Martingales

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Probability Theory 2 - MA 539

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Outline



2 Strategies and stopped martingales

3 Convergence

- Convergence in L^p
- Optional stopping theorems

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1 Definitions and first properties

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Adaptation

Context: We are given

- A probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- A filtration $\{\mathcal{F}_n; n \geq 0\}$
 - \hookrightarrow Sequence of σ -algebras such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

Definition 1.

A sequence of random variables $\{X_n; n \ge 0\}$ is adapted if:

 $X_n \in \mathcal{F}_n$.

Martingales, Supermartingales, Submartingales

Definition 2.

We consider a sequence of random variables $X = \{X_n; n \ge 0\}$ such that

•
$$\{X_n; n \ge 0\}$$
 is adapted.

2)
$$X_n \in L^1(\Omega)$$
 for all $n \ge 0$.

Then

- X is a martingale if $X_n = \mathbf{E}[X_{n+1}|\mathcal{F}_n]$.
- X is a supermartingale if $X_n \ge \mathbf{E}[X_{n+1}|\mathcal{F}_n]$.
- X is a submartingale if $X_n \leq \mathbf{E}[X_{n+1} | \mathcal{F}_n]$.

Adaptation: The data X_n only depends on information until instant n. Martingale: $n \mapsto X_n$ constant plus fluctuations. Submartingale: $n \mapsto X_n$ increasing plus fluctuations.

Supermartingale: $n \mapsto X_n$ decreasing plus fluctuations.

Random walk

Definition: Let

• { Z_i ; $i \ge 1$ } independent Rademacher r.v $\hookrightarrow \mathbf{P}(Z_i = -1) = \mathbf{P}(Z_i = 1) = 1/2$

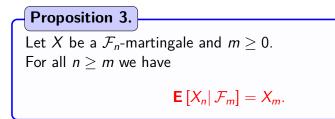
• We set $X_0 = 0$, and for $n \ge 1$,

$$X_n = \sum_{i=1}^n Z_i.$$

X is called random walk in \mathbb{Z} .

Property: X is a martingale.

Conditional expectation in the past

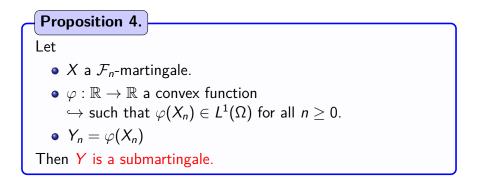


Proof: Recursive procedure.

Important corollary: Let X be a \mathcal{F}_n -martingale and $m \ge 0$. For all $n \ge m$ we have

$$\mathbf{E}[X_n] = \mathbf{E}[X_m] = \mathbf{E}[X_0].$$
(1)

Composition with a convex function



Proof: application of Jensen for conditional expectation.

Example: If X_n is a random walk, X_n^2 is a submartingale \hookrightarrow Fluctuations increase with time.

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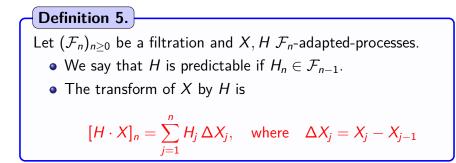
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Martingale transformation



Interpretation:

- $H \equiv$ game strategy
 - \hookrightarrow Today's decision depends on the information until yesterday
- $H \cdot X \equiv$ value if strategy H is used

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D'Alembert

Some facts about d'Alembert:

- Abandoned after birth
- Mathematician
- Contribution in fluid dynamics
- Philosopher
- Participation in 1st Encyclopedia



D'Alembert's Martingale

Example: Let $X_n = \sum_{i=1}^n \xi_i$ be a random walk. We interpret ξ_i as a gain ou a loss at *i*th iteration of the game. The filtration is $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$

Strategy: We define *H* in the following way:

•
$$H_1 = 1$$
, thus $H_1 \in \mathcal{F}_0$.
• $H_n = 2 H_{n-1} \mathbf{1}_{(\xi_{n-1}=-1)}$
Let $N = \inf\{j \ge 1; \xi_j = 1\}$. Then

$$[H \cdot X]_N = \sum_{j=1}^N H_j \, \Delta X_j = \sum_{j=1}^N H_j \, \xi_j = -\sum_{j=1}^{N-1} 2^{j-1} + 2^{N-1} = 1$$

We get an almost sure gain!

Strategies and martingales

Theorem 6.

Let

- X a martingale.
- *H* a predictable process such that $H_j \Delta X_j \in L^1$ for all *j*.

Then $H \cdot X$ is a martingale.

Interpretation: One cannot win in a fair game context \hookrightarrow Compare with d'Alembert's martingale

Proof

Main ingredients: We write

$$[H \cdot X]_{n+1} = [H \cdot X]_n + H_{n+1} (X_{n+1} - X_n).$$

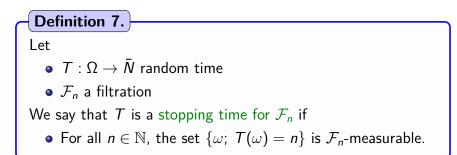
Then we use the fact that

- H is predictable
- X is a martingale

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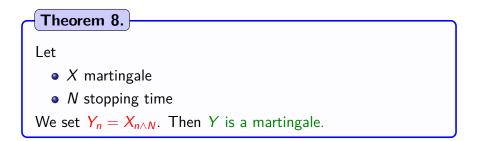
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Stopping time



Recall: basic examples are hitting times.

Stopped martingales



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Proof

Decomposition of Y: We have

$$Y_j - Y_{j-1} = (X_j - X_{j-1}) \mathbf{1}_{(j-1 < N)}.$$

Expression as transformed martingale: Set $H_j = \mathbf{1}_{(j-1 < N)}$. Then

$$Y_n = Y_0 + \sum_{j=1}^n (Y_j - Y_{j-1})$$

= $Y_0 + \sum_{j=1}^n (X_j - X_{j-1}) \mathbf{1}_{(j-1 < N)}$
= $Y_0 + \sum_{j=1}^n H_j \Delta X_j$

In addition H is predictable. Thus Y is a martingale.

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Convergence in L^2

Theorem 9.

Let X such that

- $\{X_n; n \ge 1\}$ is a martingale.
- For all *n* we have $X_n \in L^2(\Omega)$ and

$$\sup\left\{\mathbf{E}[X_n^2]; \ n \ge 0\right\} \equiv M < \infty.$$
(2)

Then

Proof

Step 1: We set $a_n = \mathbf{E}[X_n^2]$. We will show that if $n \ge m$, then

$$\mathbf{E}\left[(X_n-X_m)^2\right]=a_n-a_m.$$

Indeed,

$$\mathbf{E}[X_m X_n] = \mathbf{E} \left\{ X_m \, \mathbf{E}[X_n | \, \mathcal{F}_m] \right\} = \mathbf{E} \left[X_m^2 \right].$$

Therefore

$$\mathbf{E} \begin{bmatrix} (X_n - X_m)^2 \end{bmatrix} = \mathbf{E} \begin{bmatrix} X_n^2 \end{bmatrix} + \mathbf{E} \begin{bmatrix} X_m^2 \end{bmatrix} - 2 \mathbf{E} [X_m X_n]$$

= $\mathbf{E} \begin{bmatrix} X_n^2 \end{bmatrix} - \mathbf{E} \begin{bmatrix} X_m^2 \end{bmatrix}$
= $a_n - a_m.$

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Proof (2)

Step 2: Convergence in L^2 .

•
$$a_{n+1} - a_n = \mathbf{E}[(X_{n+1} - X_n)^2] \Longrightarrow n \mapsto a_n$$
 increasing.

- Inequality (2) \implies $(a_n)_{n\geq 0}$ bounded \implies $(a_n)_{n\geq 0}$ convergent.
- $\mathbf{E}[(X_n X_m)^2] = a_n a_m \Longrightarrow (X_n)_{n \ge 0}$ Cauchy in $L^2(\Omega)$

Conclusion: $(X_n)_{n\geq 0}$ converges in $L^2(\Omega)$ towards X_{∞} .

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Proof (3) Step 3: We have $X_n = \mathbf{E}[X_{\infty} | \mathcal{F}_n]$.

Set

$$V = |\mathbf{E}[X_{\infty}|\mathcal{F}_n] - X_n|.$$

We are reduced to show that E[V] = 0.

Computation: For $n, k \ge 0$,

$$V = |\mathbf{E}[X_{\infty}|\mathcal{F}_n] - \mathbf{E}[X_{n+k}|\mathcal{F}_n]|$$

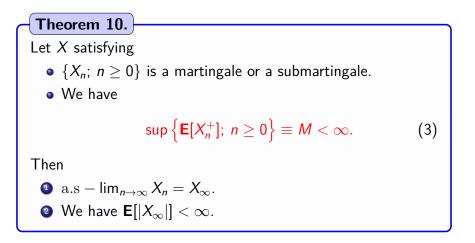
= $|\mathbf{E}[X_{\infty} - X_{n+k}|\mathcal{F}_n]| \le \mathbf{E}[|X_{\infty} - X_{n+k}||\mathcal{F}_n]$

Hence

$$\mathsf{E}[V] \leq \mathsf{E}\left[|X_{\infty} - X_{n+k}|\right] \leq \mathsf{E}^{1/2}\left[\left(X_{\infty} - X_{n+k}
ight)^2
ight]$$

We get $\mathbf{E}[V] = 0$ whenever $k \to \infty$ above.

Almost sure convergence



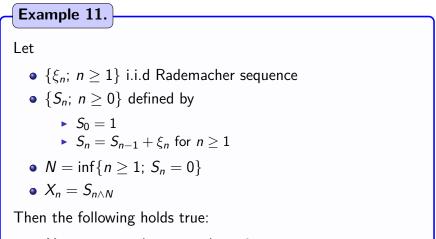
Particular cases

Particular case 1:

 $(X_n)_{n\geq 0}$ positive martingale \Longrightarrow a.s $-\lim_{n\to\infty} X_n = X_{\infty}$.

Particular case 2: $\sup \{ \mathbf{E}[X_n^2]; n \ge 0 \} \equiv M < \infty \implies \text{a.s} - \lim_{n \to \infty} X_n = X_{\infty}.$ \hookrightarrow We have both a.s and L^2 convergence.

Convergence counterexample



- X_n converges almost surely to 0
- **2** X_n does not converge in $L^1(\Omega)$

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Proof

Almost sure convergence: We have

- Theorem 8 \implies X is a martingale
- $X_n \geq 0$

Thus X_n converges almost surely to $X_\infty \ge 0$

Identification of the limit: Assume $\mathbf{P}(\Omega_k) > 0$ with k > 0 and

$$\Omega_k = \left\{ \omega; \lim_{n \to \infty} X_n(\omega) = k \right\}.$$

For $\omega \in \Omega_k$, we have the following:

• Set $n_0(\omega) = \inf\{n \ge 0; X_m(\omega) = k \text{ for } m \ge n\}.$

• For $m \ge n_0$ we have $X_{m+1} = X_m \pm 1$

This yields a contradiction. Hence $\mathbf{P}(\Omega_k) = 0$ and $X_{\infty} = 0$ a.s



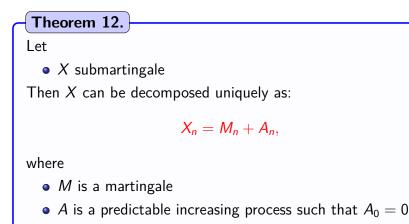
Convergence in $L^1(\Omega)$: According to (1) we have $\mathbf{E}[X_n] = \mathbf{E}[X_0] = 1$

Thus we cannot have $L^1(\Omega) - \lim_{n \to \infty} X_n = 0$

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Doob's decomposition



Proof

Expression for M and A: We wish

Therefore if $X_n = M_n + A_n$ we have

$$\mathbf{E} [X_n | \mathcal{F}_{n-1}] = \mathbf{E} [M_n | \mathcal{F}_{n-1}] + \mathbf{E} [A_n | \mathcal{F}_{n-1}] = M_{n-1} + A_n = X_{n-1} + A_n - A_{n-1}$$

We thus take:

 $A_n - A_{n-1} = \mathbf{E} [X_n | \mathcal{F}_{n-1}] - X_{n-1}, \text{ and } M_n = X_n - A_n$

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Proof (2)

Expression for A and M: recall that

 $A_n - A_{n-1} = \mathbf{E} [X_n | \mathcal{F}_{n-1}] - X_{n-1}, \text{ and } M_n = X_n - A_n$

Proof of Doob's properties: We have

- **1** A_n is increasing since **E** $[X_n | \mathcal{F}_{n-1}] \ge X_{n-1}$
- **2** $A_n \in \mathcal{F}_{n-1}$ by induction
- The martingale property for M is obtained as follows:

$$\mathbf{E} [M_n | \mathcal{F}_{n-1}] = \mathbf{E} [X_n - A_n | \mathcal{F}_{n-1}]$$

= $\mathbf{E} [X_n | \mathcal{F}_{n-1}] - A_n$
= $A_n - A_{n-1} + X_n - A_n$
= M_{n-1}

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Doob's inequality

Theorem 13.

We consider

- A submartingale X
- $\bar{X}_n \equiv \max_{m \le n} X_m^+$
- A real number $\lambda > 0$
- The set $A = \{\bar{X}_n \ge \lambda\}$

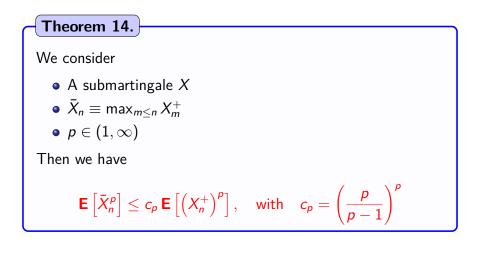
Then we have

$$\lambda \mathbf{P}(A) \leq \mathbf{E}[X_n \mathbf{1}_A] \leq \mathbf{E}[X_n^+]$$

Otherwise stated:

$$\mathbf{P}\left(\bar{X}_n \ge \lambda\right) \le \frac{\mathbf{E}\left[X_n^+\right]}{\lambda}$$

L^p maximum inequality



L^p bound for |Y|

Theorem 15.

We consider

- A martingale Y
- Y_n^{*} ≡ max_{m≤n} |Y_m|
 p ∈ (1,∞)

Then we have

$$\mathsf{E}\left[|Y_n^*|^p
ight] \leq c_p \, \mathsf{E}\left[|Y_n|^p
ight], \quad ext{with} \quad c_p = \left(rac{p}{p-1}
ight)^p$$

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Counterexample in $L^1(\Omega)$

Example 16.

As in Example 11, set

• $\{\xi_n; n \ge 1\}$ i.i.d Rademacher sequence

•
$$\{S_n; n \ge 0\}$$
 defined by

•
$$S_0 = 1$$

• $S_n = S_{n-1} + \xi_n$ for $n \ge 1$

•
$$N = \inf\{n \ge 1; S_n = 0\}$$

• $X_n = S_{n \land N}$

Then Theorem 14 is not satisfied for p = 1 and X:

I
$$\lim_{n\to\infty} \mathbf{E}[X_n] = 1$$

 $\mathbf{E}[\bar{X}_{\infty}] = \infty$

Item 1: We have already seen in Example 11 that

$$\mathbf{E}[X_n] = 1, \quad \text{for all} \quad n \ge 0.$$

Hence we trivially have

 $\lim_{n\to\infty}\mathbf{E}[X_n]=1.$

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Proof (2) Hitting times: For $x \in \mathbb{Z}$ set

$$T_x = \inf\{n \ge 0; S_n = x\}.$$

Then for a < 1 < b we have (see Section 5):

$$\mathbf{P}(T_b < T_a) = \frac{1-a}{b-a}.$$

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Item 2: Thanks to (4) we have, for all M > 1

$$\mathbf{P}\left(ar{X}_{\infty}\geq M
ight)=\mathbf{P}\left(T_{M}< T_{0}
ight)=rac{1}{M}.$$

Therefore

$$\mathbf{E}\left[\bar{X}_{\infty}\right] = \sum_{M=1}^{\infty} \frac{1}{M} = \infty.$$

Image: A matrix

Convergence in L^p

Theorem 17.

Let X and p such that

•
$$\{X_n; n \ge 1\}$$
 is a martingale.

- *p* > 1.
- For all n we have $X_n \in L^p(\Omega)$ and

 $\sup \{\mathbf{E}[|X_n|^p]; n \ge 0\} \equiv M < \infty.$

Then

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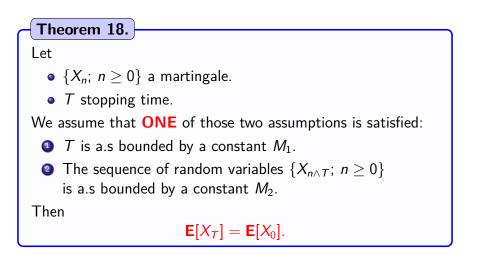
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Simple optional stopping theorem



Under Hypothesis 1: Let $\kappa \in \mathbb{N}$ such that $T \leq \kappa$ a.s.

Computation: We use the fact that $\{X_{n \wedge T}; n \leq \kappa\}$ is a martingale.

$$\mathsf{E}[X_0] = \mathsf{E}[X_{0 \wedge T}] = \mathsf{E}[X_{n \wedge T}] = \mathsf{E}[X_{\kappa \wedge T}] = \mathsf{E}[X_T]$$

Under Hypothesis 2: We set $Y_n = X_{n \wedge T}$. Then

•
$$(Y_n)_{n\geq 0}$$
 bounded martingale in L^2
 $\implies Y_n \to Y_\infty$ in L^2 and a.s. Hence $\mathbf{E}[Y_\infty] = \mathbf{E}[Y_0]$.

• We have $Y_{\infty} = X_T$ and $Y_0 = X_0$. Therefore

$$\mathbf{E}[Y_{\infty}] = \mathbf{E}[Y_0] \implies \mathbf{E}[X_T] = \mathbf{E}[X_0].$$

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Case of a submartingale

Proposition 19.

Let

- $\{X_n; n \ge 0\}$ a submartingale.
- T stopping time.

We assume that:

• T is a.s bounded by a constant M.

Then

 $\mathbf{E}[X_0] \leq \mathbf{E}[X_T] \leq \mathbf{E}[X_M].$

$L^{p}(\Omega)$ bound for stopped martingales

Proposition 20.

Let X and p such that

- $\{X_n; n \ge 1\}$ is a submartingale.
- *p* > 1.
- For all n we have $X_n \in L^p(\Omega)$ and

$$\sup \{\mathbf{E}[|X_n|^p]; n \ge 0\} \equiv M < \infty.$$

For a stopping time N we set

$$Y_n=X_{n\wedge N}.$$

Then

$$\sup \{\mathbf{E}[|Y_n|^p]; n \ge 0\} \le M.$$

Definition of a submartingale: If X_n is a submartingale then

 $|X_n|^p$ is a submartingale

Application of Proposition 19:

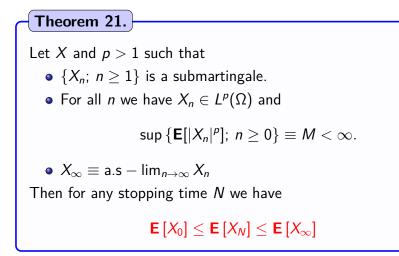
Since $N \wedge n$ is a stopping time bounded by n, we have

$$\mathbf{E}\left[|X_{N\wedge n}|^{p}\right] \leq \mathbf{E}\left[|X_{n}|^{p}\right],$$

and hence

$$\sup_{n\geq 0} \mathsf{E}\left[|X_{N\wedge n}|^{p}\right] \leq \sup_{n\geq 0} \mathsf{E}\left[|X_{n}|^{p}\right] = M$$

Optional stopping in $L^p(\Omega)$



Application of Proposition 19: for $n \ge 1$ we have

$$\mathsf{E}\left[X_{0}\right] \leq \mathsf{E}\left[X_{N \wedge n}\right] \leq \mathsf{E}\left[X_{n}\right]$$

Application of Proposition 20:

 $n \mapsto X_{N \wedge n}$ and $n \mapsto X_n$ are bounded submartingales in $L^p(\Omega)$. Thus:

$$X_{n \wedge N} \xrightarrow{\text{a.s. } L^p} X_N$$
, and $X_n \xrightarrow{\text{a.s. } L^p} X_\infty$.

Therefore taking limits in (5) we get:

$\mathsf{E}\left[X_{0}\right] \leq \mathsf{E}\left[X_{N}\right] \leq \mathsf{E}\left[X_{\infty}\right]$

Image: A matrix and a matrix

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(5)

Optional sampling: the general form

Theorem 22.

Let X, p > 1 and two stopping times M, N such that

- $\{X_n; n \ge 1\}$ is a submartingale.
- For all *n* we have $X_{n \wedge N} \in L^p(\Omega)$ and

$$\sup \{\mathbf{E}[|X_{n \wedge N}|^{p}]; n \ge 0\} \equiv A < \infty.$$

• $M \leq N$ almost surely

Then we have

 $\mathbf{E}[X_M] \leq \mathbf{E}[X_N]$, and $X_M \leq \mathbf{E}[X_N | \mathcal{F}_M]$.

Proof of $\mathbf{E}[X_M] \leq \mathbf{E}[X_N]$: Set $Y_n = X_{n \wedge N}$. Then

• Y is a submartingale satisfying the assumptions of Theorem 21 • $Y_{\infty} = X_N$

Invoking Theorem 21 we thus get

 $\mathsf{E}[Y_M] \leq \mathsf{E}[Y_\infty] \quad \Longleftrightarrow \quad \mathsf{E}[X_M] \leq \mathsf{E}[X_N].$

Proof (2)

Definition of a stopping time: For $A \in \mathcal{F}_M$ we set

$$T = M \mathbf{1}_A + N \mathbf{1}_{A^c}.$$

Then T is a stopping time. Indeed:

$$\{T \le n\} = (\{M \le n\} \cap A) \bigcup (\{N \le n\} \cap A^c)$$

= $(\{M \le n\} \cap A) \bigcup (\{N \le n\} \cap \{M \le n\} \cap A^c),$

and hence:

 $\{T \leq n\} \in \mathcal{F}_n$

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Image: A matrix

Proof (3)

Inequality involving A: For A, T as before, applying $\mathbf{E}[X_T] \leq \mathbf{E}[X_N]$ we get

$$\mathbf{E}[X_T] \leq \mathbf{E}[X_N]$$

$$\iff \mathbf{E}[X_M \mathbf{1}_A] + \mathbf{E}[X_N \mathbf{1}_{A^c}] \leq \mathbf{E}[X_N \mathbf{1}_A] + \mathbf{E}[X_N \mathbf{1}_{A^c}]$$

$$\iff \mathbf{E}[X_M \mathbf{1}_A] \leq \mathbf{E}[X_N \mathbf{1}_A]$$

Therefore, by definition of the conditional expectation we get:

$$\mathbf{E}[X_M \mathbf{1}_A] \le \mathbf{E}\{\mathbf{E}[X_N | \mathcal{F}_M] \mathbf{1}_A\}$$
(6)

Image: A matrix

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Proof (4)

Conclusion: For $k \ge 1$ we set

$$A_k = \left\{X_M - \operatorname{\mathsf{E}}\left[X_N \middle| \mathcal{F}_M
ight] \geq rac{1}{k}
ight\}.$$

Then $A_k \in \mathcal{F}_M$ and according to (6) we have

$$\mathbf{P}(A_k)=0.$$

Hence:

$$\mathbf{P}\left(X_{M}-\mathbf{E}\left[X_{N}|\,\mathcal{F}_{M}
ight]>0
ight)=\mathbf{P}\left(\cup_{k\geq1}\mathcal{A}_{k}
ight)=0$$

and thus:

$$X_M \leq \mathbf{E}\left[X_N | \mathcal{F}_M\right], \quad \text{almost surely.}$$

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