Odds

Definition 5.

Let

- $\mathbf{P}$ a probability on a sample space S
- A an event

We define the odds of A by

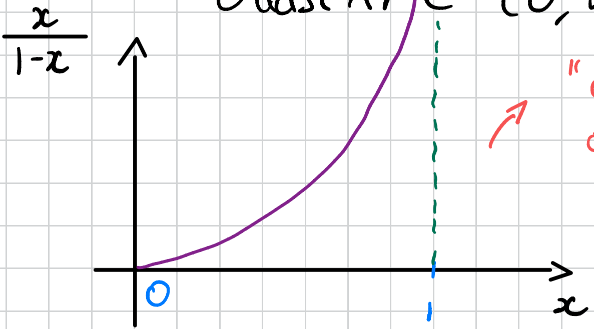
$$\frac{\mathbf{P}(A)}{\mathbf{P}(A^c)} = \frac{\mathbf{P}(A)}{1 - \mathbf{P}(A)}$$

Recall $\quad \text{Odds}(A) = \dfrac{\mathbb{P}(A)}{\mathbb{P}(A^c)} = \dfrac{\mathbb{P}(A)}{1-\mathbb{P}(A)}$ $\quad \frac{x}{1-x}$

The range of Odds is $\neq$ range of a $\mathbb{P}$

Namely $\quad \mathbb{P}(A) \in (0,1)$

$\qquad\qquad \text{Odds}(A) \in (0, \infty)$



"advantage" of odds

Main appeal of odds: They are used in practice → betting companies

Examples: What are the odds that Republicans keep the House in November) $\equiv A$

→ $\simeq 5$

But formula used by betting company was

$$\tilde{O}(A) = \frac{1 - \mathbb{P}(A)}{\mathbb{P}(A)} = \frac{1}{O(A)}$$

→ $\mathbb{P}(A) \simeq 20\%$

Meaning of odd ≈ 5 in terms of betting:

If we bet $1 on A today and A happens

→ we get $5 back

Odds and conditioning

Proposition 6.

Situation: We have

- An hypothesis H, true with probability $\mathbf{P}(H)$
- A new evidence E

Formula: The odds of H after evidence E are given by

$$\frac{\mathbf{P}(H|\, E)}{\mathbf{P}(H^c|\, E)} = \frac{\mathbf{P}(H)}{\mathbf{P}(H^c)} \, \frac{\mathbf{P}(E|\, H)}{\mathbf{P}(E|\, H^c)}$$

Proof

Inversion of conditioning: We have

$$\begin{aligned} \mathbf{P}(H|\,E) &= \frac{\mathbf{P}(E|\,H)\,\mathbf{P}(H)}{\mathbf{P}(E)} \\ \mathbf{P}(H^c|\,E) &= \frac{\mathbf{P}(E|\,H^c)\,\mathbf{P}(H^c)}{\mathbf{P}(E)} \end{aligned}$$

Conclusion:

$$\frac{\mathbf{P}(H|\,E)}{\mathbf{P}(H^c|\,E)} = \frac{\mathbf{P}(H)}{\mathbf{P}(H^c)}\,\frac{\mathbf{P}(E|\,H)}{\mathbf{P}(E|\,H^c)}$$

Example: coin tossing (1)

Situation:

- Urn contains two type A coins and one type B coin.
- When a type A coin is flipped,
  it comes up heads with probability $\frac{1}{4}$
- When a type B coin is flipped,
  it comes up heads with probability $\frac{3}{4}$
- A coin is randomly chosen from the urn and flipped

Question:

Given that the flip landed on heads
$\hookrightarrow$ What is the probability that it was a type A coin?

Example: coin tossing (2)

Model: We set

- $A =$ type A coin flipped
- $B =$ type B coin flipped
- $H =$ Head obtained

Data:

$$\mathbf{P}(A) = \frac{2}{3}, \qquad \mathbf{P}(H|\, A) = \frac{1}{4}, \qquad \mathbf{P}(H|\, B) = \frac{3}{4}$$

Aim:
Compute $\mathbf{P}(A|\, H)$

Example: coin tossing (3)

Application of Proposition 6:

$$\frac{\mathbf{P}(A|\,H)}{\mathbf{P}(B|\,H)} = \frac{\mathbf{P}(A)}{\mathbf{P}(B)}\,\frac{\mathbf{P}(H|\,A)}{\mathbf{P}(H|\,B)}$$

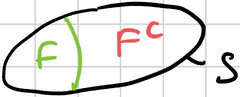
Numerical result: We get

$$\frac{\mathbf{P}(A|\,H)}{\mathbf{P}(B|\,H)} = \frac{2/3}{1/3}\,\frac{1/4}{3/4} = \frac{2}{3}$$

Therefore

$$\mathbf{P}(A|\,H) = \frac{2}{5}$$

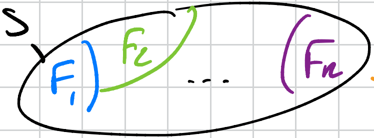
Recall: we have seen



$$\mathbb{P}(E) = \mathbb{P}(E \mid F)\,\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\,\mathbb{P}(F^c)$$

$$\mathbb{P}(F \mid E) = \frac{\mathbb{P}(E \mid F)\,\mathbb{P}(F)}{\mathbb{P}(E \mid F)\,\mathbb{P}(F) + \mathbb{P}(E \mid F^c)\,\mathbb{P}(F^c)}$$

Next step: split S into more than 2 pieces



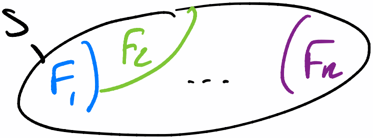
partition of the sample space

Generalization of Proposition 3

Proposition 7.

Let

- $\mathbf{P}$ a probability on a sample space S
- $F_1, \ldots, F_n$ partition of S, i.e
  - F_i mutually exclusive
  - $\cup_{i=1}^n F_i = S$
- E another event



Then we have

$$\mathbf{P}(E) = \sum_{i=1}^{n} \mathbf{P}(E|\, F_i)\, \mathbf{P}(F_i)$$

Generalization of Proposition 4

Proposition 8.

Let

- $\mathbf{P}$ a probability on a sample space S
- $F_1, \ldots, F_n$ partition of S, i.e
  - F_i mutually exclusive
  - $\cup_{i=1}^n F_i = S$
- E another event

Then we have

$$\mathbf{P}(F_j | E) = \frac{\mathbf{P}(E | F_j)\, \mathbf{P}(F_j)}{\boxed{\sum_{i=1}^n \mathbf{P}(E | F_i)\, \mathbf{P}(F_i)}}$$

$= \mathbb{P}(E)$

Example: card game (1)

Situation:

- 3 cards identical in form (say Jack)
- Coloring of the cards on both faces:
  - 1 card RR
  - 1 card BB
  - 1 card RB
- 1 card is "randomly" selected, with upper side R

Question:

What is the probability that the other side is B?

Model Define the events

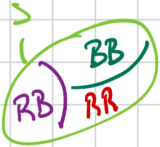
RR : we have picked the RR card
BB : " " BB "
RB : " " RB "
R : Upper side is R

The data we have is

$$\mathbb{P}(RR) = \mathbb{P}(BB) = \mathbb{P}(RB) = \frac{1}{3}$$

$$\mathbb{P}(R \mid RR) = 1 \qquad \mathbb{P}(R \mid BB) = 0$$

$$\mathbb{P}(R \mid RB) = \frac{1}{2}$$



$\mathbb{P}(RR) = \mathbb{P}(BB) = \mathbb{P}(RB) = \frac{1}{3}$

$\mathbb{P}(R \mid RR) = 1 \qquad \mathbb{P}(R \mid BB) = 0$

$\mathbb{P}(R \mid RB) = \frac{1}{2}$

We wish to compute

$$\boxed{\mathbb{P}(RB \mid R)} \overset{\text{generalized Bayes}}{=}$$

$$\frac{\mathbb{P}(R \mid RB)\,\mathbb{P}(RB)}{\mathbb{P}(R \mid RB)\,\mathbb{P}(RB) + \mathbb{P}(R \mid BB)\,\mathbb{P}(BB) + \mathbb{P}(R \mid RR)\,\mathbb{P}(RR)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{\cancel{3}}}{\frac{1}{2} \times \frac{1}{\cancel{3}} + 0 \times \frac{1}{\cancel{3}} + 1 \times \frac{1}{\cancel{3}}} \quad \boxed{= \frac{1}{3}} \quad \left(\neq \frac{1}{2}\right)$$

Example: card game (2)

Model: We define the events

- RR: chosen card is all red
- BB: chosen card is all black
- RB: chosen card is red and black
- R: upturned side of chosen card is red

Aim:
Compute $\mathbf{P}(RB|\, R)$

Example: card game (3)

Application of Proposition 8:

$$\begin{aligned}&\mathbf{P}\left(RB \mid R\right)\\&= \frac{\mathbf{P}\left(R \mid RB\right)\mathbf{P}(RB)}{\mathbf{P}\left(R \mid RR\right)\mathbf{P}(RR)+\mathbf{P}\left(R \mid RB\right)\mathbf{P}(RB)+\mathbf{P}\left(R \mid BB\right)\mathbf{P}(BB)}\end{aligned}$$

Numerical values:

$$\mathbf{P}\left(RB \mid R\right) = \frac{\frac{1}{2}\times\frac{1}{3}}{1\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}+0\times\frac{1}{3}} = \frac{1}{3}$$

$A =$ "More than 100 h" $F_j =$ "Type j chosen"

Example: disposable flashlights

$\mathbb{P}(A \mid F_1) = .7$ $\mathbb{P}(A \mid F_2) = .4$ $\mathbb{P}(A \mid F_3) = .3$

Situation: $\mathbb{P}(F_1) = .2$ $\mathbb{P}(F_2) = .3$ $\mathbb{P}(F_3) = .5$

- Bin containing 3 different types of disposable flashlights
- Proba that a type 1 flashlight will give over 100 hours of use is .7
- Corresponding probabilities for types 2 & 3: .4 and .3
- 20% of the flashlights are type 1, 30% are type 2, and 50% are type 3

Questions:

1. What is the probability that a "randomly chosen" flashlight will give more than 100 hours of use?
2. Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight, for $j = 1, 2, 3$? $\rightarrow$ $\mathbb{P}(F_j \mid A)$ $j = 1, 2, 3$

Events A: Flashlight lasts more than 100h

F_j = type j chosen $j=1,2,3$

Data

$\mathbb{P}(A|F_1) = .7$ $\mathbb{P}(A|F_2) = .4$ $\mathbb{P}(A|F_3) = .3$

$\mathbb{P}(F_1) = .2$ $\mathbb{P}(F_2) = .3$ $\mathbb{P}(F_3) = .5$

We wish to compute

$$\boxed{\mathbb{P}(A)} = \mathbb{P}(A|F_1)\,\mathbb{P}(F_1) + \mathbb{P}(A|F_2)\,\mathbb{P}(F_2) + \mathbb{P}(A|F_3)\,\mathbb{P}(F_3)$$

$$= 0.7 \times 0.2 + 0.4 \times 0.3 + 0.3 \times 0.5$$

$$\boxed{= 41\%}$$

Second question. We wish to compute

$$\mathbb{P}(F_1 \mid A) \overset{\text{Bayes}}{=} \frac{\mathbb{P}(A \mid F_1)\,\mathbb{P}(F_1)}{\mathbb{P}(A)}$$

simpler than computing
$\sum \mathbb{P}(A \mid F_j)\,\mathbb{P}(F_j)$

$$= \frac{0.7 \times 0.2}{0.41}$$

$>$
$=$
$<$
$\mathbb{P}(F_1)$?

Type 1 are better quality.
If flashlight works after 100h,
it indicates that flashlight is of type 1

$= 34\%$ ($> 20\% = \mathbb{P}(F_1)$)

Example: disposable flashlights (2)

Model: We define the events

- A: flashlight chosen gives more than 100h of use
- F_j: type j is chosen

Aim 1:
Compute $\mathbf{P}(A)$

Example: disposable flashlights (3)

Application of Proposition 7:

$$\mathbf{P}(A) = \sum_{j=1}^{3} \mathbf{P}(A|\, F_j)\; \mathbf{P}(F_j)$$

Numerical values:

$$\mathbf{P}(A) = 0.7 \times 0.2 + 0.4 \times 0.3 + 0.3 \times 0.5 = .41$$

Example: disposable flashlights (4)

Aim 2:

Compute $\mathbf{P}(F_1 | A)$

Application of Proposition 8:

$$\mathbf{P}(F_1 | A) = \frac{\mathbf{P}(A | F_1)\,\mathbf{P}(F_1)}{\mathbf{P}(A)}$$

Numerical value:

$$\mathbf{P}(F_1 | A) = \frac{0.7 \times 0.2}{0.41} = \frac{14}{41} \simeq 34\%$$

Outline

Definition of independence

Definition 9.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events

Then E and F are independent if

$$\mathbf{P}(E\,F) = \mathbf{P}(E)\,\mathbf{P}(F)$$

Notation:

$$E \text{ and } F \text{ independent denoted by } E \perp\!\!\!\perp F$$

Warning: $E \perp\!\!\!\perp F$ is not the same as E, F disjoint

The 2 notions are in fact opposite

If $E \perp\!\!\!\perp F$ AND $E \cap F = \emptyset$, then

$$\begin{cases} \mathbb{P}(E \cap F) \overset{\perp\!\!\!\perp}{=} \mathbb{P}(E)\,\mathbb{P}(F) \\ \mathbb{P}(E \cap F) \overset{EF=\emptyset}{=} \mathbb{P}(\emptyset) = 0 \end{cases}$$

Thus $\mathbb{P}(E)\,\mathbb{P}(F) = 0$

$\Rightarrow$ either $\mathbb{P}(E) = 0$ or $\mathbb{P}(F) = 0$

$\Rightarrow$ either $\boxed{E = \emptyset \text{ or } F = \emptyset}$