Outline

1. Restarting a diffusion
2. Generator of a diffusion
3. Transition densities and Kolmogorov equations
4. Summary

Equation: Shifted version,

$$X_u^{s,x} = x + \int_0^u b(s+r, X_r^{s,x})\, dr + \int_0^u \sigma(s+r)\, dW_r^s$$

Transition operator

$$\tilde{P}_{s,t}\, \phi(x) \equiv \mathbb{E}[\, \phi(X_{t-s}^{s,x})\,]$$

Note: Here we don't have the semigroup structure since diffusion is non homogeneous (b, σ depend on t)

If no dependence on t, i.e

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t,$$

we get

$$\mathbb{E}\left[\phi\left(X_{t-s}^{x}\right)\right] = \tilde{P}_{t-s}\,\phi(x)$$

where $\tilde{P}_t$ has the semigroup property

$$\tilde{P}_{t+s} = \tilde{P}_t\,\tilde{P}_s$$

Recall we have

$$\tilde{P}_{s,t}\,\phi(x) \equiv \mathbb{E}\left[\,\phi(X_{t-s}^{x})\,\right]$$

Markov property

$$\mathbb{E}\left[\,\phi(X_t)\,|\,\mathcal{F}_s\,\right] = \tilde{P}_{s,t}\,\phi(X_s)$$

Generator of X $(X_t \in \mathbb{R}^n)$

$$\mathcal{L}_t \varphi = \sum_{j=1}^{n} b^j(t,\cdot)\, \partial_{x_j} \varphi + \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(t)\, \partial^2_{x_i x_j} \varphi$$

Adjoint generator

$$\mathcal{L}_t^* \psi = -\sum_{j=1}^{n} \partial_{x_j} [\, b_j(t,\cdot)\, \psi\,] + \frac{1}{2} \sum_{i,j=1}^{n} a^{ij}(t)\, \partial^2_{x_i x_j} \psi$$

Duality

$$\int_{\mathbb{R}^n} \mathcal{L}_t \varphi \;\; \psi \; dx = \int_{\mathbb{R}^n} \varphi \, \mathcal{L}_t^* \psi \, dx$$

Chapman-Kolmogorov ($\simeq$ semigroup)

$$\tilde{P}_{s,t'} = \tilde{P}_{s,t} \circ \tilde{P}_{t,t'} \qquad s \leq t \leq t'$$

Forward equation

$$\partial_t \tilde{P}_{s,t} f = \tilde{P}_{s,t} (\mathcal{L}_t f) \,, \quad \tilde{P}_{s,s} f = f$$

Backward equation

$$\partial_s \tilde{P}_{s,t} f = - \mathcal{L}_s (\tilde{P}_{s,t} f) \,, \quad \tilde{P}_{t,t} f = f$$

Kernels (or densities) for $\tilde{P}_{s,t}$. We have

$$\tilde{P}_{s,t}\,\varphi(x) = \int_{\mathbb{R}^n} \tilde{q}_{s,t}(x,y)\,\varphi(y)\,dy$$

$\tilde{q}_{s,t}(x,y)$: kernel or density of $X^{s,x}_{t-s}$

Backward / forward eq. for $\tilde{q}$

$$\partial_s \tilde{q}_{s,t}(x,y) = -[\mathcal{L}_s \tilde{q}_{s,t}(\cdot,y)](x)$$

$$\partial_t \tilde{q}_{s,t}(x,y) = [\mathcal{L}_t^* \tilde{q}_{s,t}(x,\cdot)](y)$$