Itô's formula

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Outline

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Introduction

Aim: Differential calculus with respect to W, i.e formula of the form:

$$\delta f(W)_{st} = \sum_{j=1}^{d} \int_s^t \partial_{x_j} f(W_r)\, dW_r^j = \sum_{j=1}^{d} \int_s^t \partial_{x_j} f(W_r)\, \dot{W}_r^j\, dr$$

Problem: $\dot{W}$ nowhere defined!

Strategy:

- Define $\int_s^t u_r\, dW_r$ for piecewise constant processes
- Take limits invoking $\perp\!\!\!\perp$ of increments of W
  $\hookrightarrow$ Limit in $L^2(\Omega)$

Remark: for notational sake
$\hookrightarrow$ Computations done for real valued processes
$\hookrightarrow$ Generalization to $\mathbb{R}^d$: add indices

Elementary processes

Definition 1.

Let:

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space.
- $u = \{u_t;\ t \in [0, \tau]\}$ process defined on $(\Omega, \mathcal{F}, \mathbf{P})$

Process u is called simple if it can be decomposed as:

$$u_t = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{N-1} \xi_i \, \mathbf{1}_{(t_i, t_{i+1}]}(t),$$

with

- $N \geq 1$, $(t_1, \ldots, t_N)$ partition of $[0, \tau]$ with $t_1 = 0, t_N = \tau$
- $\xi_i \in \mathcal{F}_{t_i}$ and $|\xi_i| \leq c$ with $c > 0$

Integral of a simple process

Definition 2.

Let u simple process. Let $s, t \in [0, \tau]$ such that $s \leq t$ and:

$$t_m < s \leq t_{m+1}, \quad \text{and} \quad t_n < t \leq t_{n+1}, \quad m \leq n$$

We define $\mathcal{J}_{st}(u\, dW) =$ " $\int_s^t u_w dW_w$ " as:

$$\mathcal{J}_{st}(u\, dW) = \xi_m\, \delta W_{s t_{m+1}} + \sum_{i=m+1}^{n-1} \xi_i\, \delta W_{t_i t_{i+1}} + \xi_n\, \delta W_{t_n t}$$

Remark: For simple processes
$\hookrightarrow$ Stochastic integral and Riemann integral coincide.

Integral for simple processes: properties

Proposition 3.

Consider:

- u a simple process
- $\mathcal{J}_{st}(u\,dW)$ its stochastic integral.

Then for $\alpha, \beta \in \mathbb{R}$:

1. $\mathcal{J}_{tt}(u\,dW) = 0$
2. $\mathcal{J}_{st}((\alpha u + \beta v)\,dW) = \alpha \mathcal{J}_{st}(u\,dW) + \beta \mathcal{J}_{st}(v\,dW)$
3. $\mathbf{E}[\mathcal{J}_{st}(u\,dW)|\mathcal{F}_s] = 0$, i.e martingale property
4. $\mathbf{E}[(\mathcal{J}_{st}(u\,dW))^2] = \int_s^t \mathbf{E}[u_\tau^2]\,d\tau$
5. $\mathbf{E}[(\mathcal{J}_{st}(u\,dW))^2|\mathcal{F}_s] = \int_s^t \mathbf{E}[u_\tau^2|\mathcal{F}_s]\,d\tau$

Proof of point 5

If $i < j$, then (independence of increments of B)

$$
\begin{aligned}
&\mathbf{E}\left[\xi_i\,\delta W_{t_i t_{i+1}}\,\xi_j\,\delta W_{t_j t_{j+1}}\middle|\mathcal{F}_{t_j}\right]\\
&=\xi_i\xi_j\,\delta W_{t_i t_{i+1}}\,\mathbf{E}\left[\delta W_{t_j t_{j+1}}\middle|\mathcal{F}_{t_j}\right]\\
&=\xi_i\xi_j\,\delta W_{t_i t_{i+1}}\,\mathbf{E}\left[\delta W_{t_j t_{j+1}}\right]\\
&=0
\end{aligned}
$$

Proof of point 5 (2)

Thus:

$$
\begin{aligned}
&\mathbf{E}[(\mathcal{J}_{st}(u\,dW))^2|\mathcal{F}_s] \\
&= \mathbf{E}\left[\left(\xi_m\,\delta W_{st_m} + \sum_{i=m+1}^{n-1}\xi_i\,\delta W_{t_i t_{i+1}} + \xi_n\,\delta W_{t_n t}\right)^2 \middle| \mathcal{F}_s\right] \\
&= \mathbf{E}\left[\xi_m^2\,\delta W_{st_m}^2 + \sum_{i=m+1}^{n-1}\xi_i^2\,\delta W_{t_i t_{i+1}}^2 + \xi_n^2\,\delta W_{t_n t}^2 \middle| \mathcal{F}_s\right] \\
&= \mathbf{E}[\xi_m^2|\mathcal{F}_s](t_m - s) + \sum_{i=m+1}^{n-1}\mathbf{E}[\xi_i^2|\mathcal{F}_s](t_{i+1} - t_i) + \mathbf{E}[\xi_n^2|\mathcal{F}_s](t - t_n) \\
&= \int_s^t \mathbf{E}[u_r^2|\mathcal{F}_s]\,dr
\end{aligned}
$$

Space L_a^2

Definition 4.

We denote by $L_a^2([0,\tau])$ the set of process u such that:

1. u square integrable
2. u right continuous
3. $u_t \in \mathcal{F}_t$

The norm on L_a^2 is defined by:

$$\|u\|_{L_a^2}^2 \equiv \int_0^\tau \mathbf{E}\left[u_s^2\right] ds$$

Remark: Condition $u_t \in \mathcal{F}_t$ **and** u right continuous

$\hookrightarrow$ Ensures predictable property for u

$\hookrightarrow$ See discrete time stochastic integral

Density of simple processes in L_a^2

Proposition 5.

Let $u \in L_a^2$.
There exists a sequence $(u^n)_{n\geq 0}$ of simple processes such that:

$$\lim_{n\to\infty} \|u - u^n\|_{L_a^2} = 0.$$

Concrete approximation

Generic partition: We consider

- $\pi = \{s_0, \ldots, s_n\}$ with $0 = s_0 < \cdots < s_n = t$
- $|\pi| = \max\{|s_{j+1} - s_j|;\ 0 \le j \le n-1\}$

Proposition 6.

Let:

- $u \in L_a^2$ such that $|u_t| \le M < \infty$ for all $t \le \tau$ a.s
- Assume u is continuous (almost surely)
- $\{\pi^m;\ m \ge 1\}$ sequence of partitions of $[0, \tau]$
  $\hookrightarrow$ such that $\lim_{m\to\infty} |\pi_m| = 0$
- $u^m \equiv \sum_{s_j \in \pi^m} u_{s_j} \mathbf{1}_{[s_j, s_{j+1})}$

Then we have:

$$\lim_{m\to\infty} \|u - u^m\|_{L_a^2} = 0$$

Proof

Expression for u^m:

$$u_t^m = \sum_{s_j \in \pi^m} u_{s_j} \mathbf{1}_{[s_j, s_{j+1})}(t)$$

Properties of u^m:

1. Almost surely: $\lim_{m\to\infty} u_t^m = u_t$ for all $t \in [0, \tau]$
2. $|u_t^m| \leq Z_t$ with $Z_t \equiv M$
3. $Z \in L^2(\Omega \times [0, \tau])$

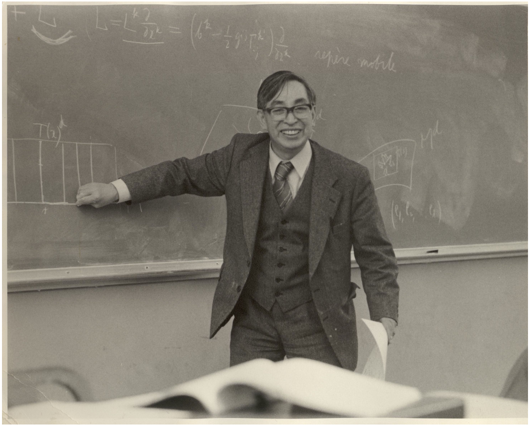
Convergence of u^m: by dominated convergence,

$$\lim_{m\to\infty} \int_0^\tau \mathbf{E}\left[|u_t - u_t^m|^2 \right] = 0$$

Kiyosi Itô (1915-2008)

Birth of Itô's integral:

- In 1938: Working for the
  $\hookrightarrow$ Japanese National Statistical Bureau
- During WW2, jobless in practice $\longrightarrow$ Stochastic integral!



Stochastic integral: extended definition

Proposition 7.

Let:

- $u \in L^2_a$
- $(u^n)_{n\geq 0}$ sequence of simple processes such that $\lim_{n\to\infty} \|u - u^n\|_{L^2_a} = 0$

Then for $s < t$:

- The sequence $(\mathcal{J}_{st}(u^n\, dW))_{n\geq 0}$ converges in $L^2(\Omega)$
- Its limit does not depend on the sequence $(u^n)_{n\geq 0}$

Proof

Cauchy sequence u^n: We know that:

$$\|u^n - u^m\|_{L^2_a} = \int_s^t \mathbf{E}\left[(u^n_w - u^m_w)^2\right] dw \xrightarrow{n\to 0} 0.$$

Cauchy sequence $\mathcal{J}_{st}(u^n)$: According to property:

$$\mathbf{E}\left[\left|\mathcal{J}_{st}\left((u^n - u^m)\ dW\right)\right|^2\right] = \int_s^t \mathbf{E}\left[|u^n_w - u^m_w|^2\right]\ dw$$

we have

$$(Z_n)_{n\geq 0} \equiv (\mathcal{J}_{st}(u^n))_{n\geq 0}$$

is a Cauchy sequence in $L^2(\Omega)$.

Stochastic integral: extended definition (2)

Definition 8.

Let $u \in L^2_a$. The stochastic integral of u with respect to W is the process $\mathcal{J}(u\,dW)$ such that for all $s < t$,

$$\mathcal{J}_{st}(u\,dW) = L^2(\Omega) - \lim_{n\to\infty} \mathcal{J}_{st}(u^n\,dW),$$

where $(u^n)_{n\geq 0}$ is an arbitrary sequence of simple processes $\hookrightarrow$ converging to u in L^2_a.

Integral of a L_a^2 process: properties

Proposition 9.

Let u a process of L_a^2, $\mathcal{J}_{st}(u\,dW)$ its stochastic integral. Then for $\alpha, \beta \in \mathbb{R}$:

1. $\mathcal{J}_{tt}(u\,dW) = 0$
2. $\mathcal{J}_{st}((\alpha u + \beta v)\,dW) = \alpha \mathcal{J}_{st}(u\,dW) + \beta \mathcal{J}_{st}(v\,dW)$
3. $\mathbf{E}[\mathcal{J}_{st}(u\,dW)|\mathcal{F}_s] = 0$, i.e martingale property
4. $\mathbf{E}[(\mathcal{J}_{st}(u\,dW))^2] = \int_s^t \mathbf{E}[u_w^2]\,dw$
5. $\mathbf{E}[(\mathcal{J}_{st}(u\,dW))^2|\mathcal{F}_s] = \int_s^t \mathbf{E}[u_w^2|\mathcal{F}_s]\,dw$

Remark: For this construction, crucial use of:

- Independence of increments for W
- L^2 convergence in probabilistic sense

Wiener integral

Proposition 10.

Let:

- W a 1-dimensional Wiener process, and $\tau > 0$.
- h^1, h^2 deterministic functions in $L^2([0,\tau])$.

Then the following stochastic integrals are well defined:

$$W(h^1) = \int_0^\tau h_r^1 \, dW_r, \qquad W(h^2) = \int_0^\tau h_r^2 \, dW_r.$$

In addition:

1. $(W(h^1), W(h^2))$ centered Gaussian vector.
2. Covariance of $W(h^1), W(h^2)$:

$$\mathbf{E}\left[W(h^1)\, W(h^2)\right] = \int_0^\tau h_r^1 \, h_r^2 \, dr.$$

Proof

Approximation: For $m \geq 1$ we set $s_j = s_j^m = \frac{j\tau}{m}$ and

$$h^{1,m} = \sum_{j=0}^{m-1} h^1_{s_j} \, \mathbf{1}_{[s_j, s_{j+1})}, \qquad h^{2,m} = \sum_{j=0}^{m-1} h^2_{s_j} \, \mathbf{1}_{[s_j, s_{j+1})}.$$

Then:

$$W(h^{1,m}) = \sum_{j=0}^{m-1} h^1_{s_j} \, \delta W_{s_j s_{j+1}}, \qquad W(h^{2,m}) = \sum_{j=0}^{m-1} h^2_{s_j} \, \delta W_{s_j s_{j+1}}.$$

Approximation properties: we have

- $(W(h^{1,m}), W(h^{2,m}))$ Gaussian vect. since W Gaussian process.
- $\mathbf{E}[W(h^{1,m}) \, W(h^{2,m})] = \int_0^\tau h^{1,m}_r \, h^{2,m}_r \, dr$.

Then limiting procedure for Gaussian vectors.

Outline

Functional set $\mathcal{C}^{1,2}$

Definition 11.

Let

- $\tau > 0$
- $f : [0, \tau] \times \mathbb{R}^d \to \mathbb{R}$

We say that $f \in \mathcal{C}^{1,2}_{b,\tau}$ if:

1. $t \mapsto f(t, x)$ is $\mathcal{C}^1([0, \tau])$ for all $x \in \mathbb{R}^d$
2. $x \mapsto f(t, x)$ is $\mathcal{C}^2(\mathbb{R}^d)$ for all $t \in [0, \tau]$
3. f and its derivatives are bounded

1-d Itô's formula

Theorem 12.

Let:

- W real-valued Brownian motion
- $f \in \mathcal{C}^{1,2}_{b,\tau}$

Then $f(t, W_t)$ can be decomposed as:

$$f(t, W_t) = f(0,0) + \int_0^t \partial_r f(r, W_r)\, dr + \int_0^t f'(r, W_r)\, dW_r + \frac{1}{2}\int_0^t f''(r, W_r)\, dr.$$

Remark:

Proof for the 1-d formula only

Multidimensional Itô's formula

Theorem 13.

Let:

- W Brownian motion, d-dimensional
- $f \in \mathcal{C}^{1,2}_{b,\tau}$
- $\Delta \equiv$ Laplace operator on $\mathbb{R}^d$, i.e $\Delta = \sum_{j=1}^{d} \partial^2_{x_j x_j}$

Then $f(t, W_t)$ can be decomposed as:

$$
\begin{aligned}
f(t, W_t) = f(0,0) &+ \int_0^t \partial_r f(r, W_r)\, dr \\
&+ \sum_{j=1}^{d} \int_0^t \partial_{x_j} f(r, W_r)\, dW_r^j + \frac{1}{2} \int_0^t \Delta f(r, W_r)\, dr
\end{aligned}
$$

Itô's formula for physicists

Simplification: We consider $f : \mathbb{R} \to \mathbb{R}$ only.

Intuition:

- Taylor expansion of $f(W)$ up to $\mathrm{o}(t)$.
- dW_t can be identified with $\sqrt{dt}$.

Heuristic computation: We get

$$\begin{aligned} df(W_t) &= f'(W_t)\, dW_t + \frac{1}{2} f''(W_t)\, (dW_t)^2 + \mathrm{o}\Big((dW_t)^2\Big) \\ &= f'(W_t)\, dW_t + \frac{1}{2} f''(W_t)\, dt + \mathrm{o}(dt)\,. \end{aligned}$$

Itô's formula is then obtained by integration.

Proof in the real case

Simplification: We consider $f : \mathbb{R} \to \mathbb{R}$ only

Generic partition: We consider

- $\pi = \{t_0, \ldots, t_n\}$ with $0 = t_0 < \cdots < t_n = t$
- $|\pi| = \max\{|t_{j+1} - t_j|;\, 0 \le j \le n-1\}$

Taylor's formula: we have

$$\begin{aligned} \delta f(W)_{0t} &= \sum_{j=0}^{n-1} \delta f(W)_{t_j t_{j+1}} \\ &= \sum_{j=0}^{n-1} f'(W_{t_j})\, \delta W_{t_j t_{j+1}} + \frac{1}{2} f''(\xi_j) \left(\delta W_{t_j t_{j+1}}\right)^2, \end{aligned}$$

where $\xi_j \in [W_{t_j}, W_{t_{j+1}}]$.

Proof in the real case (2)

Notation: We set

$$J_t^{1,\pi} = \sum_{j=0}^{n-1} f'(W_{t_j})\,\delta W_{t_j t_{j+1}}, \qquad J_t^{2,\pi} = \sum_{j=0}^{n-1} f''(\xi_j)\,\left(\delta W_{t_j t_{j+1}}\right)^2$$

Aim: Find a sequence $(\pi_m)_{m\geq 1}$ such that a.s

- $\lim_{m\to\infty} |\pi_m| = 0$
- $\lim_{m\to\infty} J_t^{1,m} = \int_0^t f'(W_s)\,dW_s$ with $J_t^{1,m} = J_t^{1,\pi_m}$
- $\lim_{m\to\infty} J_t^{2,m} = \int_0^t f''(W_s)\,ds$ with $J_t^{2,m} = J_t^{2,\pi_m}$

Proof in the real case (3)

Analysis of the term $J_t^{1,m}$: we have

$$J_t^{1,m} = \int_0^t u_s^m \, dW_s, \quad \text{with} \quad u^m \equiv \sum_{t_j \in \pi^m} u_{t_j} \mathbf{1}_{[t_j, t_{j+1})}$$

Since $u \equiv f'(W) \in L_a^2$, continuous and bounded
$\hookrightarrow$ according to Proposition 6 we get:

$$L^2(\Omega) - \lim_{m \to \infty} J_t^{1,m} = \int_0^t f'(W_s) \, dW_s$$

a.s convergence: For a subsequence $\pi_m \equiv \pi_{m_k}$ we have

$$\text{a.s} - \lim_{m \to \infty} J_t^{1,m} = \int_0^t f'(W_s) \, dW_s.$$

Proof in the real case (4)

Analysis of the term $J_t^{2,m}$, strategy: We set

$$J_t^{3,\pi} = \sum_{j=0}^{n-1} f''(W_{t_j}) \left(\delta W_{t_j t_{j+1}}\right)^2, \quad J_t^{4,\pi} = \sum_{j=0}^{n-1} f''(W_{t_j}) \left(t_{j+1} - t_j\right)$$

We will show that a.s, for a subsequence π_m with $|\pi_m| \to 0$:

$$\begin{aligned} \lim_{m\to\infty} \left| J_t^{2,\pi_m} - J_t^{3,\pi_m} \right| &= 0 \qquad (1) \\ \lim_{m\to\infty} \left| J_t^{3,\pi_m} - J_t^{4,\pi_m} \right| &= 0 \qquad (2) \\ \lim_{m\to\infty} \left| J_t^{4,\pi_m} - \int_0^t f''(W_s)\, ds \right| &= 0 \qquad (3) \end{aligned}$$

This will end the proof.

Proof in the real case (5)

Recall: we have

$$J_t^{4,\pi_m} = \sum_{j=0}^{n-1} f''(W_{t_j})\,(t_{j+1} - t_j)$$

Thus $J_t^{4,\pi_m} \equiv$ Riemann sum for $f''(W)$

Proof relation (3): Since $s \mapsto f''(W_s)$ continuous, we have

$$\lim_{m\to\infty} J_t^{4,\pi_m} = \int_0^t f''(W_s)\,ds.$$

Proof in the real case (6)

Proof relation (1): we have

$$J_t^{3,\pi} - J_t^{2,\pi} = \sum_{j=0}^{n-1} \left(f''(W_{t_j}) - f''(\xi_j) \right) \left(\delta W_{t_j t_{j+1}} \right)^2$$

Additional Hyp. for sake of simplicity: $f \in \mathcal{C}_b^3$. Then for $0 < \gamma < \frac{1}{2}$ we have:

- $\xi_j \in [W_{t_j}, W_{t_{j+1}}]$
- $|\delta W_{t_j t_{j+1}}| \leq c_{W,\gamma} |t_{j+1} - t_j|^\gamma$
  $\hookrightarrow |f''(W_{t_j}) - f''(\xi_j)| \leq c_{W,\gamma,f} \, |\pi^m|^\gamma$

Thus applying Theorem 27, Brownian Chapter:

$$\begin{aligned} \left| J_t^{3,\pi_m} - J_t^{2,\pi_m} \right| &\leq c_{W,\gamma} \, |\pi^m|^\gamma \sum_{j=0}^{n-1} \left(\delta W_{t_j t_{j+1}} \right)^2 \\ &\leq c_{W,\gamma} \, |\pi^m|^\gamma \, V_{0,t}^2(W) = c_{W,\gamma} \, t \, |\pi^m|^\gamma \xrightarrow{m \to \infty} 0 \end{aligned}$$

Proof in the real case (7)

Notation: We set

$$\delta t_j \equiv (t_{j+1} - t_j), \quad Z_j \equiv \left(\delta W_{t_j t_{j+1}}\right)^2 - \delta t_j$$

Proof relation (2), strategy:
we have

$$J_t^{4,\pi_m} - J_t^{3,\pi_m} = \sum_{j=0}^{n-1} f''(W_{t_j}) \, Z_j .$$

We proceed as follows:

1. We show $\lim_{m\to\infty} \mathbf{E}[|J_t^{4,\pi_m} - J_t^{3,\pi_m}|^2] = 0$.
2. For a subsequence $\pi_m \equiv \pi_{m_k}$ we deduce:

$$\text{a.s} - \lim_{m\to\infty} \left| J_t^{4,\pi_m} - J_t^{3,\pi_m} \right| = 0.$$

Proof in the real case (8)

Decomposition of $J_t^{4,\pi_m} - J_t^{3,\pi_m}$:

we have $J_t^{4,\pi_m} - J_t^{3,\pi_m} = \sum_{j=0}^{n-1} f''(W_{t_j})\, Z_j$. Thus:

$$\mathbf{E}\left[\left|J_t^{4,\pi_m} - J_t^{3,\pi_m}\right|^2\right] = K_t^{m,1} + K_t^{m,2}$$

with

$$\begin{aligned} K_t^{m,1} &= \sum_{j=0}^{n-1} \mathbf{E}\left[\left(f''(W_{t_j})\, Z_j\right)^2\right] \\ K_t^{m,2} &= 2 \sum_{0\le j<k\le n-1} \mathbf{E}\left[f''(W_{t_j})\, Z_j\, f''(W_{t_k})\, Z_k\right] \end{aligned}$$

Lemma: Let $X \sim \mathcal{N}(0, \sigma^2)$. Then:

$$\mathbf{E}\left[\left(X^2 - \sigma^2\right)^2\right] = 2\sigma^4$$

Proof in the real case (9)

Convergence of $K_t^{m,1}$:

We have $\delta W_{t_j t_{j+1}} \sim \mathcal{N}(0, \delta t_j)$ and $\delta W_{t_j t_{j+1}} \perp\!\!\!\perp \mathcal{F}_{t_j}$. Therefore:

$$
\begin{aligned}
K_t^{m,1} &= \sum_{j=0}^{n-1} \mathbf{E}\left[\left(f''(W_{t_j})\, Z_j\right)^2\right] \\
&= \sum_{j=0}^{n-1} \mathbf{E}\left[\left(f''(W_{t_j})\right)^2\right] \mathbf{E}\left[Z_j^2\right] \\
&= 2\sum_{j=0}^{n-1} \mathbf{E}\left[\left(f''(W_{t_j})\right)^2\right] (\delta t_j)^2 \\
&\le c_f\, |\pi^m| \sum_{j=0}^{n-1} \delta t_j = c_f\, t\, |\pi^m| \xrightarrow{m\to\infty} 0.
\end{aligned}
$$

Proof in the real case (10)

Proof relation (2): we have seen

$$\lim_{m\to\infty} K_t^{m,1} = 0.$$

In the same way, one can check that:

$$K_t^{m,2} = 0.$$

Thus:

$$\lim_{m\to\infty} \mathbf{E}[|J_t^{4,\pi_m} - J_t^{3,\pi_m}|^2] = 0.$$

This finishes the proof of (2) and of Itô's formula.

Wolfgang Doeblin

A mysterious sealed envelope:

- Son of Alfred Doeblin, escapes from Germany in 1933
- Dies during WW2 on the (short lived) French front
- In 1940, sends a sealed envelope (unsealed in 2000) containing ... Itô's formula!

Outline

Itô processes

Definition 14.

Let:

- $X : [0, \tau] \to \mathbb{R}^n$ process of L_a^2
- $a \in \mathbb{R}^n$ initial condition
- $\{b^j;\, j = 1, \ldots, n\}$ real bounded and adapted process
- $\{\sigma^{jk};\, j = 1, \ldots, n,\; k = 1, \ldots d\}$ process of L_a^2

We say that X is an Itô process if it can be decomposed as:

$$X_t^j = a^j + \int_0^t b_s^j \, ds + \int_0^t \sigma_s^{jk} \, dW_s^k,$$

for $j = 1, \ldots, n$.

Remark:

An Itô process is a particular case of semi-martingale.

Itô's formula for Itô processes

Theorem 15.

Let:

- X Itô process, defined by a, b, σ.
- $f \in \mathcal{C}^{1,2}_{b,\tau}$.

Then $f(t, X_t)$ can be decomposed as:

$$
\begin{aligned}
f(t, X_t) &= f(0, a) + \int_0^t \partial_r f(r, X_r)\, dr + \sum_{j=1}^n \int_0^t \partial_{x_j} f(r, X_r)\, b_r^j\, dr \\
&\quad + \sum_{j=1}^n \sum_{k=1}^d \int_0^t \partial_{x_j} f(r, X_r)\, \sigma_r^{jk}\, dW_r^k \\
&\quad + \frac{1}{2} \sum_{j_1, j_2=1}^n \sum_{k=1}^d \int_0^t \partial^2_{x_{j_1} x_{j_2}} f(r, X_r)\, \sigma_r^{j_1 k}\, \sigma_r^{j_2 k}\, dr.
\end{aligned}
$$

Infinitesimal generator of Brownian motion

Proposition 16.

Let:

- W a $\mathcal{F}_s$-Wiener process in $\mathbb{R}^d$.
- $s \in \mathbb{R}_+$.
- $f \in \mathcal{C}_b^2$.

Then:

$$\lim_{h \to 0} \frac{\mathbf{E}\left[\delta f(W)_{s,s+h} \middle| \mathcal{F}_s\right]}{h} = \frac{1}{2} \Delta f(W_s).$$

Proof

Recasting Itô's formula: Let

$$M_t \equiv \sum_{j=1}^{d} \int_0^t \partial_{x_j} f(W_r)\, dW_r^j.$$

Then Itô's formula can be written as:

$$\delta f(W)_{st} = \delta M_{st} + \frac{1}{2} \int_s^t \Delta f(W_r)\, dr.$$

Conditional expectation:

According to Proposition 9, M is a martingale. Thus:

$$\mathbf{E}\left[\delta f(W)_{s,s+h} \middle| \mathcal{F}_s\right] = \frac{1}{2} \int_s^{s+h} \mathbf{E}\left[\Delta f(W_r) \middle| \mathcal{F}_s\right] dr.$$

Proof (2)

Limiting procedure:

Applying dominated convergence for conditional expectation we get:

$$r \mapsto \mathbf{E}\left[\Delta f(W_r) \middle| \mathcal{F}_s\right] \quad \text{continuous.}$$

Thus:

$$\lim_{h\to 0} \frac{1}{2h} \int_s^{s+h} \mathbf{E}\left[\Delta f(W_r) \middle| \mathcal{F}_s\right] dr = \frac{1}{2} \mathbf{E}\left[\Delta f(W_s) \middle| \mathcal{F}_s\right] = \frac{1}{2} \Delta f(W_s),$$

and

$$\lim_{h\to 0} \frac{\mathbf{E}\left[\delta f(W)_{s,s+h} \middle| \mathcal{F}_s\right]}{h} = \frac{1}{2}\Delta f(W_s).$$

Extension for Itô processes

Theorem 17.

Let:

- X Itô process, defined by a, b, σ.
- $f \in \mathcal{C}_b^2$.

Then:

$$\lim_{h \to 0} \frac{\mathbf{E}\left[\delta f(X)_{s,s+h} \middle| \mathcal{F}_s \right]}{h}$$
$$= \sum_{j=1}^{n} \partial_{x_j} f(X_s)\, b_s^j + \frac{1}{2} \sum_{j_1, j_2 = 1}^{n} \sum_{k=1}^{d} \partial^2_{x_{j_1} x_{j_2}} f(X_s)\, \sigma_s^{j_1 k}\, \sigma_s^{j_2 k}.$$