = \lim_{\pi \in \Pi_{a,b}, |\pi| \to 0} \sum_{t_i, t_{i+1} \in \pi} \left| \delta f_{t_i t_{i+1}} \right| .$$

If $V_{a,b}(f) < \infty$, we say that f has finite variation on $[a, b]$.

Quadratic variation

Definition 41.

Let $a < b$ and $f : [a, b] \to \mathbb{R}$.
The quadratic variation of f on $[a, b]$ is:

$$V^2_{a,b}(f) = \lim_{\pi \in \Pi_{a,b},\, |\pi| \to 0} \sum_{t_i, t_{i+1} \in \pi} \left|\delta f_{t_i t_{i+1}}\right|^2 .$$

If $V^2_{a,b}(f) < \infty$,
$\hookrightarrow$ We say that f has a finite quadratic variation on $[a, b]$.

Remark: The limits for $V_{a,b}(f)$ and $V^2_{a,b}(f)$
$\hookrightarrow$ do not depend on the sequence of partitions

Variations of Brownian motion

Theorem 42.

Let W Wiener process.
Then almost surely W satisfies:

1. For $0 \leq a < b < \infty$ we have $V^2_{a,b}(W) = b - a$.
2. For $0 \leq a < b < \infty$ we have $V_{a,b}(W) = \infty$.

Interpretation: The paths of W have:

- Infinite variation
- Finite quadractic variation,

on any interval of $\mathbb{R}_+$.

Proof

Notations: Let $\pi = \{t_0, \ldots, t_m\} \in \Pi_{a,b}$. We set:

- $S_\pi = \sum_{k=0}^{m-1} |\delta W_{t_k t_{k+1}}|^2$.
- $X_k = |\delta W_{t_k t_{k+1}}|^2 - (t_{k+1} - t_k)$.
- $Y_k = \frac{X_k}{t_{k+1} - t_k}$.

Step 1: Show that

$$L^2(\Omega) - \lim_{|\pi| \to 0} S_\pi = b - a.$$

Decomposition: We have

$$S_\pi - (b - a) = \sum_{k=0}^{m-1} X_k$$

Proof (2)

Variance computation: The r.v X_k are centered and indep. Thus

$$\begin{aligned}\mathbf{E}\left[(S_\pi - (b-a))^2\right] &= \mathbf{Var}\left(\sum_{k=0}^{m-1} X_k\right) \\ &= \sum_{k=0}^{m-1} \mathbf{Var}(X_k) = \sum_{k=0}^{m-1} (t_{k+1}-t_k)^2\, \mathbf{Var}(Y_k)\end{aligned}$$

Since $\frac{\delta W_{t_k t_{k+1}}}{(t_{k+1}-t_k)^{1/2}} \sim \mathcal{N}(0,1)$, we get:

$$\mathbf{E}\left[(S_\pi - (b-a))^2\right] = 2\sum_{k=0}^{m-1}(t_{k+1}-t_k)^2 \leq 2|\pi|(b-a).$$

Conclusion: We have, for a subsequence π_n,

$$L^2(\Omega) - \lim_{|\pi|\to 0} S_\pi = b-a \quad \Longrightarrow \quad \text{a.s} - \lim_{n\to\infty} S_{\pi_n} = b-a.$$

Proof (3)

Step 2: Verify $V_{a,b}(W) = \infty$ for fixed $a < b$.
We proceed by contradiction. Let:

- $\omega \in \Omega_0$ such that $\mathbf{P}(\Omega_0) = 1$ and $\lim_{n\to\infty} S_{\pi_n}(\omega) = b - a > 0$.
- Assume $V_{a,b}(W(\omega)) < \infty$.

Bound on increments: Thanks to continuity of W:

1. $\sup_{n\geq 1} \sum_{k=0}^{m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| \leq c(\omega)$
2. $\lim_{n\to\infty} \max_{0\leq k\leq m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| = 0$

Thus

$$S_{\pi_n}(\omega) \leq \max_{0\leq k\leq m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| \sum_{k=0}^{m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| \longrightarrow 0.$$

Contradiction: with $\lim_{n\to\infty} S_{\pi_n}(\omega) = b - a > 0$.

Proof (4)

Step 3: Verify $V_{a,b}(W) = \infty$ for all couple $(a, b) \in \mathbb{R}_+^2$
$\hookrightarrow$ It is enough to check $V_{a,b}(W) = \infty$ for all couple $(a, b) \in \mathbb{Q}_+^2$

Recall: For all couple $(a, b) \in \mathbb{Q}_+^2$, we have found:
$\hookrightarrow$ $\exists\ \Omega_{a,b}$ s.t $\mathbf{P}(\Omega_{a,b}) = 1$ and $V_{a,b}(W(\omega)) = \infty$ for all $\omega \in \Omega_{a,b}$.

Full probability set: Let

$$\Omega_0 = \bigcap_{(a,b)\in\mathbb{Q}_+^2} \Omega_{a,b}.$$

Then:

- $\mathbf{P}(\Omega_0) = 1$.
- If $\omega \in \Omega_0$, for all couple $(a, b) \in \mathbb{Q}_+^2$ we have $V_{a,b}(W(\omega)) = \infty$.

Irregularity of W

Proposition 43.

Let:

- W Wiener process
- $\gamma > 1/2$ and $0 \le a < b$

Then almost surely W does not belong to $\mathcal{C}^\gamma([a, b])$.

Proof

Strategy: Proceed by contradiction. Let:

- $\omega \in \Omega_0$ such that $\mathbf{P}(\Omega_0) = 1$ and $\lim_{n\to\infty} S_{\pi_n}(\omega) = b - a > 0$.
- Suppose $W \in \mathcal{C}^\gamma$ with $\gamma > 1/2$, i.e $|\delta W_{st}| \leq L|t-s|^\gamma$
  $\hookrightarrow$ With L random variable.

Bound on quadratic variation: We have:

$$S_{\pi_n}(\omega) \leq L^2 \sum_{k=0}^{m_n-1} |t_{k+1} - t_k|^{2\gamma} \leq L^2 |\pi_n|^{2\gamma-1}(b-a) \longrightarrow 0.$$

Contradiction: with $\lim_{n\to\infty} S_{\pi_n}(\omega) = b - a > 0$.

Irregularity of W at each point

Theorem 44.

Let:

- W Wiener process
- $\gamma > 1/2$ and $\tau > 0$

Then

1. Almost surely the paths of W are not γ-Hölder continuous at each point $s \in [0, \tau]$.
2. In particular, W is nowhere differentiable.