1-d setting

$$X_t = \alpha + \int_0^t b(X_r)\,dr + \int_0^t \sigma(X_r)\,dW_r$$

$$X_t^{s,x} = x + \int_0^t b(X_r^{s,x})\,dr + \int_0^t \sigma(X_r^{s,x})\,dW_r^s$$

$$P_{s,t}\,\phi(x) = \mathbb{E}\left[\phi(X_t^{s,x})\right]$$

Markov: $\mathbb{E}\left[\phi(X_t) \mid \mathcal{F}_s\right] = P_{s,t}\,\phi(X_s)$

Interpretation: prediction of X_t given the past up to time s

Generator: $\mathcal{L}\phi(x) = b(x)\,\phi'(x) + \frac{1}{2}\,a(x)\,\phi''(x)$

Integrated generator identity

Lemma 9

Let b, σ Lipschitz, $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then for $u \in [0, \tau - s]$,

$$p_{s,s+u} f(x) = f(x) + \int_0^u \mathbf{E}\left[\mathcal{L} f(X_r^{s,x})\right] dr$$

Interpretation: Evolution of $\mathbf{E}[f(X_u^{s,x})]$ governed by $\mathcal{L}$
$\hookrightarrow$ Analog of the heat semigroup identity for Brownian motion

Recall : for a BM W ($W_0 = x$) starting from x,

↳ $b=0$, $\sigma=1$, $a=1$

we have (Itô)

$$f(W_t) = f(x) + \int_0^t f'(W_r)\,dW_r + \frac{1}{2}\int_0^t f''(W_r)\,dr$$

$$\Rightarrow \mathbb{E}[f(W_t)] = f(x) + \int_0^t \mathbb{E}\left[\frac{1}{2} f''(W_r)\right] dr$$

This can be read as

$$\boxed{P_{0,t} f(x) = f(x) + \int_0^t \mathbb{E}[\mathcal{L}^W f(W_r)]\,dr}$$

Proof: strategy

Two steps:

- $X^{s,x}$ is an Itô process on the restarted basis
- Itô's formula for $f(X_u^{s,x})$, then take expectations

Below: Proof for $n = d = 1$, after recalling the Itô process slides
$\hookrightarrow$ General case: same computation with indices

Recall: Itô process in dimension 1

Definition 10

Let:

- $\alpha \in \mathbb{R}$, and b bounded adapted real process
- σ real process in $L^2_a([0,\tau])$

Diff case : $b_r = b(X_r)$, $\sigma_r = \sigma(X_r)$

We say that X is an Itô process if

$$X_t = \alpha + \int_0^t b_r \, dr + \int_0^t \sigma_r \, dW_r$$

Solution of (1) with $n = d = 1$: Itô process with
$\hookrightarrow$ $b_r = b(X_r)$ and $\sigma_r = \sigma(X_r)$
$\hookrightarrow$ $b(X_\cdot)$ only continuous, which is enough for $\int_0^t b_r \, dr$

Recall: Itô's formula in dimension 1

Proposition 11

Let X Itô process defined by α, b, σ, and $f \in \mathcal{C}_b^2(\mathbb{R})$. Then:

$$f(X_t) = f(\alpha) + \int_0^t f'(X_r)\, b_r\, dr + \int_0^t f'(X_r)\, \sigma_r\, dW_r + \frac{1}{2}\int_0^t f''(X_r)\, \sigma_r^2\, dr$$

Remark: $\int_0^t f'(X_r)\, \sigma_r\, dW_r$ is a martingale

$\hookrightarrow$ Since f' bounded and $\sigma \in L_a^2([0, \tau])$

$\hookrightarrow$ Vanishing expectation

dW_t "$=$" $\sqrt{dt}$ $\quad (dW_t)^2 = dt$ $\quad \sigma^2 = a$

Recall: $X_t = \alpha + \int_0^t b(X_r)\,dr + \int_0^t \sigma(X_r)\,dW_r$

Itô: $f(X_t) = f(\alpha) + \int_0^t f'(X_r)\, b(X_r)\,dr$

$+ \int_0^t \sigma(X_r)\, f'(X_r)\,dW_r + \frac{1}{2}\int_0^t a(X_r)\, f''(X_r)\,dr$

$= f(\alpha) + \int_0^t \sigma(X_r)\, f'(X_r)\,dW_r = \mathcal{L} f(X_r)$

$+ \int_0^t \left[b(X_r)\, f'(X_r) + \frac{1}{2} a(X_r)\, f''(X_r) \right] dr$

$\Rightarrow$

$$\mathbb{E}[f(X_t)] \equiv P_{0,t} f(x)$$

$$= f(\alpha) + \int_0^t \mathbb{E}[\mathcal{L} f(X_r)]\,dr$$

↳ easy to generalize to $P_{s,t}$

Proof of the integrated identity for $n = d = 1$

Itô process: On the restarted basis, with $b_r = b(X_r^{s,x})$ and $\sigma_r = \sigma(X_r^{s,x})$

Itô's formula: Grouping the dr terms with $\mathcal{L}f = b\,f' + \frac{1}{2}\sigma^2 f''$,

$$f(X_u^{s,x}) = f(x) + \int_0^u \mathcal{L}f(X_r^{s,x})\,dr + \int_0^u f'(X_r^{s,x})\,\sigma(X_r^{s,x})\,dW_r^s$$

Expectation:

- Stochastic integral has mean 0, and $\mathbf{E}[f(X_u^{s,x})] = p_{s,s+u}f(x)$
- Fubini for the dr term: $|\mathcal{L}f(z)| \leq c_f(1 + |z|^2)$ and $\mathbf{E}[\sup_r |X_r^{s,x}|^2] < \infty$

Generator of a diffusion

Theorem 12

Let b, σ Lipschitz, $s \in [0, \tau)$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$,

$$\lim_{h \to 0^+} \frac{p_{s,s+h} f(x) - f(x)}{h} = \mathcal{L} f(x)$$

Interpretation: $\mathcal{L}$ is the infinitesimal generator of $\{p_{s,t};\ 0 \leq s \leq t \leq \tau\}$

$\hookrightarrow$ Deterministic operator, coefficients depend on the current position only

Brownian case: $p_{s,s+h} = p_h$ heat semigroup and $\mathcal{L} = \frac{1}{2}\Delta$

$\hookrightarrow$ Generator of Brownian motion, seen in Chapter 2

Proof We have seen

$$P_{s,s+h} f(x) = f(x) + \int_s^{s+h} \mathbb{E}[\mathcal{L} f(X_r^{s,x})]\, dr$$

$$\Rightarrow \frac{1}{h}\left(P_{s,s+h} f(x) - f(x)\right) = \frac{1}{h}\int_s^{s+h} \mathbb{E}[\mathcal{L} f(X_r^{s,x})]\, dr$$

If φ smooth, $\lim_{h\to 0} \frac{1}{h}\int_s^{s+h} \varphi_r\, dr = \varphi_s$

Limit on rhs as $h \to 0$

$$\lim_{h\to 0} \frac{1}{h}\int_s^{s+h} \mathbb{E}[\mathcal{L} f(X_r^{s,x})]\, dr \overset{\text{dom. convergence}}{=} \mathbb{E}[\mathcal{L} f(X_s^{s,x})]$$

($\mathcal{L} f$ unif. bounded)

$$= \mathcal{L} f(x)$$

Recall: Consider $X_t \xrightarrow{a.s.} Y$ as $t \to t_0$

Then if we can apply dominated convergence, we have

$$\mathbb{E}[X_t] \longrightarrow \mathbb{E}[Y]$$

Example: This works if $|X_t| \leq M$ a.s.

Proof

Step 1, continuity at $r = 0$: We have

$$\lim_{r \to 0^+} \mathbf{E}\left[\mathcal{L}f(X_r^{s,x})\right] = \mathcal{L}f(x)$$

- $X_r^{s,x} \to x$ a.s. (continuous paths, $X_0^{s,x} = x$), and $\mathcal{L}f$ continuous
- Domination: $|\mathcal{L}f(X_r^{s,x})| \leq c_f(1 + \sup_r |X_r^{s,x}|^2)$, integrable

Step 2, averages: Integrated identity divided by h:

$$\frac{p_{s,s+h}f(x) - f(x)}{h} = \frac{1}{h}\int_0^h \mathbf{E}\left[\mathcal{L}f(X_r^{s,x})\right] dr \xrightarrow[h \to 0^+]{} \mathcal{L}f(x)$$

$\hookrightarrow$ Average of a bounded function, continuous at $r = 0$

Evolution equation for $p_{s,t}f$

Corollary 13

Let b, σ Lipschitz, $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then $t \mapsto p_{s,t}f(x)$ is $\mathcal{C}^1$ on $[s, \tau]$, and

$$\partial_t \, p_{s,t}f(x) = \mathbf{E}\left[\mathcal{L}f\left(X_{t-s}^{s,x}\right)\right]$$

If $s=t$ $\quad \partial_t p_{s,t} f(x)|_{s=t} = \mathbb{E}[\mathcal{L}f(X_0^{s,x})]$
$= \mathcal{L}f(x)$

Interpretation:

- Right hand side is $p_{s,t}(\mathcal{L}f)(x)$, for $\mathcal{L}f$ with quadratic growth
- At $t = s$, the generator theorem: $\partial_t \, p_{s,t}f(x)|_{t=s} = \mathcal{L}f(x)$

Kolmogorov PDE

On diagonal, we get $\partial_t p_{s,t} f(x) = \mathcal{L} f(x)$

Proof

Continuity: $r \mapsto \mathbf{E}[\mathcal{L}f(X_r^{s,x})]$ continuous on $[0, \tau - s]$
$\hookrightarrow$ Continuous paths, plus the domination of Step 1 above

Integrated identity with $u = t - s$:

$$p_{s,t}f(x) = f(x) + \int_0^{t-s} \mathbf{E}\left[\mathcal{L}f(X_r^{s,x})\right] dr$$

$\hookrightarrow$ Fundamental theorem of calculus in the variable t

Towards a PDE for $p_{s,t}f$



Brownian case: $p_{s,t} = p_{t-s}$ and $\mathcal{L} = \frac{1}{2}\Delta$, so that

$$\partial_t\, p_t f = p_t\left(\tfrac{1}{2}\Delta f\right) = \tfrac{1}{2}\Delta\left(p_t f\right)$$

$\hookrightarrow$ Heat equation of Chapter 1

pred from s-h to s

pred. from s to t

General case, formally: $u(s,x) = p_{s,t}f(x)$ smooth in (s,x)

$\hookrightarrow$ $p_{s-h,t}f = p_{s-h,s}(p_{s,t}f)$, then generator theorem as $h \to 0^+$

predicting from s-h to t

$$\partial_s u + \mathcal{L}u = 0, \quad u(t,\cdot) = f$$

$\hookrightarrow$ Backward equation, proved in the next section in the additive case

Outline

1. Restarting a diffusion
2. Generator of a diffusion
3. Transition densities and Kolmogorov equations

Additive noise

Setting: $d = n$, and the equation with additive noise

Wiener integral, Gaussian

$$X_t = X_0 + \int_0^t b(r, X_r)\, dr + \int_0^t \sigma(r)\, dW_r \tag{2}$$

b depends on r -> non homogeneous coefficient

- $b : [0, \tau] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, \tau] \to \mathbb{R}^{n \times n}$) square matrix
- $X_0 \in \mathcal{F}_0$ random initial condition, with law μ law we want to sample from

Comments:

- Well posed path by path (Chapter 3)
- Forward equation of generative models, reversed in Chapter 5

Comment about how to solve the eq.

$$X_t = X_0 + \int_0^t b(r, X_r)\,dr + Z_t$$

where $Z_t = \int_0^t \sigma(r)\,dW_r$

Then let $Y_t = X_t - Z_t$. Y solves

$$Y_t = X_0 + \int_0^t b(r, Y_t + Z_t)\,dr$$

ODE with random coefficient

Smoothness assumption

Assumption:

1. b infinitely differentiable, bounded, with bounded derivatives of all orders
2. σ continuously differentiable, and $\sigma(t)$ invertible for all $t \in [0, \tau]$

Remarks:

- Far from minimal, chosen for convenience
- In force for the rest of the deck

Well posedness: b Lipschitz in x, uniformly in t
$\hookrightarrow$ Existence and uniqueness for (2)

Uniform ellipticity

$a = \sigma\sigma^* \rightarrow$ describes conditional variance

If a elliptic $\rightarrow$ noise W diffuses in every direction

Diffusion $\rightarrow$ smoothing. Thus we will get smooth densities

Lemma 14

Let σ as in the smoothness assumption, and $a(t) = \sigma(t)\sigma(t)^*$. Then a is bounded on $[0, \tau]$, and there is $\lambda > 0$ with

$$\langle a(t)\, y, y\rangle = |\sigma(t)^* y|^2 \geq \lambda\, |y|^2, \quad t \in [0, \tau],\ y \in \mathbb{R}^n$$

Interpretation:

- Noise acts in every direction, with strength bounded below
- Hypothesis of the parabolic theory giving smooth densities