Ergodic Theorems

Samy Tindel

Purdue University

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Outline

Overview

Stationary sequence: such that

$$\{X_{n+k};\, n \geq 0\} \overset{(d)}{=} \{X_n;\, n \geq 0\}.$$

Main result: law of large numbers of the type

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k) \quad \text{exists a.s}$$

Outline

Outline

1. Definitions and examples
   - Preliminaries on Markov chains
   - Examples of stationary sequences
   - Notion of ergodicity
2. Ergodic theorem
3. Recurrence

Markov chain

Definition 1.

Let $X = \{X_n;\ n \geq 0\}$ be a process.

1. X is a Markov chain if
$$\mathbf{P}\left(X_{n+1} = j \,|\, X_0 = i_0, \ldots, X_n = i_n\right) = \mathbf{P}\left(X_{n+1} = j \,|\, X_n = i_n\right)$$
for all $n \geq 0$, $i_0, \ldots, i_n, j \in E$.
2. The Markov chain is homogeneous whenever
$$\mathbf{P}\left(X_{n+1} = j \,|\, X_n = i\right) = \mathbf{P}\left(X_1 = j \,|\, X_0 = i\right)$$

Law of the Markov chain

Proposition 2.

Let X be a homogeneous Markov chain
$\hookrightarrow$ with initial law ν and transition p.

1. For $n \in \mathbb{N}$ and $i_0, \ldots, i_n \in E$, we have

$$\mathbf{P}\left(X_0 = i_0, \ldots, X_n = i_n\right) = \nu(i_0)\, p(i_0, i_1) \cdots p(i_{n-1}, i_n)$$

2. The law of X is characterized by ν and p.

Law of the Markov chain: general state space

Proposition 3.

Let X be a homogeneous Markov chain on $(S, \mathcal{S})$
$\hookrightarrow$ with initial law ν and transition p.

1. For $n \in \mathbb{N}$ and $\varphi \in \mathcal{C}_b(S)$, we have

$$\mathbf{E}\left[\varphi(X_0, \ldots, X_n)\right] = \int_{E^{n+1}} \varphi(x_0, \ldots, x_n)\nu(dx_0)\, p(x_0, x_1) \cdots p(x_{n-1}, x_n) \quad (1)$$

2. The law of X is characterized by ν and p.

A criterion for Markovianity

Proposition 4.

Let

- $Z : \Omega \to E$ random variable.
- F countable set.
- $\{Y_n;\, n \geq 1\}$ i.i.d sequence, with $Y \perp\!\!\!\perp Z$ and $Y_n \in F$.
- $f : E \times F \to E$.

We set

$$X_0 = Z, \quad \text{and} \quad X_{n+1} = f(X_n, Y_{n+1}).$$

Then X is a homogeneous Markov chain such that

$$\nu_0 = \mathcal{L}(Z), \quad \text{and} \quad p(i,j) = \mathbf{P}\left(f(i, Y_1) = j\right).$$

Remark: A converse result exists, but it won't be used.

Graph of a Markov chain

Definition 5.

Let X homogeneous Markov chain (initial law ν_0, transition p). We define a graph $\mathcal{G}(X)$ by

- $\mathcal{G}(X)$ is an oriented graph
- Vertices of $\mathcal{G}(X)$ are points of E.
- Edges of $\mathcal{G}(X)$ are defined by

$$\mathbb{V} \equiv \{(i,j);\ i \neq j,\ p(i,j) > 0\}.$$

Example

Example 6.

We consider $E = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Related graph: to be done in class

Graph and accessibility

Proposition 7.

Let X homogeneous Markov chain (initial law ν_0, transition p). Then

$$i \to j$$

iff

$$i = j \text{ or there exists an oriented path from } i \text{ to } j \text{ in } \mathcal{G}(X).$$

Minimal class

Definition 8.

An equivalence class C is minimal if:

$$\text{For all } i \in C \text{ and } j \notin C, \text{ we have } i \not\to j.$$

Example:
For Example 6, C_1, C_3 are minimal, and C_2 is not minimal.

Minimality criterions:
(i) If there exists a unique class C, it is minimal.
(ii) There exists a unique minimal class C
$\Leftrightarrow \exists!$ class C such that for all $i \in E$, we have $i \to C$.

Application: Random walk

Recurrence criterions

Theorem 9.

Let

- X homogeneous Markov chain (initial law ν_0, transition p).
- C class for the relation $\leftrightarrow$.

Then

1. If C is not minimal then it is transient
2. If C is minimal and $|E| < \infty$, then
   - C is recurrent
   - C is positive recurrent : for all $i \in C$, $\mathbf{E}_i[R_i] < \infty$.

Example

Recall: In Example 6 we had $E = \{1, 2, 3, 4, 5\}$ (hence $|E| < \infty$) and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Related classes:
$C_1 = \{1, 3\}$, $C_2 = \{2\}$ and $C_3 = \{4, 5\}$.
$\hookrightarrow$ C_1, C_3 minimal and C_2 not minimal.

Conclusion: C_1, C_3 positive recurrent, C_2 transient.

Outline

Stationary sequence

Definition 10.

Let

- $\{X_n;\, n \geq 0\}$ sequence of random variables

We say that X is stationary if for all $k \geq 1$ we have

$$\{X_{n+k};\, n \geq 0\} \stackrel{(d)}{=} \{X_n;\, n \geq 0\}.$$

iid random variables

Example 11.

Let

- $X = \{X_n;\ n \geq 0\}$ sequence of i.i.d random variables

Then X is a stationary sequence.

Markov chains

Example 12.

Let

- $X = \{X_n;\ n \geq 0\}$ Markov chain on state space S
- Transition probability: $p(x, A)$
- Hypothesis 1: unique stationary distribution π
- Hypothesis 2: $\mathcal{L}(X_0) = \pi$

Then X is a stationary sequence.

Proof:

Easy consequence of relation (1).

Trivial example of Markov chain

Example 13.

On $S = \{0, 1\}$ we take

- $p(x, 1 - x) = 1$
- $\pi(0) = \frac{1}{2}$, $\pi(1) = \frac{1}{2}$

Then

$$\mathbf{P}(X = (0, 1, 0, 1, \ldots)) = \mathbf{P}(X = (1, 0, 1, 0, \ldots)) = \frac{1}{2}$$

Rotation of the circle

Example 14.

We consider $\lambda \equiv$ Lebesgue measure and:

- $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1], \text{ Borel sets}, \lambda)$
- $\theta \in (0,1)$
- $X = \{X_n;\ n \geq 0\}$ with

$$X_n(\omega) = (\omega + n\theta) \mod 1 = \omega + n\theta - [\omega + n\theta].$$

Then X is a stationary sequence.

Proof

Law of X_1: For $\varphi \in \mathcal{C}_b(\mathbb{R})$ we have

$$\begin{aligned}
\mathbf{E}\left[\varphi(X_1)\right] &= \int_0^1 \varphi\left(x+\theta-[x+\theta]\right)\,dx \\
&= \int_0^{1-\theta} \varphi\left(x+\theta\right)\,dx + \int_{1-\theta}^1 \varphi\left(x+\theta-1\right)\,dx \\
&= \int_0^1 \varphi\left(v\right)\,dv = \mathbf{E}\left[\varphi(X_0)\right]
\end{aligned}$$

Thus $\mathcal{L}(X_1) = \mathcal{L}(X_0)$.

Proof (2)

Altenative expression for X_n: Due to the relation

$$(a+b) \mod 1 = [a \mod 1 + b \mod 1] \mod 1,$$

we have

$$\begin{aligned} X_n &= (\omega + n\theta) \mod 1 \\ &= (\omega + (n-1)\theta + \theta) \mod 1 \\ &= [(\omega + (n-1)\theta) \mod 1 + \theta \mod 1] \mod 1 \\ &= [X_{n-1} + \theta] \mod 1 \end{aligned}$$

Proof (3)

Recall: We have seen that

$$X_n = [X_{n-1} + \theta] \mod 1. \tag{2}$$

Conclusion: It is readily checked that

1. From (2) and Proposition 4, X is a Markov chain
2. The relation $\mathcal{L}(X_1) = \mathcal{L}(X_0)$ means that λ is an invariant distribution

Therefore X is stationary thanks to Example 12.

Transformation of a stationary sequence

Theorem 15.

Let

- $X = \{X_n;\ n \geq 0\}$ stationary sequence
- $g : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ measurable

We consider a sequence Y defined by

$$Y_k = g\left(\{X_{k+n};\ n \geq 0\}\right).$$

Then Y is a stationary sequence.

σ-algebra on $\mathbb{R}^{\mathbb{N}}$ (1)

Set $\mathbb{R}^{\mathbb{N}}$: We define

$$\mathbb{R}^{\mathbb{N}} = \{\omega = (\omega_j)_{j\in\mathbb{N}};\, \omega_j \in \mathbb{R}\}\,.$$

Finite dimensional set: Of the form

$$A = \left\{\omega \in \mathbb{R}^{\mathbb{N}};\, \omega_j \in B_j,\ \text{for } 1 \le j \le n\right\}, \quad \text{with} \quad B_j \in \mathcal{R},$$

where $\mathcal{R} \equiv$ Borel σ-algebra in $\mathbb{R}$.

σ-algebra on $\mathbb{R}^{\mathbb{N}}$ (2)

Definition 16.

On $\mathbb{R}^{\mathbb{N}}$ we set:

$$\mathcal{R}^{\mathbb{N}} = \sigma\,(\text{finite dimensional sets})\,.$$

Then $\mathcal{R}^{\mathbb{N}}$ is called Borel σ-algebra on $\mathbb{R}^{\mathbb{N}}$.

Remark:

Kolmogorov's extension theorem is valid on $(\mathbb{R}^{\mathbb{N}}, \mathcal{R}^{\mathbb{N}})$.

Proof of Theorem 15

Notation: For $x \in \mathbb{R}^{\mathbb{N}}$ and $k \geq 0$ we set

$$g_k(x) = g\left(\{x_{k+n};\, n \geq 0\}\right).$$

Inverse image: Let $B \in \mathcal{R}^{\mathbb{N}}$. Then

$$\left\{\omega \in \mathbb{R}^{\mathbb{N}};\ Y \in B\right\} = \left\{\omega \in \mathbb{R}^{\mathbb{N}};\ X \in A\right\},$$

where

$$A = \left\{\omega \in \mathbb{R}^{\mathbb{N}}; g(X) \in B\right\}.$$

Proof of Theorem 15 (2)

Proof of stationarity: For the generic $B \in \mathcal{R}^{\mathbb{N}}$ we have

$$\begin{aligned}
\mathbf{P}\left(\omega \in \mathbb{R}^{\mathbb{N}};\, Y \in B\right) &= \mathbf{P}\left(\omega \in \mathbb{R}^{\mathbb{N}};\, X \in A\right) \\
&= \mathbf{P}\left(\omega \in \mathbb{R}^{\mathbb{N}};\, \{X_n;\, n \geq 0\} \in A\right) \\
&= \mathbf{P}\left(\omega \in \mathbb{R}^{\mathbb{N}};\, \{X_{k+n};\, n \geq 0\} \in A\right) \\
&= \mathbf{P}\left(\omega \in \mathbb{R}^{\mathbb{N}};\, \{Y_{k+n};\, n \geq 0\} \in B\right).
\end{aligned}$$

This yields stationarity for Y.

Bernoulli shifts

Example 17.

We consider $\lambda \equiv$ Lebesgue measure and:

- $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1],\ \text{Borel sets}, \lambda)$
- $Y_0 = \omega$
- $Y = \{Y_n;\ n \geq 0\}$ where for $n \geq 1$ we have

$$Y_n(\omega) = 2Y_{n-1} \mod 1.$$

Then Y is a stationary sequence.

Proof by Markov chains

Law of Y_1: For $\varphi \in \mathcal{C}_b(\mathbb{R})$ we have

$$\begin{aligned}
\mathbf{E}\left[\varphi(Y_1)\right] &= \int_0^1 \varphi\left(2y \mod 1\right) \, dy \\
&= \int_0^{\frac{1}{2}} \varphi\left(2y\right) \, dy + \int_{\frac{1}{2}}^1 \varphi\left(2y-1\right) \, dy \\
&= \int_0^1 \varphi\left(v\right) \, dv = \mathbf{E}\left[\varphi(Y_0)\right]
\end{aligned}$$

Thus $\mathcal{L}(Y_1) = \mathcal{L}(Y_0)$.

Proof by Markov chains (2)

Proof of stationarity: We have

1. $Y_n(\omega) = 2Y_{n-1} \mod 1$
2. Thanks to proposition 4, Y is thus a Markov chain
3. The relation $\mathcal{L}(Y_1) = \mathcal{L}(Y_0)$ means that λ is an invariant distribution for Y

Therefore Y is a stationary sequence thanks to Example 12.

Proof by Theorem 15

Another representation for Y: Let

- $\{X_i;\, i \geq 1\}$ i.i.d with common law $\mathcal{B}(1/2)$
- $g(x) = \sum_{i\geq 1} x_i 2^{-(i+1)}$ defined for $x \in \{0,1\}^{\mathbb{N}}$
- $g_k(x) = g(\{x_{k+i};\, i \geq 1\})$ defined for $x \in \{0,1\}^{\mathbb{N}}$
- $Y_k = g_k(X)$

Then

$$Y_0 \sim \lambda, \quad \text{and} \quad Y_n(\omega) = 2Y_{n-1} \mod 1.$$

Stationarity of Y: We have

- X stationary
- $Y_k = g_k(X)$

Therefore Y is a stationary sequence.

Measure preserving map

Definition 18.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space
- A measurable map $\varphi : \Omega \to \Omega$

We say that φ is measure preserving if

$$\mathbf{P}\left(\varphi^{-1}(A)\right) = \mathbf{P}(A), \quad \text{for all} \quad A \in \mathcal{F}.$$

Otherwise stated, φ is measure preserving if

$$\mathbf{E}\left[g\Big(X(\varphi(\omega))\Big)\right] = \mathbf{E}\left[g(X(\omega))\right], \quad \text{for all} \quad g \in \mathcal{C}_b.$$

Measure preserving map and stationarity

Example 19.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space
- A measure preserving map $\varphi : \Omega \to \Omega$
- $X \in \mathcal{F}$ a random variable

For $n \geq 0$ we set:

$$X_n = X\left(\varphi^n(\omega)\right)$$

Then X is a stationary sequence.

Proof

Characterization with expected values:
For $g \in \mathcal{C}(\mathbb{R}^n)$ and $k \geq 1$ we have:

$$\begin{aligned}
\mathbf{E}\left[g\left(X_k, \ldots, X_{k+n}\right)\right] &= \mathbf{E}\left[g\left(X_0(\varphi^k(\omega)), \ldots, X_n(\varphi^k(\omega))\right)\right] \\
&= \mathbf{E}\left[g\left(X_0(\omega), \ldots, X_n(\omega)\right)\right] \\
&= \mathbf{E}\left[g\left(X_0, \ldots, X_n\right)\right].
\end{aligned}$$

Thus X is a stationary sequence.

Stationarity and measure preserving maps

Example 20.

Let

- Y stationary sequence with values in $S = \mathbb{R}^n$
- $\mathbf{P}$ probability measure on $(S^{\mathbb{N}}, \mathcal{R}(S^{\mathbb{N}}))$ defined by:

$$X_n(\omega) = \omega_n \quad \Longrightarrow \quad \mathcal{L}(X) = \mathcal{L}(Y)$$

We define a shift operator φ by:

$$\varphi\left(\{\omega_j; j \geq 0\}\right) = \{\omega_{j+1}; j \geq 0\}.$$

Then φ is measure preserving.

Interpretation:

Previous examples can be reduced to a measure preserving map

Two sided stationary sequence

Theorem 21.

Let

- X stationary sequence

Then X can be embedded into a two sided sequence

$$\{Y_n;\, n \in \mathbb{Z}\}.$$

Proof

Application of Kolmogorov's extension:
It is enough to define a family of probability measures

1. On $\mathbb{R}^A$ for any finite subset of $\mathbb{Z}$
2. With consistent property

Current situation: we consider Y defined by

$$\mathbf{E}\left[g\left(Y_{-m}, \ldots, Y_n\right)\right] = \mathbf{E}\left[g\left(X_0, \ldots, X_{m+n}\right)\right],$$

for all $g \in \mathcal{C}_b(\mathbb{R}^{m+n+1})$. Kolmogorov's extension applies

Outline

Setup

General setting: We are reduced to

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- A map φ preserving $\mathbf{P}$
- A random variable X
- A sequence $X_n(\omega) = X(\omega)$

Invariant set: Let $A \in \mathcal{F}$. The set A is invariant if

$$\mathbf{P}\left(A \,\Delta\, \varphi^{-1}(A)\right) = 0$$

Notation abuse: For an invariant set A, we often write

$$A = \varphi^{-1}(A)$$

Ergodicity

Definition 22.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- A map φ preserving $\mathbf{P}$
- $\mathcal{I} \equiv \sigma$-algebra of invariant events

We say that φ is ergodic if $\mathcal{I}$ is trivial, i.e:

$$A \in \mathcal{I} \quad \Longrightarrow \quad \mathbf{P}(A) \in \{0, 1\}.$$

Kolmogorov's 0-1 law

Theorem 23.

Let

- $X = \{X_n;\ n \geq 0\}$ sequence of i.i.d random variables
- For $n \geq 0$, we set $\mathcal{F}'_n = \sigma(\{X_k;\ k \geq n\})$
- Tail σ-field: $\mathcal{T} \equiv \cap_{n\geq 0}\mathcal{F}'_n$

Then $\mathcal{T}$ is trivial, i.e:

$$A \in \mathcal{T} \quad \Longrightarrow \quad \mathbf{P}(A) \in \{0, 1\}.$$

Ergodicity for i.i.d sequences

Example 24.

Let

- $X = \{X_n;\ n \geq 0\}$ sequence of i.i.d random variables
- $\varphi \equiv$ shift operator on $\Omega = \mathbb{R}^{\mathbb{N}}$

Then φ is ergodic.

Proof

Measurability of an invariant set: Let $A \in \mathcal{I}$. Then

$$A = \{\omega \in \Omega;\, \varphi(\omega) \in A\} \quad \Longrightarrow \quad A \in \mathcal{F}_1'.$$

Tail σ-field: Iterating the previous relation we get

$$A \in \mathcal{I} \quad \Longrightarrow \quad A \in \mathcal{T}$$

Hence by Kolmogorov's 0-1 law we get that $\mathcal{I}$ is trivial

Ergodicity for Markov chains

Example 25.

Let

- $X = \{X_n;\, n \geq 0\}$ MCH on countable state space S
- Transition probability: $p(x, A)$
- $\varphi \equiv$ shift operator on $\Omega = S^{\mathbb{N}}$
- Hypothesis 1: unique stationary distribution π
- Hypothesis 2: $\pi(x) > 0$ for all $x \in S$
- Hypothesis 3: $\mathcal{L}(X_0) = \pi$

Then

1. If X is not irreducible, θ is not ergodic
2. If X is irreducible, θ is ergodic

Proof

Basic Markov chains facts:

- Since $\pi(x) > 0$ for all $x \in S$, all states are recurrent
- State space decomposition:
  $S = \cup_{j \leq J} R_j$, where R_j disjoint irreducible sets

Non irreducible case: If $J \geq 2$ we have

$$X_0 \in R_j \quad \Longleftrightarrow \quad X_n \in R_j \text{ for all } n \geq 0.$$

Therefore for all $j \leq J$ we get:

- $\mathbf{1}_{(X_0 \in R_j)} = \mathbf{1}_{(X_0 \in R_j)} \mathbf{1}_{(X_1 \in R_j)} = \mathbf{1}_{(X_1 \in R_j)}$
- Iterating we get $(X_0 \in R_j) \in \mathcal{I}$
- $\pi(X_0 \in R_j) \in (0, 1)$ if $J \geq 2$

Proof (2)

General relation for $\mathbf{1}_A$:

- For $A \in \mathcal{I}$, we have $\mathbf{1}_A = \mathbf{1}_A \circ \theta_n$
- We set $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$
- We define $h(x) = \mathbf{E}_x[\mathbf{1}_A]$

Then

$$\mathbf{E}_\pi\left[\mathbf{1}_A \middle| \mathcal{F}_n\right] \stackrel{\text{invariance}}{=} \mathbf{E}_\pi\left[\mathbf{1}_A \circ \theta_n \middle| \mathcal{F}_n\right] \stackrel{\text{Markov prop}}{=} h(X_n)$$

Levy's 0-1 law: Let

- $\mathcal{F}_n \nearrow \mathcal{F}_\infty$
- $A \in \mathcal{F}_\infty$

Then a.s and in $L^1(\Omega)$ we have

$$\lim_{n\to\infty} \mathbf{E}\left[\mathbf{1}_A \middle| \mathcal{F}_n\right] = \mathbf{1}_A$$

Proof (3)

Limit of $h(X_n)$: Recall

$$\mathbf{E}_\pi\left[\mathbf{1}_A \middle| \mathcal{F}_n\right] = h(X_n), \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}\left[\mathbf{1}_A \middle| \mathcal{F}_n\right] = \mathbf{1}_A$$

Thus

$$\lim_{n\to\infty} h(X_n) = \mathbf{1}_A. \tag{3}$$

Irreducible case: If X irreducible and $\pi(y) > 0$ for all $y \in S$, then

- $h(X_n) = h(y)$ infinitely often for all $y \in S$
- According to (2) we thus have $h = \text{Cst}$

We have thus found that whenever $A \in \mathcal{I}$,

$$\mathbf{1}_A(\omega) = \text{Cst} \in \{0, 1\}$$

Ergodicity for rotations of the circle

Example 26.

We consider $\lambda \equiv$ Lebesgue measure and:

- $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1], \text{ Borel sets}, \lambda)$
- $\theta \in (0,1)$
- $X = \{X_n;\ n \geq 0\}$ with

$$X_n(\omega) = (\omega + n\theta) \mod 1 = \omega + n\theta - [\omega + n\theta].$$

- $\varphi \equiv$ shift transformation

Then the following holds true:

1. If θ is rational, then φ is not ergodic
2. If θ is irrational, then φ is ergodic

Transformation of an ergodic sequence

Theorem 27.

Let

- $X = \{X_n;\ n \geq 0\}$ ergodic sequence
- $g : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ measurable

We consider a sequence Y defined by

$$Y_k = g\left(\{X_{k+n};\ n \geq 0\}\right).$$

Then Y is an ergodic sequence.

Proof

Inverse image: Let $B \in \mathcal{R}^{\mathbb{N}}$. Recall that

$$\left\{\omega \in \mathbb{R}^{\mathbb{N}};\ Y \in B\right\} = \left\{\omega \in \mathbb{R}^{\mathbb{N}};\ X \in A\right\},$$

where

$$A = \left\{\omega \in \mathbb{R}^{\mathbb{N}}; g(X) \in B\right\}.$$

Consequence for ergodicity: If B satisfies

$$\left\{\omega \in \mathbb{R}^{\mathbb{N}};\ (Y_n)_{n\geq 0} \in B\right\} = \left\{\omega \in \mathbb{R}^{\mathbb{N}};\ (Y_{1+n})_{n\geq 0} \in B\right\}$$

Then A satisfies

$$\left\{\omega \in \mathbb{R}^{\mathbb{N}};\ (X_n)_{n\geq 0} \in A\right\} = \left\{\omega \in \mathbb{R}^{\mathbb{N}};\ (X_{1+n})_{n\geq 0} \in A\right\}$$

Proof (2)

Conclusion: Since A satisfies

$$\left\{\omega \in \mathbb{R}^{\mathbb{N}};\ (X_n)_{n\geq 0} \in A\right\} = \left\{\omega \in \mathbb{R}^{\mathbb{N}};\ (X_{1+n})_{n\geq 0} \in A\right\},$$

we have

$$\mathbf{P}\left(X \in A\right) \in \{0, 1\}.$$

Therefore:

$$\mathbf{P}\left(Y \in B\right) \in \{0, 1\}.$$

Ergodicity for Bernoulli shifts

Example 28.

We consider $\lambda \equiv$ Lebesgue measure and:

- $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1], \text{ Borel sets}, \lambda)$
- $Y_0 = \omega$
- $Y = \{Y_n;\ n \geq 0\}$ where for $n \geq 1$ we have

$$Y_n(\omega) = 2Y_{n-1} \mod 1.$$

- $\varphi \equiv$ shift transformation

Then φ is ergodic.

Proof

Representation for Y: Recall that we have defined

- $\{X_i;\, i \geq 1\}$ i.i.d with common law $\mathcal{B}(1/2)$
- $g(x) = \sum_{i\geq 1} x_i 2^{-(i+1)}$ defined for $x \in \{0,1\}^{\mathbb{N}}$
- $g_k(x) = g(\{x_{k+i};\, i \geq 1\})$ defined for $x \in \{0,1\}^{\mathbb{N}}$
- $Y_k = g_k(X)$

Then

$$Y_0 \sim \lambda, \quad \text{and} \quad Y_n(\omega) = 2Y_{n-1} \quad \text{mod } 1.$$

Ergodicity of Y: We have

- X ergodic
- $Y_k = g_k(X)$

Hence owing to Theorem 27, Y is a stationary sequence.

Outline

Birkhoff's ergodic theorem

Theorem 29.

Let

- $X \in L^1(\Omega)$
- φ a measure preserving map on Ω
- $\mathcal{I}$ the σ-field of invariant sets

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} X(\varphi^j(\omega)) = \mathbf{E}\left[X | \mathcal{I}\right], \quad \text{a.s and in } L^1(\Omega)$$

Ergodic theorem for ergodic maps

Proposition 30.

Let

- $X \in L^1(\Omega)$
- φ a measure preserving map on Ω
- Hypothesis: φ is ergodic

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} X(\varphi^j(\omega)) = \mathbf{E}[X], \quad \text{a.s and in } L^1(\Omega)$$

Maximal ergodic lemma

Lemma 31.

Let

- $X \in L^1(\Omega)$
- φ a measure preserving map on Ω
- $X_j = X(\varphi^j(\omega))$
- $S_n = \sum_{j=0}^{n-1} X_j$
- $M_k = \max\{0, S_1, \ldots, S_k\}$

Then

$$\mathbf{E}\left[X\,\mathbf{1}_{(M_k>0)}\right] \geq 0.$$

Proof

Lower bound for X: We will prove that for $n = 1, \ldots, k$

$$X(\omega) \geq S_n(\omega) - M_k(\varphi(\omega)) \tag{4}$$

Relation (4) for $n = 1$:
We have $S_1(\omega) = X(\omega)$ and $M_k(\varphi(\omega)) \geq 0$. Thus

$$X(\omega) \geq S_1(\omega) - M_k(\varphi(\omega))$$

Relation (4) for $n = 2, \ldots, k$:
We have $M_k(\varphi(\omega)) \geq S_j(\varphi(\omega))$ for $j = 1, \ldots, k$. Thus

$$X(\omega) + M_k(\varphi(\omega)) \geq X(\omega) + S_j(\varphi(\omega)) = S_{j+1}(\omega)$$

and

$$X(\omega) \geq S_{j+1}(\omega) - M_k(\varphi(\omega))$$

Proof (2)

Consequence of relation (4):

$$X(\omega) \geq \max\left\{S_1(\omega), \ldots, S_k(\omega)\right\} - M_k(\varphi(\omega))$$

Integration of the previous relation:

$$\begin{aligned}
\mathbf{E}\left[X\mathbf{1}_{(M_k>0)}\right] &\geq \int_{\{M_k>0\}} \left[\max\left\{S_1(\omega), \ldots, S_k(\omega)\right\} - M_k(\varphi(\omega))\right] d\mathbf{P} \\
&= \int_{\{M_k>0\}} \left[M_k(\omega) - M_k(\varphi(\omega))\right] d\mathbf{P}
\end{aligned}$$

Proof (3)

Conclusion: We have seen

$$\mathbf{E}\left[X\mathbf{1}_{(M_k>0)}\right] \geq \int_{\{M_k>0\}} [M_k(\omega) - M_k(\varphi(\omega))] \, d\mathbf{P}$$

In addition on $\{M_k > 0\}^c$ we have

1. $M_k(\omega) = 0$
2. $M_k(\varphi(\omega)) \geq 0$
3. Therefore $\int_{\{M_k>0\}^c} [M_k(\omega) - M_k(\varphi(\omega))] \, d\mathbf{P} \leq 0$

We thus get

$$\mathbf{E}\left[X\mathbf{1}_{(M_k>0)}\right] \geq \int [M_k(\omega) - M_k(\varphi(\omega))] \, d\mathbf{P} \overset{\varphi \text{ invariant}}{=} 0$$

Proof of Theorem 29

Reduction to $\mathbf{E}[X|\mathcal{I}] = 0$: We

- Set $\hat{X} = X - \mathbf{E}[X|\mathcal{I}]$
- Recall that $\mathbf{E}[X|\mathcal{I}]$ is invariant

Therefore we have

$$\frac{1}{n}\sum_{j=0}^{n} X(\varphi^j(\omega)) - \mathbf{E}\left[X|\mathcal{I}\right] = \frac{1}{n}\sum_{j=0}^{n}\Big(X(\varphi^j(\omega)) - \mathbf{E}\left[X|\mathcal{I}\right]\Big)$$
$$= \frac{1}{n}\sum_{j=0}^{n}\left[X - \mathbf{E}\left\{X|\mathcal{I}\right\}\right](\varphi^j(\omega)) = \frac{1}{n}\sum_{j=0}^{n}\hat{X}(\varphi^j(\omega))$$

We can thus prove Theorem 29 for X such that $\mathbf{E}[X|\mathcal{I}] = 0$

Proof of Theorem 29 (2)

Sufficient condition: We define

- $\bar{X} = \limsup_n \frac{S_n}{n}$
- $D = \{\omega;\, \bar{X}(\omega) > \varepsilon\}$ for $\varepsilon > 0$

We wish to prove that

$$\mathbf{P}(D) = 0$$

Invariance of D:
Since $\bar{X}$ is invariant, we have $D \in \mathcal{I}$

Proof of Theorem 29 (3)

Notation: We set

- $X^*(\omega) = (X(\omega) - \varepsilon)\mathbf{1}_D(\omega)$
- $X_j^* = X^*(\varphi^j(\omega))$
- $S_n^* = \sum_{j=0}^{n-1} X_j^*$
- $M_k^* = \max\{0, S_1^*, \ldots, S_k^*\}$
- $F_n = \{M_n^* > 0\}$ (increasing sequence)
- $F = \cup_{n\geq 0} F_n$

Relation between F and D: We have

$$F = \{\text{There exists } n \text{ s.t } X(\varphi^n(\omega)) > \varepsilon\} \cap D = D$$

We thus wish to prove that

$$\mathbf{P}(D) = \mathbf{P}(F) = 0$$

Proof of Theorem 29 (4)

Proof of $\mathbf{E}[X^* \mathbf{1}_D] \geq 0$: Owing to Lemma 31 we have

$$\mathbf{E}\left[X^* \mathbf{1}_{F_n}\right] \geq 0$$

By dominated convergence we get:

$$\mathbf{E}\left[X^* \mathbf{1}_D\right] = \mathbf{E}\left[X^* \mathbf{1}_F\right] \geq 0$$

Proof of $\mathbf{P}(D) = 0$: Since $D \in \mathcal{I}$ and $\mathbf{E}[X|\,\mathcal{I}] = 0$ we get

$$\begin{aligned} \mathbf{E}\left[X^* \mathbf{1}_D\right] &= \mathbf{E}\left[(X - \varepsilon)\, \mathbf{1}_D\right] \\ &= \mathbf{E}\left\{\mathbf{E}[X|\,\mathcal{I}]\, \mathbf{1}_D\right\} - \varepsilon\, \mathbf{P}(D) = -\varepsilon\, \mathbf{P}(D) \end{aligned}$$

We have seen that $\mathbf{E}[X^* \mathbf{1}_D] \geq 0$, which yields

$$\mathbf{P}(D) = 0$$

Proof of Theorem 29 (5)

Almost sure limit of $\frac{S_n}{n}$: For $k \geq 1$ we have seen that

$$\mathbf{P}(D_k) \equiv \mathbf{P}\left(\left\{\omega;\, \limsup_n \frac{S_n}{n} > \frac{1}{k}\right\}\right) = 0$$

Taking limits as $k \to \infty$ we get:

$$\mathbf{P}\left(\left\{\omega;\, \limsup_n \frac{S_n}{n} > 0\right\}\right) = 0$$

Since the same result is true for the r.v $-X$ we end up with:

$$\mathbf{P}\left(\left\{\omega;\, \limsup_n \frac{S_n}{n} = 0\right\}\right) = 1$$

Proof of Theorem 29 (6)

Truncation procedure: For the convergence in $L^1(\Omega)$ we set

$$X_M^1 = X\,\mathbf{1}_{(|X|\leq M)}, \quad \text{and} \quad X_M^2 = X\,\mathbf{1}_{(|X|>M)}$$

and

$$A_n = \frac{1}{n}\sum_{m=0}^{n-1} X(\varphi^m(\omega)) - \mathbf{E}[X|\,\mathcal{I}]$$

Then for $n \geq 1$ we have

$$\mathbf{E}\left[|A_n|\right] \leq \mathbf{E}\left[\left|A_n^1\right|\right] + \mathbf{E}\left[\left|A_n^2\right|\right],$$

where for $j = 1, 2$ we have defined:

$$A_n^j \;=\; \frac{1}{n}\sum_{m=0}^{n-1} X_M^j(\varphi^m(\omega)) - \mathbf{E}[X_M^j|\,\mathcal{I}]$$

Proof of Theorem 29 (7)

Limit for A_n^1: We have

1. $\text{a.s} - \lim_{n\to\infty} A_n^1 = 0$
2. $|A_n^1| \le 2M$

Therefore by dominated convergence we have:

$$\lim_{n\to\infty} \mathbf{E}\left[\left|A_n^1\right|\right] = 0$$

Proof of Theorem 29 (8)

Limit for A_n^2: We have

$$\begin{aligned}\mathbf{E}\left[\left|A_n^2\right|\right] &\leq \frac{1}{n}\sum_{m=0}^{n-1}\mathbf{E}\left[\left|X_M^2(\varphi^m(\omega))\right|\right] + \mathbf{E}\left[\mathbf{E}[|X_M^2|\,|\,\mathcal{I}]\right] \\ &\leq 2\,\mathbf{E}\left[|X_M^2|\right]\end{aligned}$$

In addition, by dominated convergence we have:

$$\lim_{n\to\infty}\mathbf{E}\left[|X_M^2|\right] = 0$$

Therefore

$$\lim_{n\to\infty}\mathbf{E}\left[\left|A_n^2\right|\right] = 0$$

Proof of Theorem 29 (9)

Conclusion for the $L^1(\Omega)$ convergence: We have seen

$$\lim_{n\to\infty} \mathbf{E}\left[\left|A_n^1\right|\right] = 0, \quad \text{and} \quad \lim_{n\to\infty} \mathbf{E}\left[\left|A_n^2\right|\right] = 0.$$

We thus get

$$L^1(\Omega) - \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} X_M^j(\varphi^m(\omega)) = \mathbf{E}[X_M^j | \mathcal{I}]$$

LLN for iid random variables

Example 32.

Let

- $X = \{X_n;\ n \geq 0\}$ sequence of i.i.d random variables
- Hypothesis: $X_0 \in L^1(\Omega)$
- $\bar{X}_n \equiv \frac{1}{n}\sum_{j=1}^{n} X_j$

Then

$$\lim_{n\to\infty} \bar{X}_n = \mathbf{E}[X_0], \quad \text{a.s and in } L^1(\Omega)$$

Remark:

W.r.t the usual LLN, we have obtained the $L^1(\Omega)$ convergence

Proof

Previous results: We have seen that

1. X is an ergodic sequence
2. $\mathcal{I} \subset \mathcal{T}$ and $\mathcal{T}$ is trivial

Conclusion: Theorem 29 applies and can be read as

$$\lim_{n\to\infty} \bar{X}_n = \mathbf{E}[X_0 | \mathcal{I}] = \mathbf{E}[X_0], \quad \text{a.s and in } L^1(\Omega)$$

Remark: The $L^1(\Omega)$ convergence can also be obtained as follows

- Result: If $\mathbf{P} - \lim_{n\to\infty} Y_n = Y$, then
  $\hookrightarrow L^1(\Omega) - \lim_{n\to\infty} Y_n = Y$ iff $\lim_{n\to\infty} \mathbf{E}[|Y_n|] = \mathbf{E}[|Y|]$
- Apply this result successively to $Y_n = X_n^+$ and $Y_n = X_n^-$

LLN for Markov chains

Example 33.

Let

- $X = \{X_n;\ n \geq 0\}$ MCH on countable state space S
- Hypothesis 1: unique stat. dist. π and $\mathcal{L}(X_0) = \pi$
- Hypothesis 2: $\pi(x) > 0$ for all $x \in S$
- Hypothesis 3: X irreducible
- Hypothesis 4: $f : S \to \mathbb{R}$ satisfies $f \in L^1(\pi)$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(X_j) = \sum_{x\in S} f(x)\,\pi(x), \quad \text{a.s and in } L^1(\Omega)$$

Proof

Previous results: We have seen that

1. X is an ergodic sequence
2. Therefore $f(X)$ is an ergodic sequence
3. $\mathcal{I}$ is trivial whenever X is irreducible

Conclusion: Theorem 29 applies and can be read as

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} f(X_j) = \mathbf{E}[f(X_0)|\,\mathcal{I}] = \mathbf{E}[f(X_0)], \quad \text{a.s and in } L^1(\Omega)$$

Remark: W.r.t the usual LLN for Markov chains,
we have obtained the $L^1(\Omega)$ convergence

LLN for rotation of the circle

Example 34.

We consider $\lambda \equiv$ Lebesgue measure and:

- $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1], \text{ Borel sets}, \lambda)$
- $\theta \in (0,1) \cap \mathbb{Q}^c$
- A Borel subset of $[0,1]$
- $X = \{X_n;\ n \geq 0\}$ with

$$X_n(\omega) = (\omega + n\theta) \mod 1 = \omega + n\theta - [\omega + n\theta].$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_A(X_n) = |A|, \quad \text{a.s and in } L^1(\Omega)$$

Deterministic LLN for rotation of the circle

Theorem 35.

In the context of Example 34, for $0 \leq a < b < 1$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{[a,b)}(X_n(x)) = b - a,$$

for all $x \in [0, 1)$

Proof:

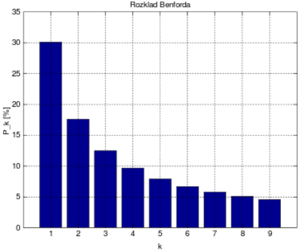
- Start from Example 34
- Additional ingredient based on density arguments

Benford's law

Proposition 36.

For $k \in \{1, \ldots, 9\}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbf{1}_{(1^{\text{st}} \text{ digit of } 2^m = k)} = \log_{10}\left(\frac{k+1}{k}\right)$$



Proof

Notation: We set

- $\theta = \log_{10}(2)$
- $A_k = [\log_{10}(k), \log_{10}(k+1))$

Expression 1 for $\log_{10}(2^m)$: we have

$$\log_{10}(2^m) = m\,\theta$$

Expression 2 for $\log_{10}(2^m)$: First digit of 2^m is k_0 iff

$$2^m = k_0 10^\alpha + k_1, \quad \text{with} \quad \alpha \geq 0,\ k_1 < 10^\alpha$$

and

$$\log_{10}(2^m) = \alpha + \log_{10}\left(k_0 + \frac{k_1}{10^\alpha}\right)$$

Proof (2)

Ergodic sequence: We have seen that first digit of 2^m is k_0 iff

$$m\,\theta = \alpha + \log_{10}\left(k_0 + \frac{k_1}{10^\alpha}\right),$$

which is equivalent to

$$m\,\theta \mod 1 \in A_k$$

Thus

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mathbf{1}_{(1^{\text{st}} \text{ digit of } 2^m = k)} = \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} \mathbf{1}_{(m\,\theta \mod 1 \in A_k)}$$

Proof (3)

Conclusion: We have

$$\theta = \log_{10}(2) \in \mathbb{Q}^c$$

We can thus apply Theorem 35 with $x = 0$ and $A = A_k$. We get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbf{1}_{(m\theta \mod 1 \in A_k)} = |A_k| = \log_{10}\left(\frac{k+1}{k}\right)$$

Outline

Number of points visited

Theorem 37.

Let

- X stationary sequence with values in $\mathbb{R}^d$
- $S_n = \sum_{j=0}^{n-1} X_j$
- $A = \{S_k \neq 0 \text{ for all } k \geq 1\}$
- $R_n = |\{S_1, \ldots, S_n\}| \equiv \mathrm{Card}(\{S_1, \ldots, S_n\})$

Then

$$\lim_{n\to\infty} \frac{R_n}{n} = \mathbf{E}\left[\mathbf{1}_A \middle| \mathcal{I}\right], \quad \text{almost surely}$$

Proof

Lower bound: We have

$$S_j(\varphi^m(\omega)) = S_{j+m}(\omega) - S_m(\omega)$$

Thus

$$S_j(\varphi^m(\omega)) \neq 0 \quad \Longleftrightarrow \quad S_{j+m}(\omega) \neq S_m(\omega)$$

and

$$\mathbf{1}_A(\varphi^m(\omega)) = \{S_{j+m}(\omega) \neq S_m(\omega) \text{ for all } j \geq 1\}$$

We have thus obtained

$$R_n \geq \sum_{m=1}^{n} \mathbf{1}_A(\varphi^m(\omega)).$$

and owing to the ergodic theorem we get

$$\liminf_{n\to\infty} \frac{R_n}{n} \geq \liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \mathbf{1}_A(\varphi^m(\omega)) = \mathbf{E}\left[\mathbf{1}_A | \mathcal{I}\right]$$

Proof (2)

Upper bound: For $k \geq 1$ set

$$A_k = \{S_j \neq 0 \text{ for all } 1 \leq j \leq k\}.$$

Then

$$\mathbf{1}_{A_k}(\varphi^m(\omega)) = \{S_j(\omega) \neq S_m(\omega) \text{ for all } m+1 \leq j \leq m+k\}$$

and

$$R_n \leq k + \sum_{m=1}^{n-k} \mathbf{1}_{A_k}(\varphi^m(\omega)).$$

Applying the ergodic theorem again we get

$$\liminf_{n\to\infty} \frac{R_n}{n} \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n-k} \mathbf{1}_{A_k}(\varphi^m(\omega)) = \mathbf{E}\left[\mathbf{1}_{A_k} \middle| \mathcal{I}\right] \tag{5}$$

Proof (3)

Limit in k: We have $k \mapsto A_k$ decreasing and

$$\bigcap_{k=1}^{\infty} A_k = A.$$

Therefore by monotone convergence we get

$$\lim_{k\to\infty} \mathbf{E}\left[\mathbf{1}_{A_k} \middle| \mathcal{I}\right] = \mathbf{E}\left[\mathbf{1}_{A} \middle| \mathcal{I}\right].$$

Thanks to (5) we have thus proved that

$$\limsup_{n\to\infty} \frac{R_n}{n} \leq \mathbf{E}\left[\mathbf{1}_{A} \middle| \mathcal{I}\right]$$

Recurrence criterion in $\mathbb{Z}$

Theorem 38.

Let

- X stationary sequence with values in $\mathbb{Z}$
- Hypothesis: $X_1 \in L^1(\Omega)$
- $S_n = \sum_{j=0}^{n-1} X_j$
- $A = \{S_k \neq 0 \text{ for all } k \geq 1\}$

The following holds true:

1. If $\mathbf{E}[X_1 | \mathcal{I}] = 0$ then $\mathbf{P}(A) = 0$
2. If $\mathbf{P}(A) = 0$ then $\mathbf{P}(S_n = 0 \text{ i.o}) = 1$

Proof

Standing assumption: $\mathbf{E}[X_1 | \mathcal{I}] = 0$, which yields

$$\lim_{n\to\infty} \frac{S_n}{n} = 0. \tag{6}$$

Relation for $\max S$: For all $K \geq 1$ we have

$$\limsup_{n\to\infty} \max_{1\leq k\leq n} \frac{|S_k|}{n} = \limsup_{n\to\infty} \max_{K\leq k\leq n} \frac{|S_k|}{n} \leq \max_{k\geq K} \frac{|S_k|}{k}$$

Taking $K \to \infty$ and thanks to (6) we get

$$\lim_{n\to\infty} \max_{1\leq k\leq n} \frac{|S_k|}{n} = 0 \tag{7}$$

Proof (2)

Consequence on R_n: In $d = 1$ we have

$$R_n = \text{Range}\left\{X_j;\, 0 \le j \le n\right\}.$$

Hence

$$R_n \le 1 + 2 \max_{1\le k\le n} |S_k|,$$

and according to (7) we get

$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{|R_n|}{n} \stackrel{\text{Theorem 37}}{=} \mathbf{E}\left[\mathbf{1}_A \middle| \mathcal{I}\right] \stackrel{\text{Relation (7)}}{=} 0$$

Conclusion for item 1:

We take expectations in previous relation, which yields

$$\mathbf{P}(A) = 0$$

Proof (3)

Notation: We set

$$F_j = \{S_i \neq 0 \text{ for } i < j \text{ and } S_j = 0\}$$
$$G_{j,k} = \{S_{j+i} - S_j \neq 0 \text{ for } i < k \text{ and } S_{j+k} - S_j = 0\}$$

Simple relations on F_j, $G_{j,k}$:

1. Since $\mathbf{P}(A) = 0$ we have $\sum_{k\geq 1} \mathbf{P}(F_k) = 1$
2. By stationarity, $\mathbf{P}(G_{j,k}) = \mathbf{P}(F_k)$
3. Hence we also have $\sum_{k\geq 1} \mathbf{P}(G_{j,k}) = 1$
4. For fixed j, the sets $G_{j,k}$ are disjoint

Proof (4)

More relations on F_j, $G_{j,k}$: We have obtained

$$\sum_{k\geq 1} \mathbf{P}\left(F_j \cap G_{j,k}\right) = \mathbf{P}\left(F_j\right), \quad \text{and} \quad \sum_{j,k\geq 1} \mathbf{P}\left(F_j \cap G_{j,k}\right) = 1$$

Conclusion on recurrence: We have

$$\bigcup_{j,k\geq 1} F_j \cap G_{j,k} = \{S_n = 0 \text{ at least two times}\}$$

Therefore

$$\mathbf{P}\left(S_n = 0 \text{ at least two times}\right) = 1$$

Generalization:

- With sets $G_{j_1,\ldots,j_m}$
- Taking $m \to \infty$

Extension by Kac

Theorem 39.

Let

- X stationary sequence with values in $(S, \mathcal{S})$
- $A \in \mathcal{S}$ and $S_n = \sum_{j=0}^{n-1} X_j$
- $T_0 = 0$ and $T_n = \inf\{m > T_{n-1};\, X_m \in A\}$
- $t_n = T_n - T_{n-1}$
- Hypothesis: $\mathbf{P}\left(X_n \in A \text{ at least once}\right) = 1$

Then the following holds true:

1. Under $\mathbf{P}(\cdot \,|\, X_0 \in A)$ the sequence (t_n) is stationary
2. We have

$$\mathbf{E}\left[T_1 \,|\, X_0 \in A\right] = \frac{1}{\mathbf{P}\left(X_0 \in A\right)}$$