

Summary - Stochastic integrals

Simple process

$$\xi_i \in \mathcal{F}_{t_i}$$

$$U_t = \sum_i \xi_i \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

For U simple

$$\int_{t_k}^{t_\ell} U_r dW_r = \sum_{i=k}^{l-1} \xi_i \{W_{t_{i+1}} - W_{t_i}\}$$

+ interpolation

We have seen

{ simple processes }

dense in $L_a^2 \equiv$ square integ. + adapted

Moreover, if $u^n \rightarrow u$ in L_a^2
with u^n simple, we have
(Itô's isometry)

$$\begin{aligned} & \mathbb{E} \left(\int_s^t (u_r^n - u_r^m) dW_r \right)^2 \\ &= \mathbb{E} \int_s^t (u_r^n - u_r^m)^2 dr \rightarrow 0 \end{aligned}$$

Thus $\int_s^t u_r^n dW_r$ converges in $L^2(\mathcal{F})$

Stochastic integral: extended definition (2)

Definition 8

Let $u \in L_a^2$. The stochastic integral of u with respect to W is the process $\mathcal{J}(u dW)$ such that for all $s < t$,

$$\mathcal{J}_{st}(u dW) = L^2(\Omega) - \lim_{n \rightarrow \infty} \mathcal{J}_{st}(u^n dW),$$

where $(u^n)_{n \geq 0}$ is an arbitrary sequence of simple processes
 $\hookrightarrow$ converging to u in L_a^2 .

Integral of a L^2_a process: properties

Proposition 9

Let u a process of L^2_a , $\mathcal{J}_{st}(u dW)$ its stochastic integral.
Then for $\alpha, \beta \in \mathbb{R}$:

- ① $\mathcal{J}_{tt}(u dW) = 0$
- ② $\mathcal{J}_{st}((\alpha u + \beta v) dW) = \alpha \mathcal{J}_{st}(u dW) + \beta \mathcal{J}_{st}(v dW)$
- ③ $\mathbf{E}[\mathcal{J}_{st}(u dW) | \mathcal{F}_s] = 0$, i.e martingale property
- ④ $\mathbf{E}[(\mathcal{J}_{st}(u dW))^2] = \int_s^t \mathbf{E}[u_w^2] dw$
- ⑤ $\mathbf{E}[(\mathcal{J}_{st}(u dW))^2 | \mathcal{F}_s] = \int_s^t \mathbf{E}[u_w^2 | \mathcal{F}_s] dw$

Remark: For this construction, crucial use of:

- Independence of increments for W
- L^2 convergence in probabilistic sense

Rmk For the approximation:

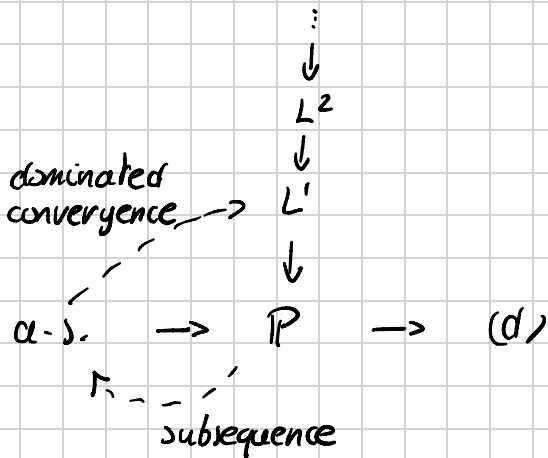
$$\int_s^t u_r^n \, dw_r \xrightarrow{L^2(\mathcal{Q})} \int_s^t u_r \, dw_r$$

We don't claim

$$\int_s^t u_r \, dw_r \xrightarrow{a.s.} \int_s^t u_r \, dw_r$$

Convergence of r.v.

a.s., $\mathbb{P}$, L^p , (d)



Wiener integral

Proposition 10

Let:

- W a 1-dimensional Wiener process, and $\tau > 0$.
- h^1, h^2 deterministic functions in $L^2([0, \tau])$.

Then the following stochastic integrals are well defined:

$$W(h^1) = \int_0^\tau h_r^1 dW_r, \quad W(h^2) = \int_0^\tau h_r^2 dW_r.$$

In addition:

- 1 $(W(h^1), W(h^2))$ centered Gaussian vector.
- 2 Covariance of $W(h^1), W(h^2)$:

$$\mathbf{E} [W(h^1) W(h^2)] = \int_0^\tau h_r^1 h_r^2 dr.$$

$= \langle h^1, h^2 \rangle_{L^2([0, \tau])}$

Proof

Approximation: For $m \geq 1$ we set $s_j = s_j^m = \frac{j\tau}{m}$ and

$$h^{1,m} = \sum_{j=0}^{m-1} h_{s_j}^1 \mathbf{1}_{(s_j, s_{j+1}]}, \quad h^{2,m} = \sum_{j=0}^{m-1} h_{s_j}^2 \mathbf{1}_{(s_j, s_{j+1}]}.$$

Then:

$$W(h^{1,m}) = \sum_{j=0}^{m-1} h_{s_j}^1 \delta W_{s_j s_{j+1}}, \quad W(h^{2,m}) = \sum_{j=0}^{m-1} h_{s_j}^2 \delta W_{s_j s_{j+1}}.$$

Approximation properties: we have

- $(W(h^{1,m}), W(h^{2,m}))$ Gaussian vect. since W Gaussian process.
- $\mathbf{E}[W(h^{1,m}) W(h^{2,m})] = \int_0^\tau h_r^{1,m} h_r^{2,m} dr.$

Then limiting procedure for Gaussian vectors.

Outline

- 1 Itô's integral
- 2 Itô's formula
- 3 Extensions and consequences of Itô's formula

Problem If x is a C^1 function on $[0, T]$,
then the usual change of variable formula reads

$$\begin{aligned} f(x_t) - f(x_0) &= \int_0^t f'(x_r) \underbrace{dx_r}_{= \dot{x}_r dr} \end{aligned} \quad \left. \begin{array}{l} \text{same for} \\ f(t, x_t) \end{array} \right\}$$

Question: what if x is not differentiable?

(like $W = \text{BM}$)

Functional set $\mathcal{C}^{1,2}$

Definition 11

Let

- $\tau > 0$
- $f : [0, \tau] \times \mathbb{R}^d \rightarrow \mathbb{R}$

We say that $f \in \mathcal{C}_{b,\tau}^{1,2}$ if:

- 1 $t \mapsto f(t, x)$ is $\mathcal{C}^1([0, \tau])$ for all $x \in \mathbb{R}^d$
- 2 $x \mapsto f(t, x)$ is $\mathcal{C}^2(\mathbb{R}^d)$ for all $t \in [0, \tau]$
- 3 f and its derivatives are bounded

1-d Itô's formula

Theorem 12

Let:

- W real-valued Brownian motion
- $f \in \mathcal{C}_{b,\tau}^{1,2}$

Then $f(t, W_t)$ can be decomposed as:

$$f(t, W_t) = f(0, 0) + \int_0^t \partial_r f(r, W_r) dr + \int_0^t f'(r, W_r) dW_r + \frac{1}{2} \int_0^t f''(r, W_r) dr.$$

Remark:

Proof for the 1-d formula only

Multidimensional Itô's formula

Theorem 13

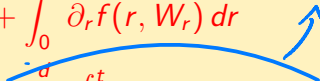
Let:

- W Brownian motion, d -dimensional
- $f \in \mathcal{C}_{b,\tau}^{1,2}$
- $\Delta \equiv$ Laplace operator on $\mathbb{R}^d$, i.e $\Delta = \sum_{j=1}^d \partial_{x_j}^2$

Then $f(t, W_t)$ can be decomposed as:

$$f(t, W_t) = f(0, 0) + \int_0^t \partial_r f(r, W_r) dr + \sum_{j=1}^d \int_0^t \partial_{x_j} f(r, W_r) dW_r^j + \frac{1}{2} \int_0^t \Delta f(r, W_r) dr$$

$= \int_0^t \langle \nabla f(r, W_r), dW_r \rangle$



Physicists' proof of Ito for $f(w_t)$:

Idea: Taylor expansion up to order $o(t)$

$$dw_t = \sqrt{dt}$$

We get

$$df(w_t) = f'(w_t) \underbrace{dw_t}_{\sqrt{dt}} + \frac{1}{2} f''(w_t) dt + o(dt)$$

Integrating, we get Ito.

Real proof. Consider a partition
of $[0, t]$:

$$\{0 = t_0 < t_1 < \dots < t_n = t\} = \pi$$

$$|\pi| = \max \{t_{i+1} - t_i\}$$

We write

$$\xi_i \in [W_{t_i}, W_{t_{i+1}}]$$

$$\begin{aligned} \delta f(W)_{0t} &= \sum_{j=0}^{n-1} \delta f(W)_{t_j, t_{j+1}} \\ &\stackrel{\text{Taylor}}{=} \sum_{j=0}^{n-1} f'(W_{t_j}) \delta W_{t_j, t_{j+1}} + \frac{1}{2} f''(\xi_j) (\delta W_{t_j, t_{j+1}})^2 \end{aligned} \quad \begin{matrix} J_t^{1, \pi} \\ J_t^{2, \pi} \end{matrix}$$

Taylor expansion . If $a < b$, $\exists \xi \in (a, b)$
s.t

$$f(b) = f(a) + f'(a)(b-a) \\ + \frac{1}{2} f''(\xi)(b-a)^2$$

Term $J_t^{',\pi}$

↗ stoch integral of a
simple process U

$$\sum_{j=0}^{n-1} f'(W_{t_j}) \delta W_{t_j, t_{j+1}} \quad) J_t^{',\pi}$$

Consider

$$U_r^\pi = \sum_{i=0}^{n-1} f'(W_{t_i}) \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

Then

$$J_t^{',\pi} = \int_0^t U_r^\pi dW_r$$

$$U_t = \sum_i \xi_i \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

For U simple

$$\int_{t_k}^{t_k} U_r dW_r = \sum_{i=k}^{l-1} \xi_i \delta W_{t_i, t_{i+1}}$$

We can easily see that

$$u^\pi \xrightarrow{L_a^2} u \equiv f'(w)$$

Therefore

$$\boxed{J_t^{1,\pi}} = \int_0^t u_r^\pi dw_r$$
$$\xrightarrow{L^2(\Omega)} \int_0^t u_r dw_r$$

$$= \int_0^t f'(w_r) dw_r$$

$$\mathbb{E}[(\Delta W_{t_j, t_{j+1}})^2] = t_{j+1} - t_j$$

Term $J_t^{2,\pi}$

$$J_t^{2,\pi} = \sum_{j=0}^{n-1} f''(\xi_j) (\Delta W_{t_j, t_{j+1}})^2$$

$$= \sum_{j=0}^{n-1} (f''(\xi_j) - f''(W_{t_j})) (\Delta W_{t_j, t_{j+1}})^2 \quad D_t^{1,\pi}$$

$$D_t^{2,\pi} \left[+ \sum_{j=0}^{n-1} f''(W_{t_j}) [(\Delta W_{t_j, t_{j+1}})^2 - (t_{j+1} - t_j)] \right]$$

$$+ \sum_{j=0}^{n-1} f''(W_{t_j}) (t_{j+1} - t_j) \quad J_t^{3,\pi}$$

Term $D_t^{1,\pi}$

Assume $f \in C_b^3$

$$\sum_{i=0}^{n-1} (f''(\bar{\xi}_i) - f''(W_{t_i})) (\Delta W_{t_i, t_{i+1}})^2$$

We have

$f \in C_b^3$

$$|f''(\bar{\xi}_i) - f''(W_{t_i})| \leq c_f |\bar{\xi}_i - W_{t_i}|$$

$$\leq c_f |\Delta W_{t_i, t_{i+1}}|$$

$$\leq c_f K(\omega) |t_{i+1} - t_i|^\delta \quad \delta < \frac{1}{2}$$

$$\leq C(\omega) |\pi|^\delta$$

We get

$$\sum_{i=0}^{n-1} (f''(\xi_i) - f''(w_i)) (\Delta W_{t_i, t_{i+1}})^2$$

$$\leq C(\omega) |\pi|^\delta \sum_{i=0}^{n-1} (\Delta W_{t_i, t_{i+1}})^2$$

$$\leq C_2(\omega) |\pi|^\delta t \xrightarrow[|\pi| \rightarrow 0]{a.s.} 0$$

Term $D_t^{2\pi}$

$= z_j$

$$\sum_{j=0}^{n-1} f''(w_{t_j}) \left[(\Delta w_{t_j, t_{j+1}})^2 - (t_{j+1} - t_j) \right]$$

$$\mathbb{E} [|D_t^{2\pi}|^2]$$

$$\leq c f \sum_{j=0}^{n-1} \mathbb{E} [z_j^2]$$

$$= \sum_{j=0}^{n-1} \mathbb{E} (f''(w_{t_j}))^2 z_j^2$$

+ crossed terms



treated by conditional
expectation + 4 increments
of w

$$Z_j = [(\Delta W_{t_j, t_{j+1}})^2 - (t_{j+1} - t_j)]$$

We have : $\Delta W_{t_j, t_{j+1}} = (t_{j+1} - t_j)^{1/2} Y$

where $Y \sim W(0,1)$. Thus

$$\boxed{Z_j} = (t_{j+1} - t_j) (Y^2 - 1) \quad = 2$$

$$\begin{aligned} \Rightarrow \sum_{j=0}^{n-1} \mathbb{E}[Z_j^2] &= \sum (t_{j+1} - t_j)^2 \mathbb{E}[(Y^2 - 1)^2] \\ &= 2 \sum_{j=0}^{n-1} (t_{j+1} - t_j)^2 \leq 2\pi \sum (t_{j+1} - t_j) \end{aligned}$$

$$= 2t \pi \xrightarrow{\pi \rightarrow 0} 0$$

Conclusion

$$D_t^{2,\pi} \xrightarrow{L^2(\mathbb{R})} 0$$

Term

$$\sum_{j=0}^{n-1} f''(w_{t_j}) (t_{j+1} - t_j) J_t^{3,\pi}$$

= Riemann sum for $\int_0^t f''(w_t) dt$

Thus

$$J_t^{3,\pi} \xrightarrow{a.s.} \int_0^t f''(w_t) dt$$

Rmk

Some of the convergences are

a.), some are $L^2(\Omega)$. But

taking subsequences π_{n_k} ,

we get Itô's formula

Itô's formula for physicists

Simplification: We consider $f : \mathbb{R} \rightarrow \mathbb{R}$ only.

Intuition:

- Taylor expansion of $f(W)$ up to $o(t)$.
- dW_t can be identified with $\sqrt{dt}$.

Heuristic computation: We get

$$\begin{aligned} df(W_t) &= f'(W_t) dW_t + \frac{1}{2} f''(W_t) (dW_t)^2 + o((dW_t)^2) \\ &= f'(W_t) dW_t + \frac{1}{2} f''(W_t) dt + o(dt). \end{aligned}$$

Itô's formula is then obtained by integration.

Proof in the real case

Simplification: We consider $f : \mathbb{R} \rightarrow \mathbb{R}$ only

Generic partition: We consider

- $\pi = \{t_0, \dots, t_n\}$ with $0 = t_0 < \dots < t_n = t$
- $|\pi| = \max\{|t_{j+1} - t_j|; 0 \leq j \leq n-1\}$

Taylor's formula: we have

$$\begin{aligned}\delta f(W)_{0t} &= \sum_{j=0}^{n-1} \delta f(W)_{t_j t_{j+1}} \\ &= \sum_{j=0}^{n-1} f'(W_{t_j}) \delta W_{t_j t_{j+1}} + \frac{1}{2} f''(\xi_j) (\delta W_{t_j t_{j+1}})^2,\end{aligned}$$

where $\xi_j \in [W_{t_j}, W_{t_{j+1}}]$.

Proof in the real case (2)

Notation: We set

$$J_t^{1,\pi} = \sum_{j=0}^{n-1} f'(W_{t_j}) \delta W_{t_j t_{j+1}}, \quad J_t^{2,\pi} = \sum_{j=0}^{n-1} f''(\xi_j) \left(\delta W_{t_j t_{j+1}} \right)^2$$

Aim: Find a sequence $(\pi_m)_{m \geq 1}$ such that a.s

- $\lim_{m \rightarrow \infty} |\pi_m| = 0$
- $\lim_{m \rightarrow \infty} J_t^{1,m} = \int_0^t f'(W_s) dW_s$ with $J_t^{1,m} = J_t^{1,\pi_m}$
- $\lim_{m \rightarrow \infty} J_t^{2,m} = \int_0^t f''(W_s) ds$ with $J_t^{2,m} = J_t^{2,\pi_m}$

Proof in the real case (3)

Analysis of the term $J_t^{1,m}$: we have

$$J_t^{1,m} = \int_0^t u_s^m dW_s, \quad \text{with} \quad u^m \equiv \sum_{t_j \in \pi^m} u_{t_j} \mathbf{1}_{(t_j, t_{j+1}]}]$$

Since $u \equiv f'(W) \in L_a^2$, continuous and bounded

$\hookrightarrow$ according to Proposition 6 we get:

$$L^2(\Omega) - \lim_{m \rightarrow \infty} J_t^{1,m} = \int_0^t f'(W_s) dW_s$$

a.s convergence: For a subsequence $\pi_m \equiv \pi_{m_k}$ we have

$$\text{a.s} - \lim_{m \rightarrow \infty} J_t^{1,m} = \int_0^t f'(W_s) dW_s.$$

Proof in the real case (4)

Analysis of the term $J_t^{2,m}$, strategy: We set

$$J_t^{3,\pi} = \sum_{j=0}^{n-1} f''(W_{t_j}) \left(\delta W_{t_j t_{j+1}} \right)^2, \quad J_t^{4,\pi} = \sum_{j=0}^{n-1} f''(W_{t_j}) (t_{j+1} - t_j)$$

We will show that a.s, for a subsequence π_m with $|\pi_m| \rightarrow 0$:

$$\lim_{m \rightarrow \infty} \left| J_t^{2,\pi_m} - J_t^{3,\pi_m} \right| = 0 \quad (1)$$

$$\lim_{m \rightarrow \infty} \left| J_t^{3,\pi_m} - J_t^{4,\pi_m} \right| = 0 \quad (2)$$

$$\lim_{m \rightarrow \infty} \left| J_t^{4,\pi_m} - \int_0^t f''(W_s) ds \right| = 0 \quad (3)$$

This will end the proof.

Proof in the real case (5)

Recall: we have

$$J_t^{4,\pi_m} = \sum_{j=0}^{n-1} f''(W_{t_j}) (t_{j+1} - t_j)$$

Thus $J_t^{4,\pi_m} \equiv$ Riemann sum for $f''(W)$

Proof relation (3): Since $s \mapsto f''(W_s)$ continuous, we have

$$\lim_{m \rightarrow \infty} J_t^{4,\pi_m} = \int_0^t f''(W_s) ds.$$

Proof in the real case (6)

Proof relation (1): we have

$$J_t^{3,\pi} - J_t^{2,\pi} = \sum_{j=0}^{n-1} \left(f''(W_{t_j}) - f''(\xi_j) \right) \left(\delta W_{t_j t_{j+1}} \right)^2$$

Additional Hyp. for sake of simplicity: $f \in \mathcal{C}_b^3$. Then for $0 < \gamma < \frac{1}{2}$ we have:

- $\xi_j \in [W_{t_j}, W_{t_{j+1}}]$
- $|\delta W_{t_j t_{j+1}}| \leq c_{W,\gamma} |t_{j+1} - t_j|^\gamma$
 $\hookrightarrow |f''(W_{t_j}) - f''(\xi_j)| \leq c_{W,\gamma,f} |\pi^m|^\gamma$

Thus applying Theorem 27, Brownian Chapter:

$$\begin{aligned} \left| J_t^{3,\pi^m} - J_t^{2,\pi^m} \right| &\leq c_{W,\gamma} |\pi^m|^\gamma \sum_{j=0}^{n-1} \left(\delta W_{t_j t_{j+1}} \right)^2 \\ &\leq c_{W,\gamma} |\pi^m|^\gamma V_{0,t}^2(W) = c_{W,\gamma} t |\pi^m|^\gamma \xrightarrow{m \rightarrow \infty} 0 \end{aligned}$$

Proof in the real case (7)

Notation: We set

$$\delta t_j \equiv (t_{j+1} - t_j), \quad Z_j \equiv \left(\delta W_{t_j t_{j+1}} \right)^2 - \delta t_j$$

Proof relation (2), strategy:

we have

$$J_t^{4,\pi_m} - J_t^{3,\pi_m} = \sum_{j=0}^{n-1} f''(W_{t_j}) Z_j.$$

We proceed as follows:

- 1 We show $\lim_{m \rightarrow \infty} \mathbf{E}[|J_t^{4,\pi_m} - J_t^{3,\pi_m}|^2] = 0$.
- 2 For a subsequence $\pi_m \equiv \pi_{m_k}$ we deduce:

$$\text{a.s.} - \lim_{m \rightarrow \infty} |J_t^{4,\pi_m} - J_t^{3,\pi_m}| = 0.$$

Proof in the real case (8)

Decomposition of $J_t^{4,\pi_m} - J_t^{3,\pi_m}$:

we have $J_t^{4,\pi_m} - J_t^{3,\pi_m} = \sum_{j=0}^{n-1} f''(W_{t_j}) Z_j$. Thus:

$$\mathbf{E} \left[\left| J_t^{4,\pi_m} - J_t^{3,\pi_m} \right|^2 \right] = K_t^{m,1} + K_t^{m,2}$$

with

$$K_t^{m,1} = \sum_{j=0}^{n-1} \mathbf{E} \left[\left(f''(W_{t_j}) Z_j \right)^2 \right]$$

$$K_t^{m,2} = 2 \sum_{0 \leq j < k \leq n-1} \mathbf{E} \left[f''(W_{t_j}) Z_j f''(W_{t_k}) Z_k \right]$$

Lemma: Let $X \sim \mathcal{N}(0, \sigma^2)$. Then:

$$\mathbf{E} \left[\left(X^2 - \sigma^2 \right)^2 \right] = 2\sigma^4$$

Proof in the real case (9)

Convergence of $K_t^{m,1}$:

We have $\delta W_{t_j t_{j+1}} \sim \mathcal{N}(0, \delta t_j)$ and $\delta W_{t_j t_{j+1}} \perp\!\!\!\perp \mathcal{F}_{t_j}$. Therefore:

$$\begin{aligned} K_t^{m,1} &= \sum_{j=0}^{n-1} \mathbf{E} \left[\left(f''(W_{t_j}) Z_j \right)^2 \right] \\ &= \sum_{j=0}^{n-1} \mathbf{E} \left[\left(f''(W_{t_j}) \right)^2 \right] \mathbf{E} \left[Z_j^2 \right] \\ &= 2 \sum_{j=0}^{n-1} \mathbf{E} \left[\left(f''(W_{t_j}) \right)^2 \right] (\delta t_j)^2 \\ &\leq c_f |\pi^m| \sum_{j=0}^{n-1} \delta t_j = c_f t |\pi^m| \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Proof in the real case (10)

Proof relation (2): we have seen

$$\lim_{m \rightarrow \infty} K_t^{m,1} = 0.$$

In the same way, one can check that:

$$K_t^{m,2} = 0.$$

Thus:

$$\lim_{m \rightarrow \infty} \mathbf{E}[|J_t^{4,\pi_m} - J_t^{3,\pi_m}|^2] = 0.$$

This finishes the proof of (2) and of Itô's formula.