MA 539 - PROBLEM LIST

DISCRETE TIME PROCESSES AND BROWNIAN MOTION

1. Gaussian vectors and CLT

Problem 1. Let $\gamma_{a,b}$ be the function:

$$\gamma_{a,b}(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} 1_{\{x>0\}},$$

where $a, b > 0$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

1.1. Show that $\gamma_{a,b}$ is a density.

1.2. Let X a random variable with density $\gamma_{a,b}$. Check, for $\lambda > 0$:

$$\mathbf{E}[e^{-\lambda X}] = \frac{1}{(1+\lambda b)^a}, \qquad \mathbf{E}[X] = ab, \qquad VarX = ab^2.$$

1.3. Let X (resp. X') a random variable with density $\gamma_{a,b}$ (resp. $\gamma_{a',b}$). We assume X and X' independent. Show that $X + X'$ admits the density $\gamma_{a+a',b}$.

1.4. Application: Let $X_1, X_2, ..., X_n, n$ i.i.d random variables, with law $\mathcal{N}(0,1)$. Show that $X_1^2 + X_2^2 + ... + X_n^2$ is Gamma distributed.

Problem 2. Let X be a random variable distributed as $\mathcal{N}_1(m, \sigma^2)$.

2.1. Assume $m = 0$. We set $Y = e^{\alpha X^2}$ with $\alpha \neq 0$. Compute $E[Y^n]$.

2.2. Find the density of $|X - 1|$ when $m = 2$ and $\sigma = 1$.

Problem 3. Let ϵ be a Rademacher random variable, that is:

$$\mathbf{P}(\epsilon = 1) = \mathbf{P}(\epsilon = -1) = 1/2.$$

Assume that ϵ is independent of X, where $X \sim \mathcal{N}_1(0,1)$.

3.1. Show that the law of ϵX is still Gaussian.

3.2. Show that $X + \varepsilon X$ is not a Gaussian variable. Deduce that the random vector $(X, \varepsilon X)$ is not a Gaussian vector.

Problem 4. Let X, Y be two independent standard Gaussian random variables.

4.1. Show that $\frac{X}{Y}$ is well-defined, and is distributed according to a Cauchy law.

4.2. If $t \geq 0$, compute $\mathbf{P}(|X| \leq t|Y|)$.

Problem 5. If (X, Y) is a centered Gaussian vector in $\mathbb{R}^2$ with $\mathbf{E}[X^2] = \mathbf{E}[Y^2] = 1$ and if $\mathbf{E}[XY] = r$ with $r \in (-1, 1)$, compute $\mathbf{P}(XY \geq 0)$. *Hint:* one can prove and use the following claim: $(X, Y) = (X,\, sX + \sqrt{1 - s^2}Z)$ with $X, Z \sim \mathcal{N}(0, 1)$ independent and $s \in (0, 1)$ to be determined. Then we invoke the result shown in Problem 4.

Problem 6. Let $X, Y \sim \mathcal{N}(0, 1)$ be two independent random variables. For all $a \in (-1, 1)$, show that:

$$\mathbf{E}\left[\exp\left(aXY\right)\right] = \mathbf{E}\left[\exp\left(\frac{a}{2}X^2\right)\right] \mathbf{E}\left[\exp\left(-\frac{a}{2}Y^2\right)\right].$$

Problem 7. Let X and Y two independent standard Gaussian random variables $\mathcal{N}(0, 1)$. We set $U = X^2 + Y^2$ and $V = \frac{X}{\sqrt{U}}$. Show that U and V are independent, and compute their law.

Problem 8. Let A be the matrix defined by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

8.1. Show that there exist a centered Gaussian vector G with covariance matrix A. The coordinates of G are denoted by X, Y and Z.

8.2. Is G a random variable with density? Compute the characteristic function of G.

8.3. Characterize the law of $U = X + Y + Z$.

8.4. Show that $(X - Y, X + Z)$ is a Gaussian vector.

8.5. Determine the set of random variables $\xi = aX + bY + cZ$, independent of U.

Problem 9. Let Q be a positive definite quadratic form defined on $\mathbb{R}^n$. We introduce a function f given as

$$f(x) = \lambda \exp\left(-\frac{Q(x)}{2}\right), \qquad x \in \mathbb{R}^n.$$

9.1. According to Q, compute the unique value λ such that f is a density. *Hint:* show that f can be seen as the density of a Gaussian vector.

9.2. Application: $n = 2$, $Q(x, y) = 3x^2 + y^2 + 2xy$.

Problem 10. Let $X = (X_1, \ldots, X_n)$ be a centered Gaussian vector with covariance matrix Id_n.

10.1. Show that the random vector $(X_1 - \bar{X}, \ldots, X_n - \bar{X})^*$ is independent of $\bar{X}$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

10.2. Deduce that the random variables $\bar{X}$ and $W = \max_{1\leq i\leq n} X_i - \min_{1\leq i\leq n} X_i$ are independent. Why is this result (somewhat) surprising?

Problem 11. A restaurant can serve 75 meals. In practice, it has been established that 20% of customers with a reservation do not show up.

11.1. The restaurant owner has accepted 90 reservations. What is the probability that more than 65 persons will come?

11.2. What is the maximal number of reservations which can be accepted if we wish to serve all customers with probability ≥ 0.9?

Problem 12. Let $(X_n; n \geq 1)$ be a sequence of i.i.d $\mathbb{R}^k$-valued random variables, which are assumed to be square integrable. In the sequel $(\xi_n; n \geq 1)$ designates a sequence of i.i.d bounded real-valued random variables. We assume that $(X_n; n \geq 1)$ is independent of $(\xi_n; n \geq 1)$ and also that either X_1 or ξ_1 is centered. We set

$$Y_n = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \xi_i X_i.$$

Show that Y_n converges in distribution as n goes to infinity. Characterize the limiting law.

Problem 13. The aim of this problem is to show that the Laplace transform characterizes probability laws on $\mathbb{R}_+$. To this aim, for all $t > 0$ and $x > 0$ we set,

$$a_n(x,t) = \int_0^{n/t} \frac{y^{n-1} x^n}{(n-1)!} e^{-yx} dy.$$

13.1. Invoking the law of large numbers (resp. central limit theorem), show that

$$\lim_{n\to\infty} a_n(x,t) = \begin{cases} \mathbf{1}_{\{x>t\}} & \text{if } x \neq t \\ \frac{1}{2} & \text{if } x = t. \end{cases}$$

13.2. Let X be a random variable with values in $\mathbb{R}_+$. We set $G(\theta) = E[e^{-\theta X}]$.

(1) Using Question 13.1, show that:

$$\lim_{n\to\infty} (-1)^n \int_0^{n/t} \frac{y^{n-1}}{(n-1)!} \frac{d^n G}{dy^n}(y) dy = \frac{1}{2} P(X = t) + P(X > t).$$

(2) Deduce that G characterizes the distribution of X.

Problem 14. Let X and Y two real valued i.i.d random variables. We assume that $\frac{X+Y}{\sqrt{2}}$ has the same law as X and Y. We also suppose that this common law admits a variance, denoted by σ^2.

14.1. Show that X is centered random variable.

14.2. Show that if X_1, X_2, Y_1 and Y_2 are independent random variables having the same law as X, then $\frac{1}{2}(X_1 + X_2 + Y_1 + Y_2)$ has the same law as X.

14.3. Applying the central limit theorem, show that X is distributed as $\mathcal{N}(0, \sigma^2)$.

Problem 15. The aim of this problem is to give an example of application for the multidimensional central limit theorem. Let $(Y_i;\, i \geq 1)$ be a sequence of i.i.d real valued random variables. We will denote by F common cumulative distribution function and $\hat{F}_n$ the empirical cumulative distribution function for the n-sample $(Y_1, \ldots, Y_n)$:

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{Y_i \leq x\}}, \quad x \in \mathbb{R}.$$

15.1. Let x a fixed real number. Show that:

- $\hat{F}_n(x)$ converges a.s. to $F(x)$, when $n \to \infty$;
- $\sqrt{n}(\hat{F}_n(x) - F(x))$ converges in law, when $n \to \infty$, to a centered Gaussian random variable with variance $F(x)(1 - F(x))$.

15.2. We will generalize this result to a multidimensional setting. Let $x_1, x_2, ..., x_d$ be a sequence of real numbers such that $x_1 < x_2 < ... < x_d$, and X_n be the random vector in $\mathbb{R}^d$, with coordinates $X_n^{(1)}, X_n^{(2)}, \cdots, X_n^{(d)}$ where:

$$X_n^{(i)} = \mathbf{1}_{\{Y_n \leq x_i\}}; \quad 1 \leq i \leq d,$$

for all $n \geq 1$. Show that:

$$\left(\sqrt{n}(F_n(x_1) - F(x_1)), \ldots, \sqrt{n}(F_n(x_d) - F(x_d))\right)$$

converges in law, when $n \to \infty$, to a centered Gaussian vector for which we will compute the covariance matrix.

2. Conditional expectation

Problem 16. Let X_1 and X_2 two independent random variables, both following a Poisson law with parameter λ. Let $Y = X_1 + X_2$. Compute

$$\mathbf{P}(X_1 = i|Y).$$

Problem 17. Let (X, Y) be a vector of $\mathbb{R}^2$, distributed uniformly over the unit disc. Compute the conditional density of X given Y.

Problem 18. Let (X, Y) be a couple of random variables with joint density

$$f(x, y) = 4y(x - y)\exp(-(x + y))\mathbf{1}_{0 \leq y \leq x}.$$

18.1. Compute $\mathbf{E}[X|Y]$.

18.2. Compute $\mathbf{P}(X < 1|Y)$.

Problem 19. We consider a head or tail type game, where the probability of getting head (resp. tail) is p (resp. $1 - p$), with $0 < p < 1$. Player A throws the dice. He wins as soon as the number of heads exceeds the number of tails by a quantity of 2. He looses if the number of tails exceeds the number of heads by a quantity of 2. The game is stopped whenever A has won or lost.

19.1. Let E_n be the event: "the game is not over after $2n$ throws", $n \geq 1$. Show that $\mathbf{P}(E_n) = r^n$ where r is a real number to be determined.

19.2. Compute the probability that the player A wins and show that the game will stop a.s.

Problem 20. Let N be a random variable with values in $\{0, 1, \ldots, n\}$; we denote by $\alpha_k = \mathbf{P}(N = k)$. We consider a sequence $(\epsilon_n; n \geq 0)$ of independent random variables, whose common law is given by $\mathbf{P}(\epsilon_0 = 1) = p; \mathbf{P}(\epsilon_0 = 0) = q$, with $p + q = 1, p > 0, q > 0$. We assume that N is independent of the family $(\epsilon_n; n \geq 1)$. We define a random variable X by the relation: $X = \sum_{k=1}^{N} \epsilon_k$.

20.1. Compute the law of X. Express

- $\mathbf{E}[X]$ in terms of $\mathbf{E}[N]$.
- $\mathbf{E}[X^2]$ in terms of $\mathbf{E}[N]$ and $\mathbf{E}[N^2]$.

20.2. Let $p' \in]0, 1[$. Determine the law of N if we wish the conditional law of N given $X = 0$ to be a binomial law $\mathcal{B}(n, p')$.

Problem 21. We consider the relations:

$$\mathbf{P}(X = 0) = \frac{1}{3}; \qquad \mathbf{P}(X = 2^n) = \mathbf{P}(X = -2^n) = \frac{2^{-n}}{3}; \qquad \forall n \geq 1. \tag{1}$$

21.1. Show that the relations (1) define a probability law for a random variable X.

21.2. Consider the following transition probability:

$$Q(0, .) = \frac{1}{2}(\delta_2 + \delta_{-2}), \qquad Q(x, .) = \frac{1}{2}(\delta_0 + \delta_{2x}), \qquad x \in \mathbb{R}^*,$$

where δ_a designates a Dirac measure in a. Let Y be a second real valued random variable such that the conditional law of Y given X is given by the transition probability Q. Show that $\mathbf{E}(Y|X) = X$ and that Y and X share the same law.

Problem 22. We note $\mathcal{B}_n$ the set of Borel sets of $\mathbb{R}^n$, and let $\mathcal{S}_n$ be the set of symmetric Borel sets A of $\mathbb{R}^n$, i.e. $-A = A$.

22.1. Show that $\mathcal{S}_n$ is a sub σ-algebra of $\mathcal{B}_n$ and that a random variable Y is $\mathcal{S}_n$-measurable if and only if $Y(-x) = Y(x)$.

22.2. We say that a probability measure P over $(\mathbb{R}^n, \mathcal{B}_n)$ is is symmetric if $\mathbf{P}(A) = \mathbf{P}(-A)$ for all A lying in $\mathcal{B}_n$. Show that if ϕ is a real valued integrable random variable defined on $(\mathbb{R}^n, \mathcal{B}_n, P)$, we have: $\mathbf{E}[\phi|\mathcal{S}_n](x) = \frac{1}{2}(\phi(x) + \phi(-x))$.

22.3. We assume $n = 1$ and we denote by X the identity application of $\mathbb{R}$ onto $\mathbb{R}$. Determine $\mathbf{E}[\phi||X|]$ and $\mathbf{E}[\phi|X^2]$.

Problem 23. Let X and Y two real valued and independent random variables, with uniform law on $[0, 1]$. We set $U = \inf\{X, Y\}$ and $V = \sup\{X, Y\}$. Compute $\mathbf{E}[U|V]$ and the best prediction of U by an affine function of V.

Problem 24. Let $G \in \mathcal{G}$. Show that

$$\mathbf{P}(G|A) = \frac{\int_G \mathbf{P}(A|\mathcal{G})dP}{\int_\Omega \mathbf{P}(A|\mathcal{G})dP}.$$

This can be seen as a general version of Baye's formula.

Problem 25. Let X and Y be two random variables such that $X \leq Y$, and $a > 0$.

25.1. Show that $\mathbf{E}[X|\mathcal{F}] \leq \mathbf{E}[Y|\mathcal{F}]$.

25.2. Show that

$$\mathbf{P}(|X| \geq a|\mathcal{F}) \leq \frac{\mathbf{E}[X^2|\mathcal{F}]}{a^2}.$$

Problem 26. Let X_1 and X_2 be two random variables such that $X_1 = X_2$ on $B \in \mathcal{F}$. Show that

$$\mathbf{E}[X_1|\mathcal{F}] = \mathbf{E}[X_2|\mathcal{F}] \qquad \text{a.s on } B.$$

Problem 27. Give an example on $\Omega = \{a, b, c\}$ for which

$$\mathbf{E}\left[\mathbf{E}[X|\mathcal{F}_1]|\mathcal{F}_2\right] \neq \mathbf{E}\left[\mathbf{E}[X|\mathcal{F}_2]|\mathcal{F}_1\right].$$

Problem 28. Let X and Y be two random variables.

28.1. Show that if X and Y are independent, then $\mathbf{E}[X|Y] = \mathbf{E}[X]$.

28.2. Give an example of random variables with values in $\{-1, 0, 1\}$ such that X and Y are not independent, in spite of the fact that $\mathbf{E}[X|Y] = \mathbf{E}[X]$.

Problem 29. Let $n \geq 1$ be a fixed integer and p_1, p_2, p_3 three positive real numbers satisfying $p_1 + p_2 + p_3 = 1$. We set:

$$p_{i,j} = n! \; \frac{p_1^i \; p_2^j \; p_3^{n-i-j}}{i! \; j! \; (n-i-j)!}$$

whenever $i + j \leq n$, and $p_{i,j} = 0$ if $i + j > n$.

29.1. Show that there exists a couple of random variables (X, Y) such that $\mathbf{P}(X = i, Y = j) = p_{i,j}$.

29.2. Determine the law of X, the law of Y and the law of Y given X, expressed as a conditional regular law.

29.3. Compute $\mathbf{E}[XY]$ thanks to the conditional regular law introduced in the previous question 29.2.

Problem 30. Let $X_1, X_2, \ldots, X_n$, n be some real valued integrable random variables, independent and equally distributed. We set $m = \mathbf{E}[X_1]$ and $S_n = \sum_{i=1}^n X_i$.

30.1. Compute $\mathbf{E}[S_n|X_i]$ for all $i, 1 \leq i \leq n$.

30.2. Compute $\mathbf{E}[X_i|S_n]$ for all $i, 1 \leq i \leq n$.

30.3. We assume now that $n = 2$ and that the random variables X_i have a common density φ. Compute the conditional density of X_i given S_2. Give a specific expression whenever the law of each X_i is an exponential law.

Problem 31. The random vector (X, Y) has a density

$$f_{X,Y}(x, y) = \frac{x}{\sqrt{2\pi}} \exp(-\frac{1}{2}x(y + x))\mathbf{1}_{\{x>0\}}\mathbf{1}_{\{y>0\}}.$$

Determine the conditional distribution of Y given X, expressed as a conditional regular law.

Problem 32. Let X and Y be two real valued random variables, such that Y follows an exponential law. We assume that given Y, X is distributed according to a Poisson law with parameter Y (given as a conditional regular law).

32.1. Compute the law of the couple (X, Y), the law of X, and the law of Y given X as a conditional regular law.

32.2. Show that $\mathbf{E}[(Y - X)^2] = 1$, conditioning first with respect to Y, then integrating with respect to Y.

Problem 33. Let $X_1, X_2, \dots, X_n$ be n real valued independent random variables, admitting a common density p.

33.1. Show that for all $i \neq j, \mathbf{P}(X_i = X_j) = 0$. In the following we set $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ for the sequence $\{X_1, X_2, \dots, X_n\}$ arranged in increasing order:

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}.$$

33.2. Show that $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ admits a density of the form:

$$f(x_1, x_2, \dots, x_n) = n!\ p(x_1)p(x_2)\cdots p(x_n)\mathbf{1}_{\{x_1<x_2<\cdots<x_n\}}$$

33.3. We assume now that the common law of the random variables X_i is the uniform law on $[a, b]$.

(1) Determine the density of $(X_{(1)}, X_{(n)})$.
(2) We set $\mu_n(a, b; x_1, x_2, \dots, x_n)$ for the density of $X_{(1)}, X_{(2)}, \dots, X_{(n)}$. Show that conditionally on $X_{(1)}$ and $X_{(n)}$ the vector $X_{(2)}, X_{(3)}, \dots, X_{(n-1)}$ admits the CRL given by $\mu_{n-2}(X_1, X_n; x_2, x_3, \dots, x_{n-1})$. Deduce that

$$\left(\frac{X_{(2)} - X_{(1)}}{X_{(n)} - X_{(1)}}, \dots, \frac{X_{(n-1)} - X_{(1)}}{X_{(n)} - X_{(1)}}\right)$$

is a random variable independent of $(X_{(1)}, X_{(n)})$ and possesses a density $\mu_{n-1}(0, 1; \cdot)$.

Problem 34. We consider a measurable space $(\Omega, \mathcal{F})$ and some families $\{\mathcal{A}_i; i \leq n\}$ of subsets of Ω such that $\mathcal{A}_i \subset \mathcal{F}$.

34.1. Show that if the $\mathcal{A}_i$ are independent and each one is a π-system, then the σ-algebras $\sigma(\mathcal{A}_i)$ are independent.

34.2. Let $\{X_i; i \leq n\}$ be a collection of real valued random variables. Show that these random variables are independent if and only if

$$\mathbf{P}(X_1 \leq x_1, \ldots, X_n \leq x_n) = \prod_{i \leq n} \mathbf{P}(X_i \leq x_i)$$

for any vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

3. Discrete time martingales

Problem 35. Let $\{X_n; n \geq 1\}$ be a martingale with respect to a filtration $\mathcal{G}_n$, and let

$$\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}.$$

Show that $\mathcal{F}_n \subset \mathcal{G}_n$ and that X_n is a $\mathcal{F}_n$-martingale.

Problem 36. We say that $f : \mathbb{R}^d \to \mathbb{R}$ is a super-harmonic function whenever $\Delta f \leq 0$. These functions satisfy:

$$f(x) \geq \int_{\partial B(x,r)} f(y) d\pi(y),$$

where $\partial B(x,r) = \{y; |x-y| = r\}$ is the boundary of the ball centered at x with radius r, and π is the surface measure of this boundary. Let $f \geq 0$ be a super-harmonic function on $\mathbb{R}^d$, and let $\{\xi_n; n \geq 1\}$ be a sequence of iid random variables, with common uniform law on $\partial B(0,1)$. We set $S_n = \xi_n + S_{n-1}$ and $S_0 = x$. Show that $X_n = f(S_n)$ is a supermartingale.

Problem 37. Let $\{\xi_n; n \geq 1\}$ be a sequence of independent random variables such that $\xi_j \in L^1(\Omega)$ and $\mathbf{E}[\xi_j] = 0$. We set

$$X_n = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \xi_{i_1} \cdots \xi_{i_k}.$$

Show that X is a martingale.

Problem 38. Let $\{X_n; n \geq 1\}$ and $\{Y_n; n \geq 1\}$ be two sub-martingales with respect to $\mathcal{F}_n$. Show that $X_n \vee Y_n$ is a sub-martingale.

Problem 39. Let $\{Y_n; n \geq 1\}$ be a iid sequence of positive random variables such that $\mathbf{E}[Y_j] = 1$, $\mathbf{P}(Y_j = 1) < 1$ and $\mathbf{P}(Y_j = 0) = 0$. We set

$$X_n = \Pi_{j \leq n} Y_j.$$

39.1. Show that X is a martingale.

39.2. Show that $\lim_{n \to \infty} X_n = 0$ a.s

Problem 40. We wish to study a branching process defined in the following way: let $\{\xi_i^n;\, i, n \geq 1\}$ be a sequence of iid random integer valued random variables. We set $Z_0 = 1$ and pour $n \geq 0$,

$$Z_{n+1} = \left(\sum_{i=1}^{Z_n} \xi_i^{n+1} \right) \mathbf{1}_{(Z_n > 0)}.$$

This process is called *Galton Watson process*, and represents the number of living individuals at each generation in various biological models. We set

$$\mathcal{F}_n = \sigma\{\xi_i^m;\, 1 \leq m \leq n,\, i \geq 1\},$$

and $\mu = \mathbf{E}[\xi_i^n]$.

40.1. Show that $\frac{Z_n}{\mu^n}$ is a $\mathcal{F}_n$-martingale.

40.2. Show that $\frac{Z_n}{\mu^n}$ converges a.s to a random variable Z_∞.

40.3. We assume now that $\mu < 1$.

(1) Show that $\mathbf{P}(Z_n > 0) \leq \mathbf{E}[Z_n]$.
(2) Show that Z_n converges in probability to 0.
(3) Show that $Z_n = 0$ for n large enough.

40.4. We assume now that $\mu = 1$ and $\mathbf{P}(\xi_i^n = 1) < 1$.

(1) Show that Z_n converges a.s to a random variable Z_∞.
(2) Suppose that $\mathbf{P}(Z_\infty = k) > 0$ for $k \geq 0$. Show that there exists $N > 0$ such that

$$\mathbf{P}(Z_n = k \text{ for all } n \geq N) > 0.$$

(3) Show that $\mathbf{P}(Z_n = k \text{ for all } n \geq N) = 0$ for all $k > 0$.
(4) Deduce that $Z_\infty = 0$.

40.5. Eventually we assume that $\mu > 1$, and we will show that

$$\mathbf{P}(Z_n > 0 \text{ for all } n \geq 0) > 0.$$

To this aim, we set $p_k = \mathbf{P}(\xi_i^n = k)$, and set

$$\phi(s) = \sum_{k=0}^{\infty} p_k s^k, \qquad s \in [0, 1].$$

Namely, ϕ is the moment generating function of ξ_i^n.

(1) Show that ϕ is increasing, convex, and that $\lim_{s \to 1} \phi'(s) = \mu$.
(2) Let $\theta_m = \mathbf{P}(Z_m = 0)$. Show that $\theta_m = \phi(\theta_{m-1})$.
(3) Invoking the fact that $\phi'(1) > 1$, show that there exists at least one root for the equation $\phi(x) = x$ in $[0, 1)$. Let ρ be the smallest of those roots.
(4) Show that ϕ is strictly convex, and deduce that ρ is the unique root of $\phi(x) = x$ in $[0, 1)$.
(5) Show that the extinction probability is such that

$$\mathbf{P}(Z_n = 0 \text{ for some } n \geq 0) = \rho < 1.$$

40.6. Galton and Watson were interested in family name survivals. Suppose that each family has exactly three children, and that the gender distribution is uniform. In 19th century England, only males could keep their family names. Compute the survival probability in this context.

Problem 41. Let $\{Y_n; n \geq 1\}$ be a sequence of independent random variables, with common Gaussian law $\mathcal{N}(0, \sigma^2)$, where $\sigma > 0$. We set $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$ and $X_n = \sum_{i=1}^n Y_i$. Recall that:

$$\mathbf{E}[\exp(uY_1)] = \exp\left(\frac{u^2\sigma^2}{2}\right).$$

We also set, for all $u \in \mathbb{R}^*$,

$$Z_n^u = \exp\left(uX_n - \frac{1}{2}nu^2\sigma^2\right).$$

41.1. Show that $\{Z_n^u; n \geq 1\}$ is a $\mathcal{F}_n$-martingale for all $u \in \mathbb{R}^*$.

41.2. We wish to study the almost sure convergence of Z_n^u for $u \in \mathbb{R}^*$.

(1) Show that for all $u \in \mathbb{R}^*$, Z_n^u converges almost surely.
(2) Show that

$$K_n \equiv \frac{1}{n}\left(uX_n - \frac{1}{2}nu^2\sigma^2\right)$$

converges almost surely, and determine its limit.
(3) Find the almost sure limit of Z_n^u for $u \in \mathbb{R}^*$.

41.3. We now study the L^1-convergence of Z_n^u, for $u \in \mathbb{R}^*$.

(1) Find $\lim_{n\to\infty} \mathbf{E}[Z_n^u]$.
(2) Is the martingale Z_n^u converging in L^1?

Problem 42. At time 1, an urn contains 1 green ball and 1 blue ball. A ball is drawn, and replaced by 2 balls of the same color as the one which has been drawn. This gives a new composition at time 2. This procedure is then repeated successively. We set Y_n for the number of green balls at time n, and write $X_n = \frac{Y_n}{n+1}$ for the proportion of green balls at time n. We also set $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

42.1. Show that $\mathbf{E}[Y_{n+1}|\mathcal{F}_n] = (Y_n + 1)X_n + Y_n(1 - X_n)$.

42.2. Show that $\{X_n; n \geq 1\}$ is a $\mathcal{F}_n$-martingale, which converges almost surely to a random variable U.

42.3. By means of the dominated convergence theorem, show that for all $k \geq 1$, we have $\lim_{n\to\infty} \mathbf{E}[X_n^k] = \mathbf{E}[U^k]$.

42.4. Fix $k \geq 1$. We set, for $n \geq 1$,

$$Z_n = \frac{Y_n(Y_n + 1)\dots(Y_n + k - 1)}{(n + 1)(n + 2)\dots(n + k)}.$$

(1) Let us define the random variables $\mathbf{1}_{\{Y_{n+1}=Y_n\}}$ and $\mathbf{1}_{\{Y_{n+1}=Y_n+1\}}$. Relying on these quantities, show that $\{Z_n; n \geq 1\}$ is a $\mathcal{F}_n$-martingale.

(2) Express the almost sure limit of Z_n as a function of the random variable U.
(3) Compute the value of $\mathbf{E}[U^k]$.
(4) Show that these moments are those of the law $\mathcal{U}([0,1])$.

Problem 43. Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with respect to a filtration $\mathcal{F}_n$. We assume that there exists a constant $M > 0$ such that for all $n \geq 1$

$$\mathbf{E}\left[|X_n - X_{n-1}| \,\middle|\, \mathcal{F}_{n-1}\right] \leq M \quad a.s.$$

43.1. Show that if $(V_n)_{n\geq 1}$ is a predictable (i.e V_n is $\mathcal{F}_{n-1}-$measurable) process taking positive values, then we have

$$\sum_{n=1}^{\infty} V_n \mathbf{E}\left[|X_n - X_{n-1}|\middle|\mathcal{F}_{n-1}\right] \leq M \sum_{n=1}^{\infty} V_n.$$

43.2. Let ν be an integrable stopping time. Show that X_ν is integrable, and that $X_{\nu\wedge p}$ converges to X_ν in L^1. Deduce that $\mathbf{E}(X_\nu) = \mathbf{E}(X_0)$.
Hint : write

$$X_\nu - X_{\nu\wedge p} = \sum_{n=1}^{\infty} \mathbf{1}_{\{\nu\wedge p < n \leq \nu\}} \left(X_n - X_{n-1}\right).$$

43.3. Show that if $\nu_1 \leq \nu_2$ are two stopping times with ν_2 integrable, then $\mathbf{E}[X_{\nu_2}] = \mathbf{E}[X_{\nu_1}]$.

Problem 44. Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables with a common law given by $\mathbf{P}(Y_n = 1) = p = 1 - \mathbf{P}(Y_n = -1) = 1 - q$. We define $(S_n)_{n\in\mathbb{N}}$ by $S_0 = 0$ and $S_n = \sum_{k=1}^n Y_k$.

44.1. We assume that $p = q = \frac{1}{2}$. We set $T_a = \inf\{n \geq 0,\ S_n = a\} \quad (a \in \mathbb{Z}^*)$. Show that $\mathbf{E}(T_a) = +\infty$.

44.2. Let $T = T_{a,b} = \inf\{n \geq 0,\ S_n = -a \text{ or } S_n = b\} \quad (a, b \in \mathbb{N})$. Using the value of $\mathbf{E}(S_T)$, compute the probability of the event $(S_T = -a)$.

44.3. Show that $Z_n = S_n^2 - n$ is a martingale, and from the value of $\mathbf{E}(Z_T)$ compute $\mathbf{E}(T)$.

44.4. We assume that $p > q$ and we set $\mu = \mathbf{E}(Y_k)$. Show that

$$X_n = S_n - n\mu \quad \text{and} \quad U_n = \left(\frac{q}{p}\right)^{S_n}$$

are martingales. Deduce the value of $\mathbf{P}(S_T = -a)$ and $\mathbf{E}(T)$.

4. Discrete models in finance

Problem 45. This problem is concerned with the Cox, Ross and Rubinstein model: we consider a unique risky asset whose price at time n is called R_n, as well as a non risky asset with price $S_n = (1 + r)^n$. We assume the following about R_n: between time n and $n + 1$ the relative variation of price is either a or b, with $-1 < a < b$. Otherwise stated:

$$R_{n+1} = (1 + a)R_n \quad \text{or} \quad R_{n+1} = (1 + b)R_n, \quad n = 0, \ldots, N - 1.$$

The natural space for all possible results is thus $\Omega = \{1+a, 1+b\}^N$, and we also consider $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$. The set Ω is equipped with a probability $\mathbf{P}$ such that all singletons of Ω have a non zero probability. Set $T_n = \frac{R_n}{R_{n-1}}$, and note that $\mathcal{F}_n = \sigma(T_1, \ldots, T_n)$.

45.1. Show that the actualized price $\tilde{R}_n$ is a $\mathbf{P}$-martingale if and only if $\mathbf{E}[T_{n+1}|\mathcal{F}_n] = 1 + r$.

45.2. Deduce that, in order to have a viable market, the rate r should satisfy the condition $r \in (a, b)$.

45.3. Give an example of possible arbitrage whenever $r \notin (a, b)$.

45.4. We assume in the remainder of the problem that $r \in (a, b)$, and we set $p = \frac{b-r}{b-a}$. Show that $\tilde{R}_n$ is a martingale under $\mathbf{P}$ if and only if the random variables T_j are independent, equally distributed, with a common law given by:

$$\mathbf{P}(T_j = 1 + a) = p = 1 - \mathbf{P}(T_j = 1 + b).$$

Then prove that the market is complete.

45.5. Let C_n (resp. P_n) be the value at time n, of a European call (resp. put).

(1) Show that
$$C_n - P_n = R_n - K(1 + r)^{-(N-n)}.$$
This general relation is known as *call-put parity*.

(2) Show that C_n can be written as $C_n = c(n, R_n)$, where c is a function which will be expressed thanks to the constants K, a, b, p.

45.6. Show that a perfect hedging strategy for a call is defined by a quantity $H_n = \Delta(n, R_{n-1})$ of risky asset which should be held at time n, where Δ is a function which can be expressed in terms of c.

Problem 46. In this problem we consider a *multinomial* Cox, Ross and Rubinstein model: the unique risky asset has a price R_n at time n, and the non risky asset price is given by $S_n = (1 + r)^n$. We assume the following for the risky asset price: between time n and $n + 1$ the relative variation of price belongs to the set $\{a_1, a_2, \ldots, a_k\}$, with $k \geq 3$ and $-1 < a_1 < a_2 < \ldots < a_k$. Otherwise stated:

$$R_{n+1} = (1 + a_j)R_n \quad \text{with} \quad j \in \{1, 2, \ldots, k\}, \quad n = 0, \ldots, N - 1.$$

The natural space for all possible results is thus $\Omega = \{1 + a_1, \ldots, 1 + a_k\}^N$, and we also consider $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$. The set Ω is equipped with a probability $\mathbf{P}$ such that all singletons of Ω have a non zero probability. Set $T_n = \frac{R_n}{R_{n-1}}$, and note that $\mathcal{F}_n = \sigma(T_1, \ldots, T_n)$. We set

$$p_{n,j} = \mathbf{P}\left(T_n = 1 + a_j\right), \quad j \in \{1, 2, \ldots, k\}, \quad n = 0, \ldots, N - 1.$$

46.1. Show that the actualized price $\tilde{R}_n$ is a $\mathbf{P}$-martingale if and only if $\mathbf{E}[T_{n+1}|\mathcal{F}_n] = 1 + r$.

46.2. Deduce that, in order to have a viable market, the rate r should satisfy the condition $r \in [a_1; a_k]$.

46.3. Give an example of possible arbitrage whenever $r < a_1$.

46.4. We assume in the remainder of the problem that

$$r = \frac{1}{k}\sum_{j=1}^{k} a_j.$$

Let $\mathcal{Q}$ be the set of probability measures $\mathbf{Q}$ on Ω satisfying:

(i) Under $\mathbf{Q}$, the family $\{T_n; n \le N-1\}$ is a family of i.i.d random variables.
(ii) $\tilde{R}_n$ is a $\mathbf{Q}$-martingale.

(1) Let $\mathbf{Q}^{(1)}$ be the probability on Ω defined by: the family of random variables $\{T_n; n \le N-1\}$ is a family of i.i.d random variables of common law given by:

$$Q^{(1)}\left(T_n = 1 + a_j\right) = \frac{1}{k}, \quad j \in \{1, 2, \ldots, k\}.$$

Show that $Q^{(1)} \in \mathcal{Q}$.
(2) Show that $\mathcal{Q}$ is an infinite set.
(3) Show that the market is incomplete.

46.5. We now work under the probability $\mathbf{Q}^{(1)}$. Let C_n be the value of a European call with strike K and maturity N. Show that C_n can be written under the form $C_n = c(n, R_n)$, where c is a function which will be expressed thanks to $K, a_1, \ldots, a_k$. Note that a multinomial law can be used here. This law can be defined as follows: we consider an urn with a proportion p_j of balls of type j, for $j \in \{1, \ldots, k\}$, with $\sum_{j=1}^{k} p_j = 1$. We draw n times from this urn and we call X_j the number of balls of type j obtained in this way. Then for for any tuple of integers $(n_1, \ldots, n_j)$ such that $\sum_{j=1}^{k} n_j = n$, we have

$$\mathbf{P}\left(X_1 = n_1, \ldots, X_k = n_k\right) = \frac{n!}{\prod_{j=1}^{k} n_j!}\prod_{j=1}^{k} p_j^{n_j}.$$

La law of the vector $(X_1, \ldots, X_k)$ ia called multinomial law with parameters $(n, k, p_1, \ldots, p_k)$.

5. Markov chains

Problem 47. Let $(X_n; n \ge 0)$ be a Markov chain on $(E, \mathcal{E})$, with transition probability π. We set $\mathcal{F}_n = \sigma(X_0, X_1, ..., X_n)$. A function $f : (E, \mathcal{E}) \longrightarrow (\mathbb{R}, \mathcal{R})$ is said to be invariant or harmonic for π if

$$\sum_E |f(y)|\pi(x, y) < K \qquad \text{and} \qquad \sum_E f(y)\pi(x, y) = f(x),$$

pour tout x de E.

47.1. Let f be a harmonic function such that $\sum_E |f(y)|\nu(y) < \infty$, where ν is the law of X_0. Show that $(f(X_n); n \ge 0)$ is a $\mathcal{F}_n$- martingale.

47.2. Suppose that E is a finite set which contains two absorbing states a and b: $\pi(a, a) = \pi(b, b) = 1$. We set $\sigma(x) = \inf\{n \ge 0; X_n = x\}$ and $T = \sigma(a) \wedge \sigma(b)$.

(1) Show: $\mathbf{P}(\{X_n = c; \forall n \ge 0\} | X_0 = c) = 1$ whenever $c = a$ or b.

(2) Suppose that $\pi(x, \{a, b\}) > 0$, for all $x \in E$. We set:

$$\begin{aligned} \epsilon &= \inf\{\pi(x, \{a, b\}); x \in E, x \neq a, x \neq b\} \\ A_n &= \{X_i \neq a, X_i \neq b, \text{ for all } 0 \leq i \leq n\}. \end{aligned}$$

Show that $\mathbf{P}(A_n | X_0 = x) \leq (1 - \epsilon)^n$. Deduce that $\mathbf{P}_x(T < \infty) = 1$ for all x in E.

47.3. We suppose that $\mathbf{P}_x(T < \infty) = 1$ for all x in E. Let f be a π-harmonic function, bounded and such that $f(a) \neq f(b)$. Compute the absorption probabilities in a and b:

$$\rho_{x,a} = \mathbf{P}(X_T = a | X_0 = x) \ , \quad \rho_{x,b} = \mathbf{P}(X_T = b | X_0 = x)$$

47.4. Study the particular case where $E = \{1, 2, 3, ..., n\}, \pi(1, 1) = 1, \pi(n, n) = 1, \pi(i, i + 1) = \pi(i, i - 1) = 1/2$ for $2 \leq i \leq n - 1$, with $n \geq 3$.

Problem 48. Draw the graph and classify the states for the Markov chains whose transition probabilities are given below:

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Problem 49. We consider a game according to the following rules: assume that the player possesses an initial amount equal to \$$z$, with $z \in \mathbb{N}$. At a given time n, he draws randomly (independently of the past) an integer $\ell \in \{1, 2, 3\}$. Then

- If $\ell = 1$, the player does not win or loose anything.
- If $\ell = 2$, the player's fortune is doubled.
- Si $\ell = 3$, the entire fortune is lost.

We denote by X_n the player's fortune at time n.

49.1. Show that for all $n \in \mathbb{N}$, X_{n+1} can be written as

$$X_{n+1} = 2X_n \mathbf{1}_{(\xi_{n+1}=2)} + X_n \mathbf{1}_{(\xi_{n+1}=1)},$$

where the random variables $\{\xi_k; k \geq 1\}$ are independent, with uniform law on $\{1, 2, 3\}$.

49.2. Show that $\{X_n; n \geq 0\}$ is a Markov chain.

49.3. Compute the transition matrix of X_n, denoted by $\pi = (\pi_{i,j})$.

49.4. We designate by $\pi_{i,j}^{(n)}$ the elements of the matrix π^n. By means of a recursion procedure, show that for all $i \in \mathbb{N}$ and $n \geq 1$ we have:

$$\pi_{i,j}^{(n)}(i,\, 2^j i) = \frac{\binom{n}{j}}{3^n} \quad \text{for } j \in \{0, \dots, n\}, \qquad \text{and } \pi_{i,j}^{(n)}(i, 0) = 1 - \left(\frac{2}{3}\right)^n.$$

49.5. Determine the nature of each state.

49.6. Let σ_0 be the first non zero instant for which the player's fortune is 0. Compute, for every initial state z and for all $n \in \mathbb{N}$, the quantity $\mathbf{P}_z(\sigma_0 = n)$. Show that σ_0 follows a geometric law with parameter $\frac{1}{3}$.

Problem 50. In this problem, X_n designates the number of particles at instant n in a given volume V. We assume that in the interval $[n, n+1)$, each of the X_n particles has a probability p (with $0 < p < 1$) of escaping from the volume V. We also assume that a random number of particles, following a Poisson law with parameter λ, gets into V. We suppose independence of all the random phenomena considered previously.

50.1. Show that (X_n) is a Markov chain with transition probability given by:

$$Q(x, y) = \sum_{k=(x-y)_+}^{x} \frac{C_x^k}{(y - x + k)!} p^k (1-p)^{x-k} \lambda^{y-x+k} e^{-\lambda}$$

50.2. Compute $\mathbf{E}[e^{itX_1} | X_0 = x]$ in a direct way.

50.3. Deduce the expression of the characteristic function of X_1 when X_0 follows a Poisson law $\mathcal{P}(\theta)$. Then show that $\mathcal{P}(\lambda/p)$ is an invariant measure.

50.4. Show that (X_n) is an irreducible and positive recurrent Markov chain.

50.5. What is the limit of $\overline{X}_n$, as n goes to ∞?

Problem 51. Consider a Markov chain on the vertices of a triangle (we denote $E = \{1, 2, 3\}$ this state space), defined as follows: at each instant one moves in the trigonometric sense with probability α and in the other direction with probability $1 - \alpha$, where $0 < \alpha < 1$.

51.1. Write the transition matrix p of this chain.

51.2. Compute the stationary distribution ν.

51.3. Show that for all $x, y \in E$, we have $\lim_{n\to\infty} p^n(x, y) = \nu(y)$. We shall use the following criterion: let X be an irreducible recurrent Markov chain with transition $\hat{p}$ on a finite state space E. If there exists a $m \geq 1$ satisfying $\hat{p}^m(x, y) > 0$ for all $x, y \in E$, then $\lim_{n\to\infty} \hat{p}^n(x, y) = \mu(y)$, where μ is the invariant probability of X.

51.4. Compute, for any initial distribution μ,

$$\lim_{n\to\infty} \mathbf{P}_\mu (X_n = 1, X_{n+1} = 2), \quad \text{and} \quad \lim_{n\to\infty} \mathbf{P}_\mu (X_n = 2, X_{n+1} = 1).$$

51.5. For which values of α do we get a reversible stationary law?

6. Ergodic theorems

Problem 52. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, a map φ preserving $\mathbf{P}$, a random variable X. We consider the class $\mathcal{I}$ of invariant events.

52.1. Show that $\mathcal{I}$ is a σ-field.

52.2. Show that $X \in \mathcal{I}$ if and only if $X \circ \varphi = X$ almost surely.

Problem 53. Consider a centered stationary sequence X with autocovariance function $c(m) = \mathbf{E}[X_k X_{k+m}]$.

53.1. Show that

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{2}{n^2}\sum_{j=1}^{n}\sum_{i=0}^{j-1} c(i) - \frac{c(0)}{n}.$$

53.2. We now assume that $\lim_{j\to\infty}\sum_{i=0}^{j-1} c(i) = \sigma^2$. Prove that

$$\lim_{n\to\infty} \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \sigma^2.$$

Problem 54. We consider the rotation of the circle context: take $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the Borel σ-algebra and λ stands for the Lebesgue measure. Let $\theta \in (0,1)$. Then we define $X = \{X_n;\, n \geq 0\}$ with

$$X_n(\omega) = (\omega + n\theta) \mod 1 = \omega + n\theta - [\omega + n\theta].$$

The natural shift related to this sequence is denoted by φ.

54.1. If $\theta = \frac{m}{n}$ for two integers such that $m < n$, show that φ is not ergodic by considering the set

$$A = \bigcup_{k=0}^{n-1}\left(B + \frac{k}{n}\right), \quad \text{where} \quad B \subset \left[0, \frac{1}{n}\right).$$

54.2. We now consider $\theta \in \mathbb{Q}^c$ and we wish to prove that φ is ergodic.

(1) Let $f \in L^2([0,1))$ with Fourier series decomposition $f = \sum_{k\in\mathbb{Z}} c_k e^{2\pi\imath kx}$. Show that $f \circ \varphi = f$ if and only if $f = 0$.
(2) Deduce that φ is ergodic.

Problem 55. Let $n \geq 1$ and X a stationary sequence of random variables. We consider a sequence Y with $(Y_1, \ldots, Y_n) = (X_1, \ldots, X_n)$ and such that the blocks $(Y_{nk+1}, \ldots, Y_{n(k+1)})$ are i.i.d. Let ν be a random variable independent of X, Y and uniformly distributed over $\{1, \ldots, n\}$. We consider the sequence Z defined by $Z_m = Y_{\nu+m}$ for $m \geq 1$.

55.1. Show that Z is stationary and ergodic.

Problem 56. Let X be a sequence of independent identically distributed random variables with zero mean and unit variance. For $n \geq 1$ let

$$Y_n = \sum_{i=0}^{\infty} \alpha_i X_{n+i},$$

where the α_i are constants satisfying $\sum_{i=0}^{\infty} \alpha_i^2 < \infty$.

56.1. Use the martingale convergence theorem to show that the above summation converges almost surely and in mean square.

56.2. Prove that $\lim_{n\to\infty} \bar{Y}_n = 0$, almost surely and in $L^1(\Omega)$.

7. Brownian motion

Problem 57. Let B be a standard Brownian motion.

57.1. Compute, for all couple (s,t), the quantities $\mathbf{E}[B_t|\mathcal{F}_s]$ and $\mathbf{E}[B_s B_t^2]$ (we do not assume $s \leq t$ here).

57.2. Compute $\mathbf{E}[B_t^2 B_s^2]$.

57.3. What is the law of $B_t + B_s$?

57.4. Compute $\mathbf{E}[\mathbf{1}_{(B_t \leq 0)}]$ and $\mathbf{E}[B_t^2 \, \mathbf{1}_{(B_t \leq 0)}]$.

57.5. Compute $\mathbf{E}[\int_0^t \mathrm{e}^{B_s} ds]$ and $\mathbf{E}[e^{\alpha B_t} \int_0^t \mathrm{e}^{\gamma B_s} ds]$ for $\alpha, \gamma > 0$.

Problem 58. For any continuous bounded function $f : \mathbb{R} \to \mathbb{R}$ and all $0 \leq u \leq t$, show that $\mathbf{E}[f(B_t)] = \mathbf{E}[f(G\sqrt{u} + B_{t-u})]$ with a random variable $G \sim \mathcal{N}(0,1)$ independent of B_{t-u}.

Problem 59. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function whose second derivative has at most exponential growth. Show that

$$\mathbf{E}[f(x + B_t)] = f(x) + \frac{1}{2} \int_0^t \mathbf{E}[f''(x + B_s)] \, ds \, .$$

Hint: One can use the following Gaussian integration by parts formula: let $N \sim \mathcal{N}(0,1)$ and $\psi \in \mathcal{C}^1$ with exponential growth. Then $\mathbf{E}[N \, \psi(N)] = \mathbf{E}[\psi'(N)]$. The Gaussian integration by parts should be proved first.

Problem 60. Consider a standard Brownian motion B. For all $\lambda, \mu \in \mathbb{R}$, compute

$$\mathbf{E}\left[\left(\mu B_1 + \lambda \int_0^1 B_u du\right)^2\right].$$

Problem 61. Show that the integral $\int_0^1 \left|\frac{B_s}{s}\right|^\alpha ds$ is finite almost surely if $\alpha < 2$.

8. Gaussian processes

Problem 62. Let $(X_n, n \geq 1)$ a sequence of centered Gaussian random variables, converging in law to a random variable X. Show that X is also a centered Gaussian random variable. Deduce that the process $Y = \{Y_t, t \geq 0\}$ given by $Y_t = \int_0^t B_u du$ is Gaussian. Compute its expected value and its covariance function.

Problem 63. We define the Brownian bridge by $Z_t = B_t - tB_1$ for $0 \leq t \leq 1$.

63.1. Show that Z is a Gaussian process independent of B_1. Give its law, that is its mean and its covariance function.

63.2. Show that the process $\tilde{Z}$, with $\tilde{Z}_t = Z_{1-t}$, has the same law as Z.

63.3. Show that the process Y, with $Y_t = (1-t)B_{\frac{t}{1-t}}$, $0 < t < 1$, has the same law as Z.

9. Continuous time martingales

Problem 64. Among the following processes, what are those who enjoy the martingale property? *Hint:* use the Fubini type relation $\mathbf{E}[\int_0^t B_u du | \mathcal{F}_s] = \int_0^t \mathbf{E}[B_u | \mathcal{F}_s]\, du$.

64.1. $M_t = B_t{}^3 - 3\int_0^t B_s\, ds$?

64.2. $Z_t = B_t{}^3 - 3tB_t$?

64.3. $X_t = tB_t - \int_0^t B_s\, ds$?

64.4. $Y_t = t^2 B_t - 2\int_0^t B_s ds$?

Problem 65. Let $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_1)$. Check that B is not a $\mathcal{G}_t$-martingale. *Hint:* get a contradiction, showing that if B is a $\mathcal{G}_t$-martingale, then $\mathbf{E}[B_t|B_1] = \mathbf{E}[B_s|B_1]$ for $0 \leq s, t \leq 1$.

Problem 66. Let $Z = \{Z_t, t \geq 0\}$ le process defined par $Z_t = B_t - \int_0^t \frac{B_s}{s}\, ds$.

66.1. Show that Z is a Gaussian process.

66.2. Compute the expected value and the covariance function of Z. Deduce that Z is a Brownian motion.

66.3. Show that Z is not a $\mathcal{F}_t^B$-martingale, where $(\mathcal{F}_t^B)$ is the natural filtration of B. *Hint:* compute $\mathbf{E}[Z_t - Z_s | \mathcal{F}_s^B]$ for $0 \leq s < t$.

66.4. Deduce that $\mathcal{F}^Z \subset \mathcal{F}^B$, but $\mathcal{F}^Z \neq \mathcal{F}^B$.

Problem 67. Let ϕ be a bounded adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and M a $(\mathcal{F}_t)$-martingale. We set

$$Y_t = M_t - \int_0^t \phi_s ds, \quad t \in [0, T].$$

Prove that

$$Y_t = E\left[\int_t^T \phi_s ds + Y_T \mid \mathcal{F}_t\right], \quad t \in [0,T]. \tag{2}$$

In the other direction, if Y satisfies (2) with a bounded adapted process ϕ, show that M defined by

$$M_t = Y_t + \int_0^t \phi_s ds, \quad t \in [0,T],$$

is a martingale.

Problem 68. Let $(M_t)_{t\geq 0}$ be a square integrable $\mathcal{F}_t$-martingale (that is such that $M_t \in L^2$ for all t).

68.1. Show that $\mathbf{E}[(M_t - M_s)^2|\mathcal{F}_s] = \mathbf{E}[{M_t}^2|\mathcal{F}_s] - {M_s}^2$ for $t > s$

68.2. Deduce that $\mathbf{E}[(M_t - M_s)^2] = \mathbf{E}[{M_t}^2] - \mathbf{E}[{M_s}^2]$ for $t > s$

68.3. Consider the function Φ defined by $\Phi(t) = \mathbf{E}[{M_t}^2]$. Check that Φ is increasing.

Problem 69. Show that if M is a $\mathcal{F}_t$-martingale, it is also a martingale with respect to its natural filtration $\mathcal{G}_t = \sigma(M_s, s \leq t)$.

Problem 70. Let τ be a positive random variable defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Show that $Z_t = \mathbf{P}(\tau \leq t|\mathcal{F}_t)$ is a sub-martingale.

Problem 71. Let X be a centered process with independent increments, such that for all $n \in \mathbb{N}^*$ and any $0 < t_1 < t_2 < \ldots < t_n$, the random variables X_{t_1}, $X_{t_2} - X_{t_1}$, $\ldots$, $X_{t_n} - X_{t_{n-1}}$ are independent. In addition, we assume that X is integrable, and that $(\mathcal{F}_t)$ is the natural filtration of X. Show that X is a martingale. If we further suppose that X is square integrable, show that $X_t^2 - \mathbf{E}[X_t^2]$ is also a $(\mathcal{F}_t)$-martingale.

10. Hitting times

In this section, a designates a real number and T_a is the random time defined by $T_a = \inf\{t \geq 0 : B_t = a\}$.

Problem 72. Show that T_a is a stopping time. Compute $\mathbf{E}[e^{-\lambda T_a}]$ for all $\lambda \geq 0$. Show that $\mathbf{P}(T_a < \infty) = 1$ and that $\mathbf{E}[T_a] = \infty$.

Problem 73. Prove (avoid computations) that for $b > a > 0$, the random variable $T_b - T_a$ is independent of T_a. Deduce that the process $(T_a)_{a\geq 0}$ has independent and stationary increments.

Problem 74. Let $a < 0 < b$ and $T = T_a \wedge T_b$. Compute $\mathbf{P}(T_a < T_b)$ and $\mathbf{E}[T]$.
Hint: Apply the optional sampling theorem to B_t and $B_t^2 - t$.

Problem 75. Find an expression for $Z_t = \mathbf{P}(T_a > 1|\mathcal{F}_t)$ for $0 \le t \le 1$ and $a > 0$. Recall that $\sup_{u \le t} B_u \overset{(d)}{=} |B_t|$.

Problem 76. Let $I = -\inf_{s \le T_1} B_s$. Show that I has a density given by $f_I(x) = \frac{1}{(1+x)^2}\mathbf{1}_{[0,+\infty[}(x)$.
Hint: Use $\{I \le x\} = \{T_1 < T_{-x}\}$.

Problem 77. Let $T_1 = \inf\{t \ge 0 : \; B_t = 1\}$. Use a Brownian scaling in order to show the following identities in law:

(1) $T_1 \overset{(d)}{=} \frac{1}{S_1^2}$, with $S_1 = \sup(B_u, u \le 1)$;
(2) $T_a \overset{(d)}{=} a^2 T_1$.

11. Wiener integral

Problem 78. In this problem we consider the process X defined by $X_t = \int_0^t (\sin s)\, dB_s$.

78.1. Show that, for each $t \ge 0$, the random variable X_t is well defined.

78.2. Show that $X = (X_t)_{t \ge 0}$ is a Gaussian process. Compute its expected value and its covariance function.

78.3. Compute $\mathbf{E}[X_t|\mathcal{F}_s]$ for $s, t \ge 0$.

78.4. Show that $X_t = (\sin t)B_t - \int_0^t (\cos s) B_s\, ds$ for all $t \ge 0$.

Problem 79. Let X be the process defined on $(0,1)$ by: $X_t = (1-t)\int_0^t \frac{dB_s}{1-s}$.

79.1. Show that X satisfies:

$$X_0 = 0 \quad \text{and} \quad dX_t = \frac{X_t}{t-1}dt + dB_t, \quad t \in (0,1).$$

79.2. Show that X is a Gaussian process. Compute its expected value and its covariance function.

79.3. Show that $\lim_{t \uparrow 1} X_t = 0$ in $L^2(\Omega)$.

12. Itô's formula

Problem 80. Write the following processes as Itô processes, specifying their drift and their diffusion coefficient.

(1) $X_t = {B_t}^2$
(2) $X_t = t + \exp(B_t)$;
(3) $X_t = {B_t}^3 - 3tB_t$;
(4) $X_t = (B_t + t)\exp(-B_t - t/2)$;
(5) $X_t = \exp(t/2)\,\sin(B_t)$.

Problem 81. Let X and Y defined by:

$$X_t = \exp\left(\int_0^t a(s)ds\right), \quad \text{et} \quad Y_t = Y_0 + \int_0^t \left[b(s)\exp\left(-\int_0^s a(u)du\right)\right] dB_s,$$

where $a, b : \mathbb{R} \to \mathbb{R}$ are bounded functions. We set $Z_t = X_t Y_t$. Show that $dZ_t = a(t)Z_t dt + b(t)dB_t$.

Problem 82. Let Z be the process given by $Z_t = t\,X_t\,Y_t$, where X and Y are defined by:

$$dX_t = f(t)\,dt + \sigma(t)dB_t, \quad \text{and} \quad dY_t = \eta(t)dB_t.$$

Find an expression for dZ_t.

Problem 83. Show that $Y = (Y_t)_{t\geq 0}$ defined by $Y_t = \sin(B_t) + \frac{1}{2}\int_0^t \sin(B_s)\,ds$ is a martingale. Find its expected value and its variance.

Problem 84. Let us assume that the following system admits a solution (X, Y):

$$\begin{cases} X_t &= x + \int_0^t Y_s\,dB_s \\ Y_t &= y - \int_0^t X_s\,dB_s \end{cases}, \quad t \geq 0.$$

Show that ${X_t}^2 + {Y_t}^2 = (x^2 + y^2)e^t$ for all $t \geq 0$.

Problem 85. We define Y and Z in the following way for $t \geq 0$:

$$Y_t = \int_0^t e^s\,dB_s, \quad \text{and} \quad Z_t = \int_0^t Y_s dB_s.$$

Give an expression for $\mathbf{E}[Z_t]$, $\mathbf{E}[Z_t^2]$ and $\mathbf{E}[Z_t Z_s]$ for $s, t \geq 0$.

Problem 86. Let σ be an adapted continuous process in $L^2(\Omega\times\mathbb{R})$, and let $X_t = \int_0^t \sigma_s\,dB_s - \frac{1}{2}\int_0^t \sigma_s^2\,ds$. We set $Y_t = \exp(X_t)$ and $Z_t = Y_t^{-1}$.

86.1. Give an explicit expression for the dynamics of Y, that is dY_t.

86.2. Show that Y is a local martingale on $[0, T]$ for all $T > 0$. If $\sigma = 1$, show that Y is a martingale on $[0, T]$ for all $T > 0$. Compute $\mathbf{E}[Y_t]$ in this case.

86.3. Compute dZ_t.

Problem 87. Let a, b, c, z be real valued constants, and let Z be the process defined by:

$$Z_t = e^{(a-c^2/2)t+cB_t}\left(z + b\int_0^t e^{-(a-c^2/2)s-cB_s}ds\right),\ t \geq 0.$$

Give a simple expression for dZ_t.

Problem 88. Let $(X_t)_{t\geq 0}$ be a process satisfying $X_t = x + \int_0^t a_s ds + \int_0^t \sigma_s dB_s$ for $t \geq 0$. In the previous formula, x is a real number, a is a continuous process satisfying $\int_0^t |a_s|ds < \infty$ for all $t \geq 0$, and σ is an adapted continuous process verifying $\int_0^t \mathbf{E}[\sigma_s^2]ds < \infty$ for all $t \geq 0$. We wish to show that if $X \equiv 0$, then $x = 0$, $a \equiv 0$ and $\sigma \equiv 0$.

88.1. Apply Itô's formula to $Y_t = \exp(-X_t^2)$.

88.2. Prove the claim.

Problem 89. Let X be an Itô process. A function s is called scale function for X if $s(X)$ is a local martingale. Determine the scale functions of the following processes:

(1) $B_t + \nu t$;
(2) $X_t = \exp(B_t + \nu t)$;
(3) $X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s$.

Problem 90. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}_b^1$ function.

90.1. Construct a function $\psi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ (expressed as an expected value) such that, for $t \in [0, 1]$, we have $\mathbf{E}[f(B_1)|\mathcal{F}_t] = \psi(t, B_t)$.

90.2. Write Itô's formula for ψ and simplify as much as possible.

90.3. Show that, for all $t \in [0, 1]$ we have:

$$\mathbf{E}[f(B_1)|\mathcal{F}_t] = \mathbf{E}[f(B_1)] + \int_0^t \mathbf{E}[f'(B_1)|\mathcal{F}_s]dB_s.$$

Problem 91. Let S be the solution of: $dS_t = rS_tdt + S_t\sigma(t, S_t)dB_t$, $t \in [0, T]$, where r is a constant and where $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function $\mathcal{C}^{1,1}$ with bounded derivatives.

91.1. Show that $\mathbf{E}[\Phi(S_T)|\mathcal{F}_t]$ is a martingale (as a function of t) for any bounded measurable function Φ.

In the sequel, we admit that $\mathbf{E}[\Phi(S_T)|\mathcal{F}_t] = \mathbf{E}[\Phi(S_T)|S_t]$ for all $t \in [0, T]$ (Markov property for S).

91.2. Let $\varphi(t, x)$ be the function defined by $\varphi(t, S_t) = \mathbf{E}[\Phi(S_T)|S_t]$ (the existence of φ is admitted). Write dZ_t with $Z_t = \varphi(t, S_t)$.

91.3. Invoking the fact that $\varphi(t, S_t)$ is a martingale, and admitting that φ is $C^{1,2}$, show that for all $t > 0$ and all $x > 0$ we have:

$$\frac{\partial\varphi}{\partial t}(t, x) + rx\frac{\partial\varphi}{\partial x}(t, x) + \frac{1}{2}\sigma^2(t, x)x^2\frac{\partial^2\varphi}{\partial x^2}(t, x) = 0.$$

What is the value of $\varphi(T, x)$?

Problem 92. Let B be a d-dimensional Brownian motion. We consider an open bounded subset G of $\mathbb{R}^d$, and $\tau = \inf\{t \geq 0;\ B_t \notin G\}$. We denote by K the diameter of G, i.e $K = \sup\{|x-y|;\ x, y \in G\}$.

92.1. Show that there exists $\varepsilon = \varepsilon_K \in (0,1)$ such that $\mathbf{P}_x(\tau \leq 1) \geq \varepsilon$ for all $x \in G$.

92.2. Deduce that there exists $\rho = \rho_K \in (0,1)$ such that $\mathbf{P}_x(\tau > k) \leq \rho^k$ for all $k \geq 1$ and $x \in G$.

92.3. Deduce that $\mathbf{E}_x[\tau^p] < \infty$ for all $p \geq 1$. In particular, show that $\tau < \infty$ $\mathbf{P}_x$-almost surely for all $x \in G$.

92.4. Let $\varphi \in \mathcal{C}^2(\bar{G})$ be a harmonic function, i.e such that $\Delta\varphi = 0$ sur G. Prove that $\mathbf{E}_x[\varphi(B_\tau)] = \varphi(x)$.

92.5. We now seek some harmonic functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ having the form $\varphi(x) = f(|x|^2)$ with $f : \mathbb{R} \to \mathbb{R}$.

(1) Prove that f is solution of the differential equation $f''(y) = -\frac{d}{2y} f'(y)$ for $y > 0$.
(2) Deduce the following form for radial harmonic functions:

$$\varphi(x) = \begin{cases} x & \text{if } d = 1 \\ \ln(|x|) & \text{if } d = 2 \\ |x|^{2-d} & \text{if } d \geq 3 \end{cases}$$

Problem 93. We now consider a particular case of Problem 92, namely we fix the dimension $d = 1$.

93.1. Let $a < x < b$ and $\tau = \inf\{t \geq 0;\ B_t \notin (a,b)\}$. Show that

$$\mathbf{P}_x\left(B_\tau = a\right) = \frac{b-x}{b-a}, \qquad \mathbf{P}_x\left(B_\tau = b\right) = \frac{x-a}{b-a}.$$

93.2. For $x \in \mathbb{R}$ we set $T_x = \inf\{t \geq 0;\ B_t = x\}$. Prove that $\mathbf{P}_x(T_y < \infty) = 1$ for all $x, y \in \mathbb{R}$.

93.3. Let now $s > 0$ and $x, y \in \mathbb{R}$. Show that $\mathbf{P}_x(B_t = y \text{ for a } t \geq s) = 1$.

93.4. Let $\mathcal{T}_y$ be the random set given by the points such that $B_t = y$. Applying Markov's property, show that $\mathbf{P}_x(\mathcal{T}_y \text{ unbounded}) = 1$.

Problem 94. The situation of Problem 92 is now particularized to the dimension $d = 2$. For $r > 0$ we set $S_r = \inf\{t \geq 0;\ |B_t| = r\}$.

94.1. Let $x \in \mathbb{R}^2$ such that $0 < r < |x| < R$. Prove that

$$\mathbf{P}_x\left(S_r < S_R\right) = \frac{\ln(R) - \ln(|x|)}{\ln(R) - \ln(r)}.$$

94.2. Invoking the same kind of arguments as in Problem 93, show that B is recurrent, that is for any couple $x, y \in \mathbb{R}^2$ and $r > 0$ we have $\mathbf{P}_x(T_{B(y,r)} < \infty) = 1$.

94.3. Whenever $x \neq 0$, show that $\mathbf{P}_x(T_0 < \infty) = 0$, i.e the 2-dimensional Brownian motion does not hit points. *Hint:* for a fixed $R > 0$ we have

$$(T_0 < S_R) \subset \bigcap_{n \geq 1} \left(S_{1/n} < S_R\right).$$

Problem 95. Consider now the case of a Brownian motion in dimension $d \geq 3$. For $r > 0$ we set $S_r = \inf\{t \geq 0;\ |B_t| = r\}$.

95.1. Let $x \in \mathbb{R}^d$ such that $|x| > r > 0$. Prove that $\mathbf{P}_x(S_r < \infty) = (r/|x|)^{d-2}$.

95.2. Let $A_n = \{|B_t| > n^{1/2} \text{ for all } t \geq S_n\}$. Show that $\mathbf{P}_x(\limsup_n A_n) = 1$ for all $x \in \mathbb{R}^d$.

95.3. Show that $\mathbf{P}_x$-almost surely we have $\lim_{t\to\infty} |B_t| = \infty$, for all $x \in \mathbb{R}^d$.

13. Geometrical Brownian motion

Problem 96. Let S satisfying the following stochastic differential equation:

$$dS_t = S_t(b\,dt + \sigma\,dB_t), \quad S_0 = 1, \tag{3}$$

where b and σ are constants. Let $\tilde{S}_t = e^{-bt} S_t$.

96.1. Show that $(\tilde{S}_t)_{t\geq 0}$ is a martingale. Deduce the value of $\mathbf{E}[S_t]$ and $\mathbf{E}[S_t|\mathcal{F}_s]$ for any couple (t, s).

96.2. Give an expression for the drift term and the diffusion coefficient of $\frac{1}{S}$.

96.3. Show that $S_t = \exp[(b - \frac{1}{2}\sigma^2)t + \sigma B_t]$ satisfies (3), and that

$$S_T = S_t \exp[(b - \frac{1}{2}\sigma^2)(T - t) + \sigma(B_T - B_t)]$$

for all $T \geq t$.

96.4. Let L be a process verifying $dL_t = -L_t\theta_t\,dB_t$ where θ_t is an adapted continuous process in $L^2(\Omega \times \mathbb{R})$. We set $Y_t = S_t L_t$. Compute dY_t.

96.5. Let ζ_t be defined by

$$d\zeta_t = -\zeta_t(r\,dt + \theta_t\,dB_t)$$

Show that $\zeta_t = L_t e^{-rt}$. Compute $d(\zeta^{-1})_t$ and then $d(S\,\zeta)_t$. How can we choose θ in such a way that ζS is a martingale?

Problem 97. Let f be a bounded measurable function and $S = (S_t)_{t\geq 0}$ be a process verifying the equation

$$dS_t = S_t(r - f_t)dt + \sigma dB_t, \quad S_0 = x \in \mathbb{R}.$$

The following questions are independent.

97.1. Show that $e^{-rt}S_t + \int_0^t f_s\, e^{-rs} S_s\, ds$ is a local martingale.

97.2. Show that

$$S_t = x\mathrm{e}^{rt-\int_0^t f_u\,du} + \sigma \int_0^t \mathrm{e}^{r(t-s)-\int_s^t f_u\,du} dB_s$$

is a possible expression for S. In the sequel, we work with this formula for S.

97.3. Compute the expected value and the variance of S_t for $t \geq 0$.

97.4. Let $T, K > 0$. Compute $\mathbf{E}[(S_T - K)_+]$ whenever f is constant.