Outline

1. Restarting a diffusion
2. Generator of a diffusion
3. Transition densities and Kolmogorov equations

Setting

Data:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\tau]}, \mathbf{P})$ stochastic basis
- W d-dimensional Brownian motion
- $b, \sigma^1, \dots, \sigma^d : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz, $\alpha \in \mathbb{R}^n$

Equation: X unique solution in $L^2(\Omega; \mathcal{C}([0,\tau]))$ of $\mathcal{H}_\tau$

$$X_t = \alpha + \int_0^t b(X_r)\, dr + \sum_{k=1}^{d} \int_0^t \sigma^k(X_r)\, dW_r^k \tag{1}$$

Aim: Markov property for X, as for Brownian motion

$\hookrightarrow$ Future after time s driven by the increments of W after s

Restarted Brownian motion

Notation: For $s \in [0, \tau)$ we set

$$\begin{aligned} W_u^s &= W_{s+u} - W_s, \quad u \in [0, \tau - s] \\ \mathcal{G}^s &= \sigma\{W_u^s;\ u \in [0, \tau - s]\} \end{aligned}$$

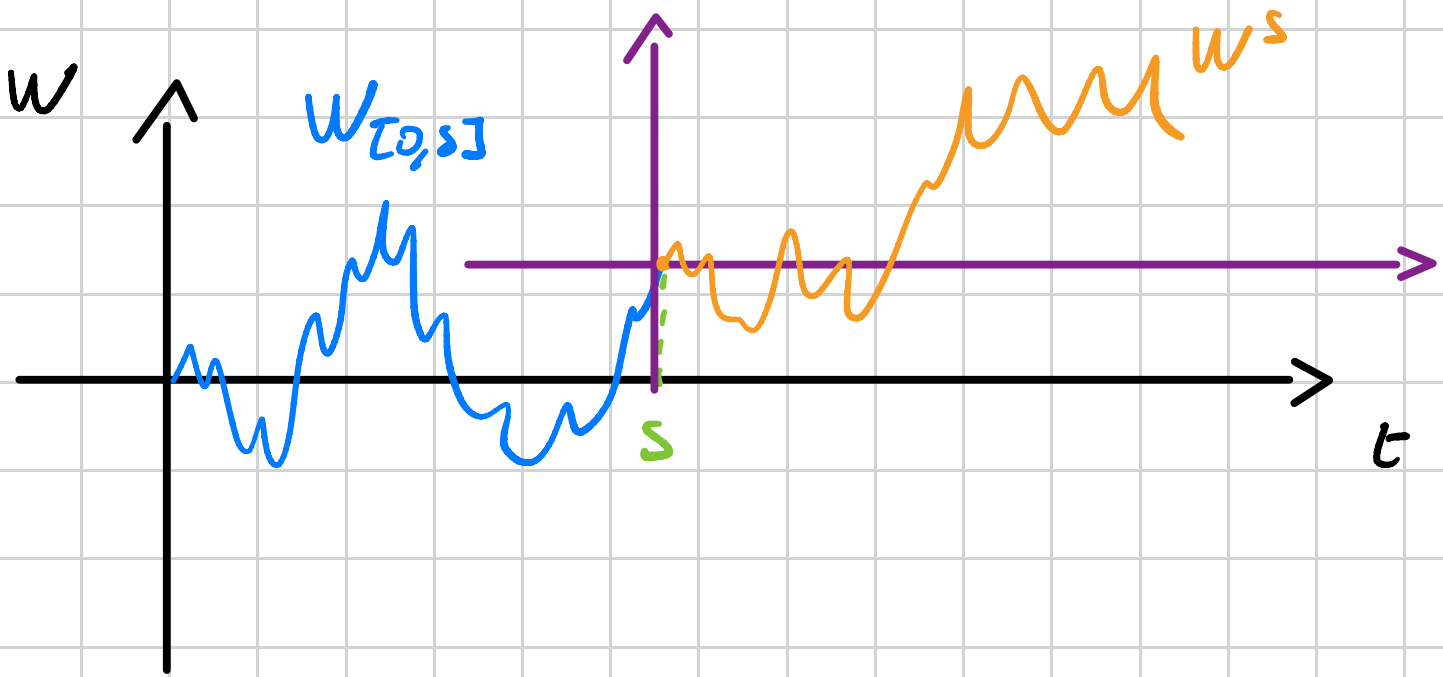
$\hookrightarrow$ Restarted filtration: $(\mathcal{F}_{s+u})_{u \in [0, \tau - s]}$

Lemma 1

Let $s \in [0, \tau)$. Then:

- *W^s is a Brownian motion with respect to $(\mathcal{F}_{s+u})_{u \in [0, \tau - s]}$*
- *$\mathcal{G}^s$ and $\mathcal{F}_s$ are independent*

$$W_u^s = W_{s+u} - W_s$$



We have $w^s \perp F_s$

Restarted diffusion

Same sde as before, driven by W^s

Notation: For $s \in [0, \tau)$ and $x \in \mathbb{R}^n$,
$X^{s,x} = \{X_u^{s,x};\ u \in [0, \tau - s]\}$ solution of

$$X_u^{s,x} = x + \int_0^u b(X_r^{s,x})\, dr + \sum_{k=1}^{d} \int_0^u \sigma^k(X_r^{s,x})\, dW_r^{s,k}$$

next question: better expression for this integral

Well posedness:

- Same equation as (1), on the restarted basis, driven by W^s
- Existence and uniqueness theorem applies verbatim
- $X^{s,x}$ adapted to $(\mathcal{F}_{s+u})_{u \in [0,\tau-s]}$, and $X^{0,\alpha} = X$

Markov property for BM

Decompose W_{s+u}, for $s, u \geq 0$

$$W_{s+u} = W_s + \underbrace{(W_{s+u} - W_s)}_{\equiv W_u^s}$$

Prop: $W^s \perp\!\!\!\perp \mathcal{F}_s$

Then

$$\mathbb{E}[\phi(W_{s+u}) \mid \mathcal{F}_s]$$

$$= \mathbb{E}[\phi(\underbrace{W_s}_{\in \mathcal{F}_s} + \underbrace{W_u^s}_{\perp\!\!\!\perp \mathcal{F}_s}) \mid \mathcal{F}_s]$$

freezing

$$= \mathbb{E}[\phi(x + W_u^s)] \mid_{x = W_s} = \overbrace{P_u \phi(W_s)}^{\text{heat semigroup}}$$

$\Rightarrow$ Markov

Aim: generalize the previous slide to a diffusion process

Restarted diffusion after s, starting from x

$$X_u^{s,x} = x + \int_0^u b(X_r^{s,x})\,dr + \int_0^u \sigma(X_r^{s,x})\,dW_r^s$$

restarted BM

Next step: simple change of variable for the stochastic integral

Restarting the Itô integral

Lemma 2

Let

- $v \in L^2_a([0, \tau - s])$ *on the restarted basis*
- $\bar{v}_r = v_{r-s}\, \mathbf{1}_{(s,\tau]}(r)$

Then for $u \in [0, \tau - s]$,

$$\int_0^u v_r \, dW_r^s = \int_s^{s+u} \bar{v}_r \, dW_r$$

Meaning: Integrals against W^s on $[0, u]$

- Are integrals against W on $[s, s + u]$
- Needed to restart equation (1) at time s

"Proof" We write $W_r^s = W_{s+r} - W_s$

$$\int_0^u v_r \, dW_r^s = \int_0^u v_r \, d[W_{s+r} - W_s]$$

(W_s: constant)

$$= \int_0^u v_r \, dW_{s+r}$$

cv: $t = s + r$, $\bar{v}_t = v_{t-s}$

$$= \int_s^{s+u} v_{t-s} \, dW_t$$

$$= \int_s^{s+u} \bar{v}_t \, dW_t$$

Justification of cv: for simple processes then take limits

Proof

Formal derivation: $dW_r^s = dW_{s+r}$

$\hookrightarrow$ Then change of variable $t = s + r$:

$$\int_0^u v_r \, dW_{s+r} = \int_s^{s+u} v_{t-s} \, dW_t = \int_s^{s+u} \bar{v}_t \, dW_t$$

Rigorous proof: No change of variables inside an Itô integral

- Check the identity for simple processes v
  $\hookrightarrow$ finite sums of increments of W
- Pass to the limit in $L^2(\Omega)$
  $\hookrightarrow$ by density of simple processes in L_a^2

Independence from the past

If $Z \perp\!\!\!\perp \mathcal{F}$, $\mathbb{E}[Z \mid \mathcal{F}] = \mathbb{E}[Z]$

Lemma 3

Let $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $u_1, \ldots, u_m \in [0, \tau - s]$. Then:

$$\left(X_{u_1}^{s,x}, \ldots, X_{u_m}^{s,x}\right) \perp\!\!\!\perp \mathcal{F}_s$$

In particular, for $\Phi : \mathbb{R}^n \to \mathbb{R}$ bounded measurable and $u \in [0, \tau - s]$,

$$\mathbf{E}\left[\Phi(X_u^{s,x}) \,|\, \mathcal{F}_s\right] = \mathbf{E}\left[\Phi(X_u^{s,x})\right]$$

Interpretation: Analog of $W_{s+u} - W_s \perp\!\!\!\perp \mathcal{F}_s$ for the diffusion
$\hookrightarrow$ This is why the equation is restarted from a deterministic point x

"Proof" Write $W^s_r = W_{s+r} - W_s$

$$X_u^{s,x} = x + \int_0^u b(X_r^{s,x})\,dr + \int_0^u \sigma(X_r^{s,x})\,dW_r^s$$

We have that W^s is a BM
Thus $X^{s,x}$ solves an equation driven by a BM

If b, σ smooth we have seen that there is a strong solution: there exists a measurable F such that

$$X_u^{s,x} = F(x, W^s)$$

Conclusion We have seen

$$X_u^{3x} = F(x, W^3)$$

$W^3 \perp\!\!\!\perp F_3$

$$\Rightarrow \boxed{X_u^{3x} \perp\!\!\!\perp F_3}$$

Same is true for a vector

$$(X_{u_1}^{3x}, \ldots, X_{u_k}^{3x})$$

Proof

Measurability: $X^{s,x}$ is a strong solution on the restarted basis
$\hookrightarrow X_u^{s,x}$ measurable with respect to $\sigma(W_r^s;\, r \leq u) \subset \mathcal{G}^s$

Independence: $\mathcal{G}^s \perp\!\!\!\perp \mathcal{F}_s$, by the lemma on restarted Brownian motion
$\hookrightarrow (X_{u_1}^{s,x}, \ldots, X_{u_m}^{s,x}) \perp\!\!\!\perp \mathcal{F}_s$

Conditional expectation: For $Z \perp\!\!\!\perp \mathcal{F}_s$ integrable,

$$\mathbf{E}\left[Z \,|\, \mathcal{F}_s\right] = \mathbf{E}\left[Z\right]$$

$\hookrightarrow$ Apply to $Z = \Phi(X_u^{s,x})$

Restart property

Proposition 4

Let $s \in [0, \tau)$ and X solution of (1). Then for $u \in [0, \tau - s]$,

$$X_{s+u} = X_s + \int_0^u b(X_{s+r})\, dr + \sum_{k=1}^{d} \int_0^u \sigma^k(X_{s+r})\, dW_r^{s,k}$$

Interpretation: After time s, X solves the restarted equation
$\hookrightarrow$ Started from the random point $X_s \in \mathcal{F}_s$, driven by W^s

Proof Write

$$X_{s+u} = \alpha + \int_0^{s+u} b(X_r)dr + \int_0^{s+u} \sigma(X_r)dW_r$$

$$X_s = \alpha + \int_0^{s} b(X_r)dr + \int_0^{s} \sigma(X_r)dW_r$$

$$\Rightarrow X_{s+u} - X_s = \int_s^{s+u} b(X_r)dr + \int_s^{s+u} \sigma(X_r)dW_r$$

$$= d(W_r - W_s)$$

CV: $r = s + v$

$$\Rightarrow \boxed{X_{s+u} = X_s + \int_0^u b(X_{s+v})dv + \int_0^u \sigma(X_{s+v})\,dW_v^s}$$

Informal proof of Markov

$$X_{s+u} = X_s + \int_0^u b(X_{s+v})\,dv + \int_0^u \sigma(X_{s+v})\,dW_v^s$$

we still have a strong solution

$$\Rightarrow \quad X_{s+u} = F(X_s, W^s)$$

$$\Rightarrow \quad \mathbb{E}[\phi(X_{s+u}) \mid \mathcal{F}_s]$$

$$= \mathbb{E}[\phi(F(\underbrace{X_s}_{\in \mathcal{F}_s}, \underbrace{W^s}_{\perp\!\!\!\perp \mathcal{F}^s}) \mid \mathcal{F}_s]$$

freezing

$$= \mathbb{E}[\phi(F(x, W^s))] \,\big|_{x = X_s}$$

Markov

Proof

Increment: Subtract (1) at times $s+u$ and s:

$$X_{s+u} - X_s = \int_s^{s+u} b(X_r)\, dr + \sum_{k=1}^{d} \int_s^{s+u} \sigma^k(X_r)\, dW_r^k$$

Lebesgue term: Change of variables $r = s + r'$, path by path

Stochastic term: Restarted Itô integral, with $v_r = \sigma^k(X_{s+r})$ and $\bar{v}_r = \sigma^k(X_r)$ on $(s, \tau]$

$$\int_s^{s+u} \sigma^k(X_r)\, dW_r^k = \int_0^u \sigma^k(X_{s+r})\, dW_r^{s,k}$$

Transition operators

Definition 5

Let:

- $0 \leq s \leq t \leq \tau$
- $\Phi : \mathbb{R}^n \to \mathbb{R}$ bounded measurable

The transition operator $p_{s,t}$ is defined by

$$p_{s,t}\Phi(x) = \mathbf{E}\left[\Phi\left(X_{t-s}^{s,x}\right)\right], \quad x \in \mathbb{R}^n$$

Interpretation:

- Average of Φ over the solution restarted from x at time s
- Generalizes the heat semigroup of Chapter 1

$$X_u^{s,x} = x + \int_0^u b(X_r)dr + \int_0^t \sigma(X_r)dW_r^s$$

Properties of $P_{s,t}$

$$P_{s,t}\,\phi(x) = \mathbb{E}\left[\phi\left(X_{t-s}^{s,x}\right)\right]$$

Then

$$\boxed{P_{s,s}\,\phi(x)} = \mathbb{E}\left[\phi\left(X_0^{s,x}\right)\right]$$

$$= \mathbb{E}\left[\phi(x)\right] \boxed{= \phi(x)}$$

Hyp: $|\phi(x)| \le M$

$$\boxed{\left|P_{s,t}\,\phi(x)\right|} = \left|\mathbb{E}\left[\phi\left(X_{t-s}^{s,x}\right)\right]\right|$$

$$\le \mathbb{E}\left[\left|\phi\left(X_{t-s}^{s,x}\right)\right|\right] \boxed{\le M}$$

Transition operators: properties

Properties:

- $p_{s,s}\Phi = \Phi$, since $X_0^{s,x} = x$
- $\|p_{s,t}\Phi\|_\infty \leq \|\Phi\|_\infty$
- Φ bounded Lipschitz $\Longrightarrow$ $p_{s,t}\Phi$ bounded Lipschitz
  $\hookrightarrow$ stability of $X^{s,x}$ in x

Brownian case: $b = 0$ and $\sigma = \mathrm{Id}$, so that

- $X_u^{s,x} = x + W_u^s$
- $p_{s,t} = p_{t-s}$, heat semigroup of Chapter 1

Markov property

Theorem 6

Let:

- b, σ *Lipschitz,* X *solution of* (1)
- $0 \leq s \leq t \leq \tau$, *and* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *bounded measurable*

Then $p_{s,t}\Phi$ *is measurable, and*

$$\mathbf{E}\left[\Phi(X_t) \,|\, \mathcal{F}_s\right] = p_{s,t}\Phi(X_s)$$

Interpretation:

Future after s predicted with X_s only

$\hookrightarrow$ Solve (1) restarted from X_s, then average

Markov property: idea of proof

Step 1, X_s with countably many values x_i: Set $A_i = (X_s = x_i) \in \mathcal{F}_s$
$\hookrightarrow$ $Z = \sum_i \mathbf{1}_{A_i} X^{s,x_i}$ solves the restarted equation from X_s
$\hookrightarrow$ $Z = X_{s+\cdot}$ by uniqueness, and independence from the past gives

$$\mathbf{E}\left[\Phi(X_t) \,|\, \mathcal{F}_s\right] = \sum_i \mathbf{1}_{A_i} \mathbf{E}\left[\Phi(X^{s,x_i}_{t-s})\right] = p_{s,t}\Phi(X_s)$$

Step 2, general X_s: Discretize X_s on dyadic cubes
$\hookrightarrow$ Stability with respect to the initial condition, for Φ Lipschitz

Step 3, general Φ: Monotone class theorem, as for Brownian motion

Outline

1. Restarting a diffusion
2. Generator of a diffusion
3. Transition densities and Kolmogorov equations

Aim : we have seen

$$\mathbb{E}[\phi(X_t) \mid \mathcal{F}_s] = P_{s,t}\,\phi(X_s)$$

In fact if $b = b(x)$, $\sigma = \sigma(x)$, the diffusion is homogeneous. In that case

$$\boxed{\mathbb{E}[\phi(X_t) \mid \mathcal{F}_s] = \tilde{P}_{t-s}\,\phi(X_s)}$$

Then we want to differentiate this relation wrt time

Diffusion matrix

Definition 7

Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ the coefficients of the SDE. The diffusion matrix is

$$a = \sigma\sigma^* : \mathbb{R}^n \to \mathbb{R}^{n\times n}, \quad \text{that is} \quad a^{j_1 j_2}(x) = \sum_{k=1}^{d} \sigma^{j_1 k}(x)\,\sigma^{j_2 k}(x)$$

$\sigma\sigma^T$

Properties:

- $a(x)$ symmetric and nonnegative, since $\langle a(x)y, y\rangle = |\sigma(x)^* y|^2$
- a Lipschitz on bounded sets whenever σ is Lipschitz

Generator

$X_t = x + \int_0^t b(X_r)\,dr + \int_0^t \sigma(X_r)\,dW_r$

Definition 8

Let:

- b, σ coefficients of the SDE, $a = \sigma\sigma^*$ the diffusion matrix
- $f \in \mathcal{C}^2(\mathbb{R}^n)$

The generator $\mathcal{L}$ of the diffusion is defined by

depends on noisy part

$$\mathcal{L}f(x) = \sum_{j=1}^{n} b^j(x)\,\partial_{x_j} f(x) + \frac{1}{2}\sum_{j_1,j_2=1}^{n} a^{j_1 j_2}(x)\,\partial^2_{x_{j_1}x_{j_2}} f(x)$$

depends on drift

Interpretation: Second order differential operator

$\hookrightarrow$ First order part from the drift, second order part from the noise

Example if $n = d = 1$. Then

$$X_t = \alpha + \int_0^t b(X_r)\,dr + \int_0^t \sigma(X_r)\,dW_r$$

Then $a(x) = \sigma^2(x)$ and

$$\mathcal{L}f(x) = b(x) f'(x) + \frac{1}{2} a(x) f''(x)$$

If now $b = 0$, $\sigma = 1$, we get

$$X_t = \alpha + W_t \quad \text{and}$$

$$\mathcal{L}f(x) = \frac{1}{2} f''(x)$$

we had seen: $\frac{1}{2}\Delta$ is the generator of BM

Generator: examples

Brownian motion: $b = 0$, $\sigma = \mathrm{Id}$, hence $a = \mathrm{Id}$ and

$$\mathcal{L}f = \frac{1}{2}\sum_{j=1}^{n} \partial^2_{x_j x_j} f = \frac{1}{2}\Delta f$$

$\hookrightarrow$ Operator of the heat equation seen in Chapter 1

Case $n = d = 1$: $b, \sigma : \mathbb{R} \to \mathbb{R}$, hence $a = \sigma^2$ and

$$\mathcal{L}f = b\, f' + \frac{1}{2}\sigma^2\, f''$$

$\hookrightarrow$ Setting of the proof below