

Brownian motion

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Stochastic calculus and Machine Learning - MA 598

Mostly taken from *Brownian Motion and Stochastic Calculus*
by I. Karatzas and S. Shreve

Outline

- 1 Stochastic processes
- 2 Definition and construction of the Wiener process
- 3 First properties
- 4 Martingale property
- 5 Markov property
- 6 Pathwise properties

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Stochastic processes

Definition 1.

Let:

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space.
- $I \subset \mathbb{R}_+$ interval.
- $\{X_t; t \in I\}$ family of random variables, $\mathbb{R}^n$ -valued

Then:

- 1 If $\omega \mapsto X_t(\omega)$ measurable, X is a **stochastic process**
- 2 $t \mapsto X_t(\omega)$ is called a path
- 3 X is continuous if its paths are continuous a.s.

Modifications of processes

Definition 2.

Let X and Y be two processes on $(\Omega, \mathcal{F}, \mathbf{P})$.

- ① X is a modification of Y if

$$\mathbf{P}(X_t = Y_t) = 1, \quad \text{for all } t \in I$$

- ② X and Y are non-distinguishable if

$$\mathbf{P}(X_t = Y_t \text{ for all } t \in I) = 1$$

Remarks:

- (i) Relation (2) implicitly means that $(X_t = Y_t \text{ for all } t \in I) \in \mathcal{F}$
- (ii) (2) is much stronger than (1)
- (iii) If X and Y are continuous, $(2) \iff (1)$

Filtrations

Filtration: Increasing sequence of σ -algebras, i.e

$\hookrightarrow$ If $s < t$, then $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.

Interpretation: $\mathcal{F}_t$ summarizes an information obtained at time t

Negligible sets: $\mathcal{N} = \{F \in \mathcal{F}; \mathbf{P}(F) = 0\}$

Complete filtration: Whenever $\mathcal{N} \subset \mathcal{F}_t$ for all $t \in I$

Stochastic basis: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbf{P})$ with a complete $(\mathcal{F}_t)_{t \in I}$

Remark: Filtration $(\mathcal{F}_t)_{t \in I}$ can always be thought of as complete

$\hookrightarrow$ One replaces $\mathcal{F}_t$ by $\hat{\mathcal{F}}_t = \sigma(\mathcal{F}_t, \mathcal{N})$

Adaptation

Definition 3.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbf{P})$ stochastic basis
- $\{X_t; t \in I\}$ stochastic process

We say that X is $\mathcal{F}_t$ -adapted if for all $t \in I$:

$$X_t : (\Omega, \mathcal{F}_t) \longrightarrow (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)) \text{ is measurable}$$

Remarks:

(i) Let $\mathcal{F}_t^X = \sigma\{X_s; s \leq t\}$ the **natural filtration** of X .

$\hookrightarrow$ Process X is always $\mathcal{F}_t^X$ -adapted.

(ii) A process X is $\mathcal{F}_t$ -adapted iff $\mathcal{F}_t^X \subset \mathcal{F}_t$

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Definition of the Wiener process

Notation: For a function f , $\delta f_{st} \equiv f_t - f_s$

Definition 4.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{W_t; t \geq 0\}$ stochastic process, $\mathbb{R}$ -valued

We say that W is a **Wiener process** if:

- 1 $W_0 = 0$ almost surely
- 2 Let $n \geq 1$ and $0 = t_0 < t_1 < \dots < t_n$. The increments $\delta W_{t_0 t_1}, \delta W_{t_1 t_2}, \dots, \delta W_{t_{n-1} t_n}$ are independent
- 3 For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, t - s)$
- 4 W has continuous paths almost surely

Illustration: chaotic path

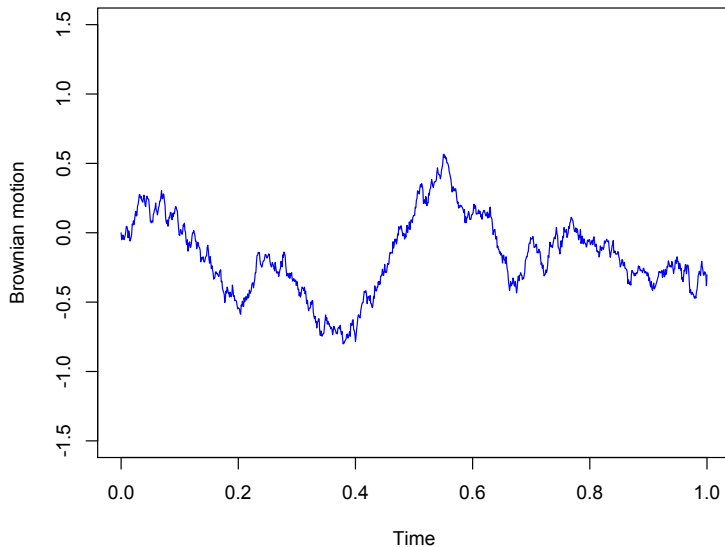
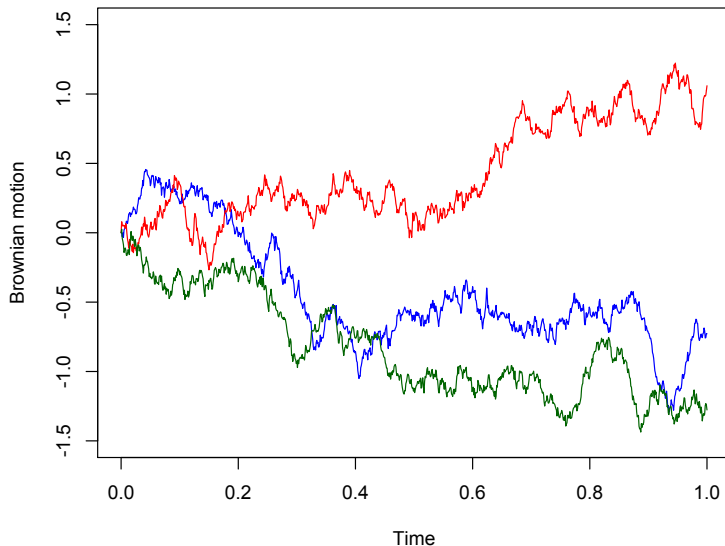


Illustration: random path



Existence of the Wiener process

Theorem 5.

There exists a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which one can construct a Wiener process.

Classical constructions:

- Kolmogorov's extension theorem
- Limit of a renormalized random walk
- Lévy-Ciesilski's construction

Haar functions

Definition 6.

We define a family of functions $\{h_k : [0, 1] \rightarrow \mathbb{R}; k \geq 0\}$:

$$h_0(t) = \mathbf{1}$$

$$h_1(t) = \mathbf{1}_{[0, 1/2]}(t) - \mathbf{1}_{(1/2, 1]}(t),$$

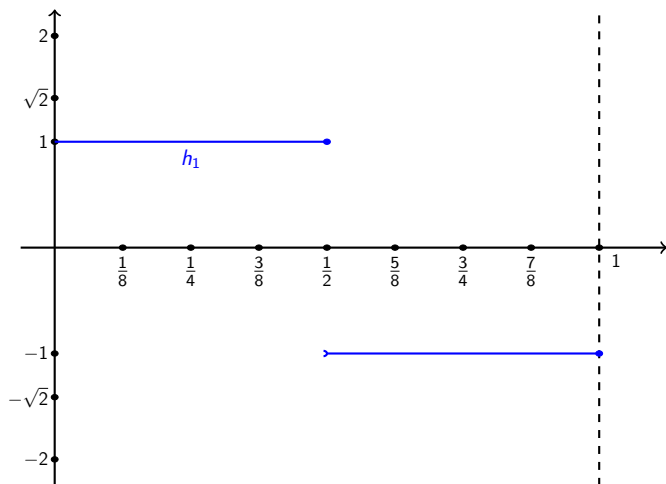
and for $n \geq 1$ and $2^n \leq k < 2^{n+1}$:

$$h_k(t) = 2^{n/2} \mathbf{1}_{\left[\frac{k-2^n}{2^n}, \frac{k-2^n+1/2}{2^n}\right]}(t) - 2^{n/2} \mathbf{1}_{\left(\frac{k-2^n+1/2}{2^n}, \frac{k-2^n+1}{2^n}\right]}(t)$$

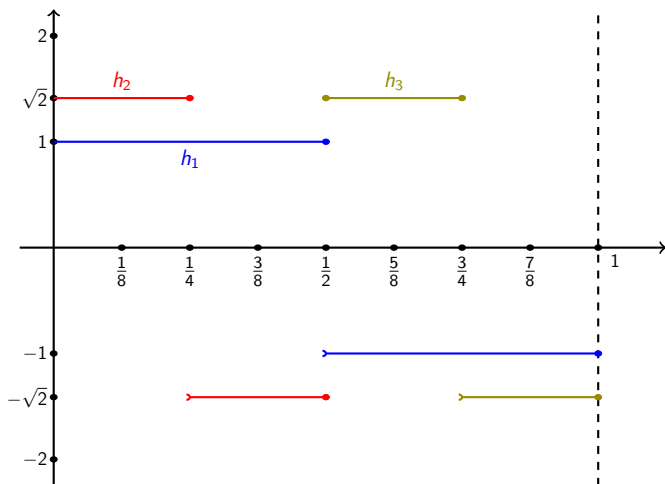
Lemma 7.

The functions $\{h_k : [0, 1] \rightarrow \mathbb{R}; k \geq 0\}$ form an orthonormal basis of $L^2([0, 1])$.

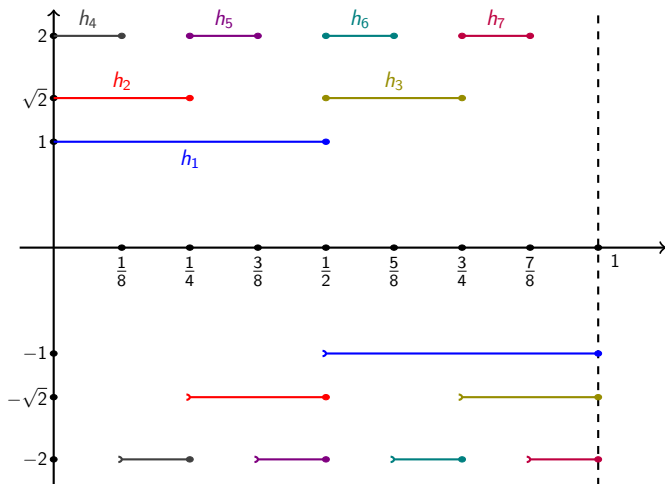
Haar functions: illustration



Haar functions: illustration



Haar functions: illustration



Proof

Norm: For $2^n \leq k < 2^{n+1}$, we have

$$\int_0^1 h_k^2(t) dt = 2^n \int_{\frac{k-2^n}{2^n}}^{\frac{k-2^n+1}{2^n}} dt = 1.$$

Orthogonality: If $k < l$, we have two situations:

(i) $\text{Supp}(h_k) \cap \text{Supp}(h_l) = \emptyset$.

Then trivially $\langle h_k, h_l \rangle_{L^2([0,1])} = 0$

(ii) $\text{Supp}(h_l) \subset \text{Supp}(h_k)$.

Then if $2^n \leq k < 2^{n+1}$ we have:

$$\langle h_k, h_l \rangle_{L^2([0,1])} = \pm 2^{n/2} \int_0^1 h_l(t) dt = 0.$$

Proof (2)

Complete system: Let $f \in L^2([0, 1])$ s.t. $\langle f, h_k \rangle = 0$ for all k .

$\hookrightarrow$ We will show that $f = 0$ almost everywhere.

Step 1: Analyzing the relations $\langle f, h_k \rangle = 0$

$\hookrightarrow$ We show that $\int_s^t f(u) du = 0$ for dyadic r, s .

Step 2: Since $\int_s^t f(u) du = 0$ for dyadic r, s , we have

$$f(t) = \partial_t \left(\int_0^t f(u) du \right) = 0, \quad \text{almost everywhere,}$$

according to Lebesgue's derivation theorem.

Schauder functions

Definition 8.

We define a family of functions $\{s_k : [0, 1] \rightarrow \mathbb{R}; k \geq 0\}$:

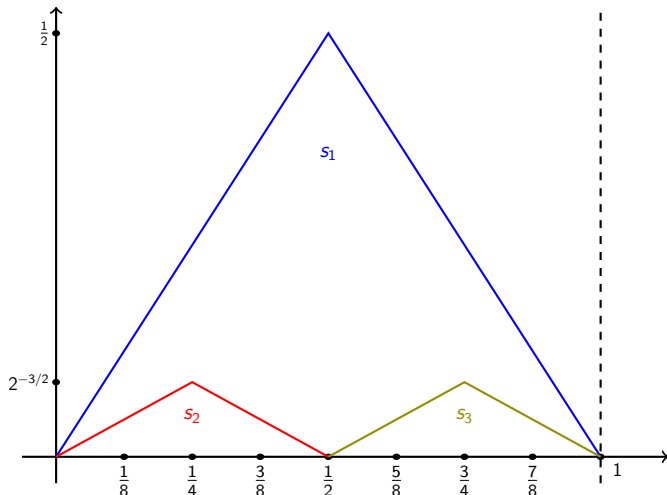
$$s_k(t) = \int_0^t h_k(u) du$$

Lemma 9.

Functions $\{s_k : [0, 1] \rightarrow \mathbb{R}; k \geq 0\}$ satisfy for $2^n \leq k < 2^{n+1}$:

- 1 $\text{Supp}(s_k) = \text{Supp}(h_k) = [\frac{k-2^n}{2^n}, \frac{k-2^n+1}{2^n}]$
- 2 $\|s_k\|_\infty = \frac{1}{2^{n/2+1}}$

Schauder functions: illustration



Gaussian supremum

Lemma 10.

Let $\{X_k; k \geq 1\}$ i.i.d sequence of $\mathcal{N}(0, 1)$ r.v. We set:

$$M_n \equiv \sup \{|X_k|; 1 \leq k \leq n\}.$$

Then

$$M_n = O\left(\sqrt{\ln(n)}\right) \quad \text{almost surely}$$

Proof

Gaussian tail: Let $x > 0$. We have:

$$\begin{aligned}\mathbf{P}(|X_k| > x) &= \frac{2}{(2\pi)^{1/2}} \int_x^\infty e^{-\frac{z^2}{4}} e^{-\frac{z^2}{4}} dz \\ &\leq c_1 e^{-\frac{x^2}{4}} \int_x^\infty e^{-\frac{z^2}{4}} dz \leq c_2 e^{-\frac{x^2}{4}}.\end{aligned}$$

Application of Borel-Cantelli: Let $A_k = (|X_k| \geq 4(\ln(k))^{1/2})$.

According to previous step we have:

$$\mathbf{P}(A_k) \leq \frac{c}{k^4} \implies \sum_{k=1}^{\infty} \mathbf{P}(A_k) < \infty \implies \mathbf{P}(\limsup A_k) = 0$$

Conclusion: ω -a.s there exists $k_0 = k_0(\omega)$ such that

$$\hookrightarrow |X_k(\omega)| \leq 4[\ln(k)]^{1/2} \text{ for } k \geq k_0.$$

Concrete construction on $[0, 1]$

Proposition 11.

Let

- $\{s_k; k \geq 0\}$ Schauder functions family
- $\{X_k; k \geq 0\}$ i.i.d sequence of $\mathcal{N}(0, 1)$ random variables.

We set:

$$W_t = \sum_{k \geq 0} X_k s_k(t).$$

Then W is a Wiener process on $[0, 1]$

$\hookrightarrow$ In the sense of Definition 4.

Proof: uniform convergence

Step 1: Show that $\sum_{k \geq 0} X_k s_k(t)$ converges

$\hookrightarrow$ Uniformly in $[0, 1]$, almost surely.

$\hookrightarrow$ This also implies that W is continuous a.s

Problem reduction: See that for all $\varepsilon > 0$

$\hookrightarrow$ there exists $n_0 = n_0(\omega)$ such that for all $n_0 \leq m < n$ we have:

$$\left\| \sum_{k=2^m}^{2^n-1} X_k s_k \right\|_{\infty} \leq \varepsilon.$$

Proof: uniform convergence (2)

Useful bounds:

- ① Let $\eta > 0$. We have (Lemma 10):

$$|X_k| \leq c k^\eta, \quad \text{with } c = c(\omega)$$

- ② For $2^p \leq k < 2^{p+1}$, functions s_k have disjoint support. Thus

$$\left\| \sum_{k=2^p}^{2^{p+1}-1} s_k \right\|_\infty \leq \frac{1}{2^{\frac{p}{2}+1}}.$$

Proof: uniform convergence (3)

Uniform convergence: for all $t \in [0, 1]$ we have:

$$\begin{aligned} \left| \sum_{k=2^m}^{2^n} X_k s_k(t) \right| &\leq \sum_{p \geq m} \sum_{k=2^p}^{2^{p+1}-1} |X_k| s_k(t) \\ &\leq \sum_{p \geq m} \left(\sup_{2^p \leq \ell \leq 2^{p+1}-1} |X_\ell| \right) \left| \sum_{k=2^p}^{2^{p+1}-1} s_k(t) \right| \\ &\leq c_1 \sum_{p \geq m} \frac{1}{2^{p(\frac{1}{2}-\eta)}} \\ &\leq \frac{c_2}{2^{m(\frac{1}{2}-\eta)}}, \end{aligned}$$

which shows uniform convergence.

Proof: law of δW_{rt}

Step 2: Show that $\delta W_{rt} \sim \mathcal{N}(0, t - s)$ for $0 \leq r < t$.

Problem reduction: See that for all $\lambda \in \mathbb{R}$,

$$\mathbf{E} \left[e^{i\lambda \delta W_{rt}} \right] = e^{-\frac{(t-r)\lambda^2}{2}}.$$

Recall: $\delta W_{rt} = \sum_{k \geq 0} X_k(s_k(t) - s_k(r))$

Computation of a characteristic function:

Invoking independence of X_k 's and dominated convergence,

$$\begin{aligned} \mathbf{E} \left[e^{i\lambda \delta W_{rt}} \right] &= \prod_{k \geq 0} \mathbf{E} \left[e^{i\lambda X_k(s_k(t) - s_k(r))} \right] \\ &= \prod_{k \geq 0} e^{-\frac{\lambda^2 (s_k(t) - s_k(r))^2}{2}} = e^{-\frac{\lambda^2}{2} \sum_{k \geq 0} (s_k(t) - s_k(r))^2} \end{aligned}$$

Proof: law of δW_{rt} (2)

Inner product computation: For $0 \leq r < t$ we have

$$\sum_{k \geq 0} s_k(r) s_k(t) = \sum_{k \geq 0} \langle h_k, \mathbf{1}_{[0,r]} \rangle \langle h_k, \mathbf{1}_{[0,t]} \rangle = \langle \mathbf{1}_{[0,r]}, \mathbf{1}_{[0,t]} \rangle = r.$$

Thus:

$$\sum_{k \geq 0} [s_k(t) - s_k(r)]^2 = t - r.$$

Computation of a characteristic function (2): We get

$$\mathbf{E} \left[e^{i\lambda \delta W_{rt}} \right] = e^{-\frac{\lambda^2}{2} \sum_{k \geq 0} (s_k(t) - s_k(r))^2} = e^{-\frac{(t-r)\lambda^2}{2}}.$$

Proof: increment independence

Simple case: For $0 \leq r < t$, we show that $W_r \perp\!\!\!\perp \delta W_{rt}$

Computation of a characteristic function: for $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{aligned}\mathbf{E} \left[e^{i(\lambda_1 W_r + \lambda_2 \delta W_{rt})} \right] &= \prod_{k \geq 0} \mathbf{E} \left[e^{i X_k [\lambda_1 s_k(r) + \lambda_2 (s_k(t) - s_k(r))]} \right] \\ &= e^{-\frac{1}{2} \sum_{k \geq 0} [\lambda_1 s_k(r) + \lambda_2 (s_k(t) - s_k(r))]^2} = e^{-\frac{1}{2} [\lambda_1^2 r + \lambda_2^2 (t-r)]}\end{aligned}$$

Conclusion: We have, for all $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\mathbf{E} \left[e^{i(\lambda_1 W_r + \lambda_2 \delta W_{rt})} \right] = \mathbf{E} \left[e^{i \lambda_1 W_r} \right] \mathbf{E} \left[e^{i \lambda_2 \delta W_{rt}} \right],$$

and thus $W_r \perp\!\!\!\perp \delta W_{rt}$.

Effective construction on $[0, \infty)$

Proposition 12.

Let:

- For $k \geq 1$, a space $(\Omega_k, \mathcal{F}_k, \mathbf{P}_k)$
 $\hookrightarrow$ On which a Wiener process W^k on $[0, 1]$ is defined
- $\bar{\Omega} = \prod_{k \geq 1} \Omega_k$, $\bar{\mathcal{F}} = \otimes_{k \geq 1} \mathcal{F}_k$, $\bar{\mathbf{P}} = \otimes_{k \geq 1} \mathbf{P}_k$

We set $W_0 = 0$ and recursively:

$$W_t = W_n + W_{t-n}^{n+1}, \quad \text{if } t \in [n, n+1].$$

Then W is a Wiener process on $\mathbb{R}_+$

$\hookrightarrow$ Defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}})$.

Partial proof

Aim: See that $\delta W_{st} \sim \mathcal{N}(0, t - s)$

$\hookrightarrow$ with $m \leq s < m + 1 \leq n \leq t < n + 1$

Decomposition of δW_{st} : We have

$$\delta W_{st} = \sum_{k=1}^n W_1^k + W_{t-n}^{n+1} - \left(\sum_{k=1}^m W_1^k + W_{s-m}^{m+1} \right) = Z_1 + Z_2 + Z_3,$$

with

$$Z_1 = \sum_{k=m+2}^n W_1^k, \quad Z_2 = W_1^{m+1} - W_{s-m}^{m+1}, \quad Z_3 = W_{t-n}^{n+1}.$$

Law of δW_{st} : The Z_j 's are independent centered Gaussian.

Thus $\delta W_{st} \sim \mathcal{N}(0, \sigma^2)$, with:

$$\sigma^2 = n - (m + 1) + 1 - (s - m) + t - n = t - s.$$

Wiener process in $\mathbb{R}^n$

Definition 13.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{W_t; t \geq 0\}$ $\mathbb{R}^n$ -valued stochastic process

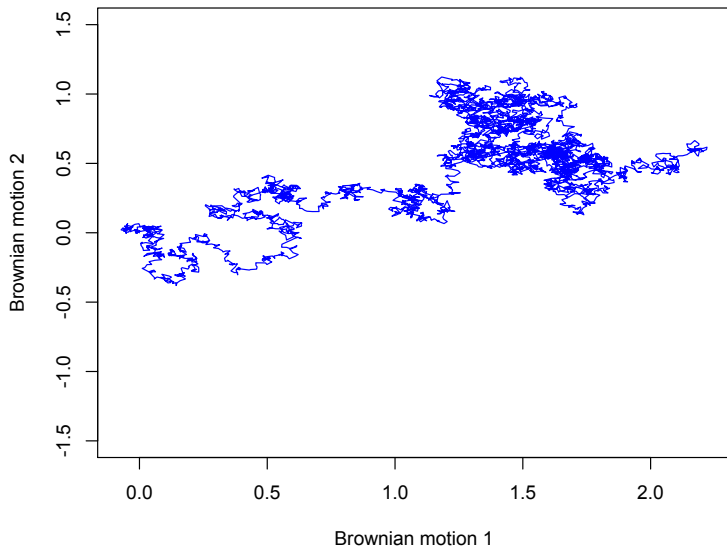
We say that W is a Wiener process if:

- 1 $W_0 = 0$ almost surely
- 2 Let $n \geq 1$ and $0 = t_0 < t_1 < \dots < t_n$. The increments
$$\delta W_{t_0 t_1}, \delta W_{t_1 t_2}, \dots, \delta W_{t_{n-1} t_n}$$
 are independent
- 3 For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, (t-s)\text{Id}_{\mathbb{R}^n})$
- 4 W continuous paths, almost surely

Remark: One can construct W

$\hookrightarrow$ from n independent real valued Brownian motions.

Illustration: 2-d Brownian motion



Wiener process in a filtration

Definition 14.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis
- $\{W_t; t \geq 0\}$ stochastic process with values in $\mathbb{R}^n$

We say that W is a Wiener process
with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ if:

- 1 $W_0 = 0$ almost surely
- 2 Let $0 \leq s < t$. Then $\delta W_{st} \perp\!\!\!\perp \mathcal{F}_s$.
- 3 For $0 \leq s < t$ we have $\delta W_{st} \sim \mathcal{N}(0, (t-s)\text{Id}_{\mathbb{R}^n})$
- 4 W has continuous paths almost surely

Remark: A Wiener process according to Definition 13
is a Wiener process in its natural filtration.

Brown

Some facts about Brown:

- Scottish, Lived 1773-1858
- Pioneer in the use of microscope
- Detailed description of cell nucleus
- Observed 2-d Brownian motion
↪ Pollen particles in water



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Gaussian property

Definition 15.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{X_t; t \geq 0\}$ stochastic process, with values in $\mathbb{R}$

We say that X is a Gaussian process if
for all $0 \leq t_1 < \dots < t_n$ we have:

$(X_{t_1}, \dots, X_{t_n})$ Gaussian vector.

Proposition 16.

Let W be a real valued Brownian motion .
Then W is a Gaussian process.

Proof

Notation: For $0 = t_0 \leq t_1 < \dots < t_n$ we set

- $X_n = (W_{t_1}, \dots, W_{t_n})$
- $Y_n = (\delta W_{t_0 t_1}, \dots, \delta W_{t_{n-1} t_n})$

Vector Y_n : Thanks to independence of increments of W
 $\hookrightarrow Y_n$ is a Gaussian vector.

Vecteur X_n : There exists $M \in \mathbb{R}^{n,n}$ such that $X_n = MY_n$
 $\hookrightarrow X_n$ Gaussian vector

Covariance matrix: We have $\mathbf{E}[W_s W_t] = s \wedge t$. Thus

$$(W_{t_1}, \dots, W_{t_n}) \sim \mathcal{N}(0, \Gamma_n), \quad \text{with} \quad \Gamma_n^{ij} = t_i \wedge t_j.$$

Consequence of Gaussian property

Characterization of a Gaussian process:

Let X Gaussian process. The law of X is characterized by:

$$\mu_t = \mathbf{E}[X_t], \quad \text{and} \quad \rho_{s,t} = \mathbf{Cov}(X_s, X_t).$$

Another characterization of Brownian motion:

Let W real-valued Gaussian process with

$$\mu_t = 0, \quad \text{and} \quad \rho_{s,t} = s \wedge t.$$

Then W is a Brownian motion.

Brownian scaling

Proposition 17.

Let

- W real-valued Brownian motion.
- A constant $a > 0$.

We define a process W^a by:

$$W_t^a = a^{-1/2} W_{at}, \quad \text{for } t \geq 0.$$

Then W^a is a Brownian motion.

Proof:

Gaussian characterization of Brownian motion.

Canonical space

Proposition 18.

Let $E = \mathcal{C}([0, \infty); \mathbb{R}^n)$. We set:

$$d(f, g) = \sum_{k \geq 1} \frac{\|f - g\|_{\infty, k}}{2^k (1 + \|f - g\|_{\infty, k})}$$

where

$$\|f - g\|_{\infty, k} = \sup \{|f_t - g_t|; t \in [0, k]\}.$$

Then E is a separable complete metric space.

Borel σ -algebra on E

Proposition 19.

Let $E = \mathcal{C}([0, \infty); \mathbb{R}^n)$. For $m \geq 1$ we consider:

- $0 \leq t_1 < \dots < t_m$
- $A_1, \dots, A_m \in \mathcal{B}(\mathbb{R}^n)$

Let $\mathcal{A}$ be the σ -algebra generated by rectangles:

$$R_{t_1, \dots, t_m}(A_1, \dots, A_m) = \{x \in E; x_{t_1} \in A_1, \dots, x_{t_m} \in A_m\}.$$

Then $\mathcal{A} = \mathcal{B}(E)$, Borel σ -algebra on E .

Wiener measure

Proposition 20.

Let

- W $\mathbb{R}^n$ -valued Wiener process, defined on $(\Omega, \mathcal{F}, \mathbf{P})$.
- $T : (\Omega, \mathcal{F}) \rightarrow (E, \mathcal{A})$, such that $T(\omega) = \{W_t(\omega); t \geq 0\}$.

Then:

- 1 The application T is measurable
- 2 Let $\mathbf{P}_0 = \mathbf{P} \circ T^{-1}$, measure on $(E, \mathcal{A})$.
 $\mathbf{P}_0$ is called **Wiener measure**.
- 3 Under $\mathbf{P}_0$, the canonical process ω can be written as:

$$\omega_t = W_t, \quad \text{where } W \text{ Brownian motion.}$$

Proof

Inverse image of rectangles: We have

$$T^{-1}(R_{t_1, \dots, t_m}(A_1, \dots, A_m)) = (W_{t_1} \in A_1, \dots, W_{t_m} \in A_m) \in \mathcal{F}.$$

Conclusion: T measurable, since $\mathcal{A}$ generated by rectangles.

Einstein

Einstein and Brownian motion:

- In 1905, 4 revolutionary papers
 - ↔ While Einstein was employee at Patent Office in Zurich
- One of the 4 papers explains BM
 - ↔ Action of water molecules on pollen
- Lead to evidence for atoms
- First elements about transition kernel of Brownian motion



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Martingale property

Definition 21.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis
- $\{X_t; t \geq 0\}$ stochastic process, with values in $\mathbb{R}^n$

We say that X is a $\mathcal{F}_t$ -martingale if

- 1 $X_t \in L^1(\Omega)$ for all $t \geq 0$.
- 2 X is $\mathcal{F}_t$ -adapted.
- 3 $\mathbf{E}[\delta X_{st} | \mathcal{F}_s] = 0$ for all $0 \leq s < t$.

Proposition 22.

Let W a $\mathcal{F}_t$ -Brownian motion.
Then W is a $\mathcal{F}_t$ -martingale.

Stopping time

Definition 23.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis.
- S random variable, with values in $[0, \infty]$.

We say that S is a stopping time if for all $t \geq 0$ we have:

$$(S \leq t) \in \mathcal{F}_t$$

Interpretation 1: If we know $X_{[0,t]}$

$\hookrightarrow$ One also knows if $S \leq t$ or $S > t$

Interpretation 2: $S \equiv$ instant for which one stops playing

$\hookrightarrow$ Only depends on information up to current time.

Typical examples of stopping time

Proposition 24.

Let:

- X process with values in $\mathbb{R}^d$, $\mathcal{F}_t$ -adapted and continuous.
- G open set in $\mathbb{R}^d$.
- F closed set in $\mathbb{R}^d$.

We set:

$$T_G = \inf \{t \geq 0; X_t \in G\}, \quad T_F = \inf \{t \geq 0; X_t \in F\}.$$

Then:

- T_F is a stopping time.
- T_G is a stopping time when X is a Brownian motion.

Proof for T_G

First aim: prove that for $t > 0$ we have

$$(T_G < t) \in \mathcal{F}_t \quad (1)$$

Problem reduction for (1): We show that

$$(T_G < t) = \bigcup_{s \in \mathbb{Q} \cap [0, t)} (X_s \in G). \quad (2)$$

Since $\bigcup_{s \in \mathbb{Q} \cap [0, t)} (X_s \in G) \in \mathcal{F}_t$, this proves our claim.

First inclusion for (2):

$$\bigcup_{s \in \mathbb{Q} \cap [0, t)} (X_s \in G) \subset (T_G < t): \text{ trivial.}$$

Proof for T_G (2)

Second inclusion for (2): If $T_G < t$, then

- There exist $s < t$ such that $X_s \in G$. We set $X_s \equiv x$.
- Let $\varepsilon > 0$ such that $B(x, \varepsilon) \in G$

Then:

- There exists $\delta > 0$ such that $X_r \in B(x, \varepsilon)$ for all $r \in (s - \delta, s + \delta)$.
- In particular, there exists $q \in \mathbb{Q} \cap (s - \delta, s]$ such that $X_q \in G$.

Since

$$(\mathbb{Q} \cap (s - \delta, s]) \subset (\mathbb{Q} \cap [0, t]),$$

we have the second inclusion.

Proof for T_G (3)

Optional times: We say that $T : \Omega \rightarrow [0, \infty]$ is an optional time if

$$(T < t) \in \mathcal{F}_t.$$

Remark: Relation (1) proves that T_G is optional.

Optional times and stopping times:

- A stopping time is an optional time.
- An optional time satisfies $(T \leq t) \in \mathcal{F}_{t+} \equiv \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$.
- When X is a Brownian motion, $\mathcal{F}_{t+} = \mathcal{F}_t$ (by Markov prop.).
- When X is a Brownian motion, **optional time = stopping time**.

Conclusion: When X is a Brownian motion, T_G is a stopping time.

Simple properties of stopping times

Proposition 25.

Let S and T two stopping times. Then

① $S \wedge T$

② $S \vee T$

are stopping times.

Proposition 26.

If T is a deterministic time ($T = n$ almost surely)

$\hookrightarrow$ then T is a stopping time.

Information at time S

Definition 27.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis.
- S stopping time.

The σ -algebra $\mathcal{F}_S$ is defined by:

$$\mathcal{F}_S = \{A \in \mathcal{F}; [A \cap (S \leq t)] \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Interpretation:

$\mathcal{F}_S \equiv$ Information up to time S .

Optional sampling theorem

Theorem 28.

Let

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ stochastic basis.
- S, T two stopping times, with $S \leq T$.
- X continuous martingale.

Hypothesis:

- $\{X_{t \wedge T}; t \geq 0\}$ **uniformly integrable** martingale.

Then:

$$\mathbf{E}[X_T | \mathcal{F}_S] = X_S.$$

In particular:

$$\mathbf{E}[X_T] = \mathbf{E}[X_S] = \mathbf{E}[X_0].$$

Remarks

Strategy of proof:

- One starts from known discrete time result.
- X is approximated by a discrete time martingale

$$Y_m \equiv X_{t_m}, \quad \text{with} \quad t_m = \frac{m}{2^n}.$$

Checking the assumption: Set $Y_t = X_{t \wedge T}$.

$\{Y_t; t \geq 0\}$ uniformly integrable martingale in following cases:

- $|Y_t| \leq M$ with M deterministic constant independent of t .
- $\sup_{t \geq 0} \mathbf{E}[|Y_t|^2] \leq M$.
- $\sup_{t \geq 0} \mathbf{E}[|Y_t|^p] \leq M$ with $p > 1$.

Example of stopping time computation

Proposition 29.

Let:

- B standard Brownian motion, with $B_0 = 0$.
- $-a < 0 < b$
- $T_a = \inf\{t \geq 0 : B_t = -a\}$ and
 $T_b = \inf\{t \geq 0 : B_t = b\}$.
- $T = T_a \wedge T_b$.

Then:

$$\mathbf{P}(T_a < T_b) = \frac{b}{b+a}, \quad \text{and} \quad \mathbf{E}[T] = ab.$$

Proof

Optional sampling for $M_t = B_t$: yields

$$\mathbf{P}(T_a < T_b) = \frac{b}{b+a}.$$

Optional sampling for $M_t = B_t^2 - t$: yields, for a constant $\tau > 0$,

$$\mathbf{E}[B_{T \wedge \tau}^2] = \mathbf{E}[T \wedge \tau].$$

Limiting procedure: by dominated and monotone convergence,

$$\mathbf{E}[B_T^2] = \mathbf{E}[T].$$

Conclusion: we get

$$\mathbf{E}[T] = \mathbf{E}[B_T^2] = a^2 \mathbf{P}(T_a < T_b) + b^2 \mathbf{P}(T_b < T_a) = ab.$$

Bachelier

Bachelier and Brownian motion:

- Born 1870, son of a wine merchant
 $\hookrightarrow$ Used to business
- In 1900 (predates Einstein) PhD thesis
 $\hookrightarrow$ Theory of speculation
- Mathematical description of BM
 $\hookrightarrow$ Application to stock pricing
- Not well perceived by academic establishment



Outline

- 1 Stochastic processes
- 2 Definition and construction of the Wiener process
- 3 First properties
- 4 Martingale property
- 5 Markov property**
- 6 Pathwise properties

Wiener measure indexed by $\mathbb{R}^d$

Proposition 30.

Let:

- $x \in \mathbb{R}^d$.

There exists a probability measure $\mathbf{P}_x$ on $(E, \mathcal{A})$ such that:

$\hookrightarrow$ Under $\mathbf{P}_x$ the canonical process ω can be written as:

$$\omega_t = x + W_t, \quad \text{where } W \text{ Brownian motion.}$$

Notations:

- We consider $\{\mathbf{P}_x; x \in \mathbb{R}^d\}$.
- Expected value under $\mathbf{P}_x$: $\mathbf{E}_x$.

Shift on paths

Definition 31.

Let

- $E = \mathcal{C}([0, \infty); \mathbb{R}^d)$, equipped with Borel $\mathcal{A} \equiv \sigma$ -algebra.
- $t \geq 0$.

We set:

$$\theta_t : E \rightarrow E, \quad \{\omega_s; s \geq 0\} \mapsto \{\omega_{t+s}; s \geq 0\}$$

Shift and future: Let $Y : E \rightarrow \mathbb{R}$ measurable.

$\hookrightarrow$ Then $Y \circ \theta_s$ depends on future after s .

Example: For $n \geq 1$, f measurable and $0 \leq t_1 < \dots < t_n$,

$$Y(\omega) = f(\omega_{t_1}, \dots, \omega_{t_n}) \quad \implies \quad Y \circ \theta_s = f(W_{s+t_1}, \dots, W_{s+t_n}).$$

Markov property

Theorem 32.

Let:

- W Wiener process, with values in $\mathbb{R}^d$.
- $Y : E \rightarrow \mathbb{R}$ bounded measurable.
- $s \geq 0$.

Then:

$$\mathbf{E}_x[Y \circ \theta_s | \mathcal{F}_s] = \mathbf{E}_{W_s}[Y].$$

Interpretation:

Future after s can be predicted with value of W_s only.

Pseudo-proof

Very simple function: Consider $Y \equiv f(W_t)$, let $Y \circ \theta_s = f(W_{s+t})$.
For $0 \leq s < t$, independence of increments for W gives

$$\begin{aligned}\mathbf{E}_x [Y \circ \theta_s | \mathcal{F}_s] &= \mathbf{E}_x [f(W_{s+t}) | \mathcal{F}_s] = p_t f(W_s) \\ &= \mathbf{E}_{W_s} [f(W_t)] = \mathbf{E}_{W_s} [Y],\end{aligned}$$

with

$$p_h : \mathcal{C}(\mathbb{R}^d) \rightarrow \mathcal{C}(\mathbb{R}^d), \quad p_h f(x) \equiv \int_{\mathbb{R}^d} f(y) \frac{\exp\left(-\frac{|y-x|^2}{2h}\right)}{(2\pi h)^{d/2}} dy.$$

Extension:

- 1 Random variable $Y = f(W_{t_1}, \dots, W_{t_n})$.
- 2 General random variable: by π - λ -systems.

π -systems and λ -systems

π -system: Let $\mathcal{P}$ family of sets in Ω . $\mathcal{P}$ is a π -system if:

$$A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}$$

λ -system: Let $\mathcal{L}$ family of sets in Ω . $\mathcal{L}$ is a λ -system if:

- 1 $\Omega \in \mathcal{L}$
- 2 If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$
- 3 If for $j \geq 1$ we have:
 - ▶ $A_j \in \mathcal{L}$
 - ▶ $A_j \cap A_i = \emptyset$ if $j \neq i$

Then $\bigcup_{j \geq 1} A_j \in \mathcal{L}$

Dynkin's π - λ lemma

Lemma 33.

Let $\mathcal{P}$ and $\mathcal{L}$ such that:

- $\mathcal{P}$ is a π -system
- $\mathcal{L}$ is a λ -system
- $\mathcal{P} \subset \mathcal{L}$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$

Links with analysis

Heat semi-group: We have set $p_t f(x) = \mathbf{E}_x[f(W_t)]$. Then:

- The family $\{p_t; t \geq 0\}$ is a semi-group of operators.
- Generator of the semi-group: $\frac{\Delta}{2}$, with $\Delta \equiv$ Laplace operator.

Feynman-Kac formula: Let $f \in \mathcal{C}_b(\mathbb{R}^d)$ and PDE on $\mathbb{R}^d$:

$$\partial_t u(t, x) = \frac{1}{2} \Delta u(t, x), \quad u(0, x) = f(x).$$

Then

$$u(t, x) = \mathbf{E}_x[f(W_t)] = p_t f(x)$$

Strong Markov property

Theorem 34.

Let:

- W Wiener process in $\mathbb{R}^d$.
- $Y : \mathbb{R}_+ \times E \rightarrow \mathbb{R}$ bounded measurable.
- S stopping time.

Then:

$$\mathbf{E}_x[Y_S \circ \theta_S | \mathcal{F}_S] \mathbf{1}_{(S < \infty)} = \mathbf{E}_{W_S}[Y_S] \mathbf{1}_{(S < \infty)}.$$

Particular case: If S finite stopping time a.s. we have

$$\mathbf{E}_x[Y_S \circ \theta_S | \mathcal{F}_S] = \mathbf{E}_{W_S}[Y_S]$$

Reflection principle

Theorem 35.

Let:

- W real-valued Brownian motion.
- $a > 0$.
- $T_a = \inf\{t \geq 0; W_t = a\}$.

Then:

$$\mathbf{P}_0(T_a < t) = 2 \mathbf{P}_0(W_t > a).$$

Intuitive proof

Independence: If W reaches a for $s < t$
 $\hookrightarrow W_t - W_{T_a} \perp\!\!\!\perp \mathcal{F}_{T_a}$.

Consequence:

$$\mathbf{P}_0(T_a < t, W_t > a) = \frac{1}{2} \mathbf{P}_0(T_a < t)$$

Furthermore:

$$(W_t > a) \subset (T_a < t) \implies \mathbf{P}_0(T_a < t, W_t > a) = \mathbf{P}_0(W_t > a).$$

Thus:

$$\mathbf{P}_0(T_a < t) = 2 \mathbf{P}_0(W_t > a).$$

Rigorous proof

Reduction: We have to show

$$\mathbf{P}_0(T_a < t, W_t > a) = \frac{1}{2} \mathbf{P}_0(T_a < t)$$

Functional: We set (with $\inf \emptyset = \infty$)

$$S = \inf \{s < t; W_s = a\}, \quad Y_s(\omega) = \mathbf{1}_{(s < t, \omega(t-s) > a)}.$$

Then:

- ① $(S < \infty) = (T_a < t).$
- ② $Y_S \circ \theta_S = \mathbf{1}_{(S < t)} \mathbf{1}_{W_t > a} = \mathbf{1}_{(T_a < t)} \mathbf{1}_{W_t > a}.$

Rigorous proof (2)

Application of strong Markov:

$$\mathbf{E}_0[Y_S \circ \theta_S | \mathcal{F}_S] \mathbf{1}_{(S < \infty)} = \mathbf{E}_{W_S}[Y_S] \mathbf{1}_{(S < \infty)} = \varphi(W_S, S), \quad (3)$$

with

$$\varphi(x, s) = \mathbf{E}_x[\mathbf{1}_{W_{t-s} > a}] \mathbf{1}_{(s < t)}.$$

Conclusion: Since $W_S = a$ if $S < \infty$ and $\mathbf{E}_a[\mathbf{1}_{W_{t-s} > a}] = \frac{1}{2}$,

$$\varphi(W_S, S) = \frac{1}{2} \mathbf{1}_{(S < t)}.$$

Taking expectations in (3) we end up with:

$$\mathbf{P}_0(T_a < t, W_t > a) = \frac{1}{2} \mathbf{P}_0(T_a < t).$$

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Hölder continuity

Definition 36.

Let

$$f : [a, b] \longrightarrow \mathbb{R}^n.$$

We say that f is γ -Hölder if $\|f\|_\gamma < \infty$ with:

$$\|f\|_\gamma = \sup_{s, t \in [a, b], s \neq t} \frac{|\delta f_{st}|}{|t - s|^\gamma}.$$

Remark:

- 1 $\|\cdot\|_\gamma$ is a semi-norm.
- 2 Notation: $\mathcal{C}^\gamma$

Regularity of Brownian motion

Theorem 37.

Let

- $\tau > 0$
- W Wiener process on $[0, \tau]$
- $\gamma \in (0, 1/2)$

Then there exists a version $\hat{W}$ of W such that almost surely:
 $\hookrightarrow$ The paths of $\hat{W}$ are γ -Hölder.

Remark: $\hat{W}$ and W are usually denoted in the same way.

Kolmogorov's criterion

Theorem 38.

Let $X = \{X_t; t \in [0, \tau]\}$ process defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that:

$$\mathbf{E} [|\delta X_{st}|^\alpha] \leq c |t - s|^{1+\beta}, \quad \text{for } s, t \in [0, \tau], \quad c, \alpha, \beta > 0$$

Then there exists a modification $\hat{X}$ of X satisfying
 $\hookrightarrow$ Almost surely $\hat{X} \in \mathcal{C}^\gamma$ for all $\gamma < \beta/\alpha$, i.e:

$$\mathbf{P} \left(\omega; \|\hat{X}(\omega)\|_\gamma < \infty \right) = 1.$$

Proof of Theorem 37

Law of δB_{st} : We have $\delta B_{st} \sim \mathcal{N}(0, t - s)$.

Moments of δB_{st} :

Using expression for the moments of $\mathcal{N}(0, 1)$

$\hookrightarrow$ for $m \geq 1$, we have

$$\mathbf{E} \left[|\delta B_{st}|^{2m} \right] = c_m |t - s|^m \quad \text{i.e.} \quad \mathbf{E} \left[|\delta B_{st}|^{2m} \right] = c_m |t - s|^{1+(m-1)}$$

Application of Kolmogorov's criterion:

B is γ -Hölder for $\gamma < \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}$

Taking limits $m \rightarrow \infty$, the proof is finished.

Levy's modulus of continuity

Theorem 39.

Let $\tau > 0$ and W Wiener process on $[0, \tau]$.
Then almost surely W satisfies:

$$\limsup_{\delta \rightarrow 0^+} \sup_{0 \leq s < t \leq \tau, |t-s| \leq \delta} \frac{|\delta W_{st}|}{(2\delta \ln(1/\delta))^{1/2}} = 1.$$

Interpretation: W has Hölder-regularity $= \frac{1}{2}$ at each point
 $\hookrightarrow$ up to a logarithmic factor.

Variations of a function

Interval partitions: Let $a < b$ two real numbers.

- We denote by π a set $\{t_0, \dots, t_m\}$ with $a = t_0 < \dots < t_m = b$
We say that π is a **partition** of $[a, b]$.
- Write $\Pi_{a,b}$ for the set of partitions of $[a, b]$.

Definition 40.

Let $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$. The **variation** of f on $[a, b]$ is:

$$V_{a,b}(f) = \lim_{\pi \in \Pi_{a,b}, |\pi| \rightarrow 0} \sum_{t_i, t_{i+1} \in \pi} |\delta f_{t_i t_{i+1}}|.$$

If $V_{a,b}(f) < \infty$, we say that f has **finite variation** on $[a, b]$.

Quadratic variation

Definition 41.

Let $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$.

The **quadratic variation** of f on $[a, b]$ is:

$$V_{a,b}^2(f) = \lim_{\pi \in \Pi_{a,b}, |\pi| \rightarrow 0} \sum_{t_i, t_{i+1} \in \pi} |\delta f_{t_i, t_{i+1}}|^2.$$

If $V_{a,b}^2(f) < \infty$,

$\hookrightarrow$ We say that f has a **finite quadratic variation** on $[a, b]$.

Remark: The limits for $V_{a,b}(f)$ and $V_{a,b}^2(f)$

$\hookrightarrow$ do not depend on the sequence of partitions

Variations of Brownian motion

Theorem 42.

Let W Wiener process.

Then almost surely W satisfies:

- ① For $0 \leq a < b < \infty$ we have $V_{a,b}^2(W) = b - a$.
- ② For $0 \leq a < b < \infty$ we have $V_{a,b}(W) = \infty$.

Interpretation: The paths of W have:

- Infinite variation
- Finite quadratic variation,

on any interval of $\mathbb{R}_+$.

Proof

Notations: Let $\pi = \{t_0, \dots, t_m\} \in \Pi_{a,b}$. We set:

- $S_\pi = \sum_{k=0}^{m-1} |\delta W_{t_k t_{k+1}}|^2.$
- $X_k = |\delta W_{t_k t_{k+1}}|^2 - (t_{k+1} - t_k).$
- $Y_k = \frac{X_k}{t_{k+1} - t_k}.$

Step 1: Show that

$$L^2(\Omega) - \lim_{|\pi| \rightarrow 0} S_\pi = b - a.$$

Decomposition: We have

$$S_\pi - (b - a) = \sum_{k=0}^{m-1} X_k$$

Proof (2)

Variance computation: The r.v X_k are centered and indep. Thus

$$\begin{aligned}\mathbf{E} \left[(S_\pi - (b - a))^2 \right] &= \mathbf{Var} \left(\sum_{k=0}^{m-1} X_k \right) \\ &= \sum_{k=0}^{m-1} \mathbf{Var} (X_k) = \sum_{k=0}^{m-1} (t_{k+1} - t_k)^2 \mathbf{Var} (Y_k)\end{aligned}$$

Since $\frac{\delta W_{t_k t_{k+1}}}{(t_{k+1} - t_k)^{1/2}} \sim \mathcal{N}(0, 1)$, we get:

$$\mathbf{E} \left[(S_\pi - (b - a))^2 \right] = 2 \sum_{k=0}^{m-1} (t_{k+1} - t_k)^2 \leq 2|\pi|(b - a).$$

Conclusion: We have, for a subsequence π_n ,

$$L^2(\Omega) - \lim_{|\pi| \rightarrow 0} S_\pi = b - a \implies \text{a.s.} - \lim_{n \rightarrow \infty} S_{\pi_n} = b - a.$$

Proof (3)

Step 2: Verify $V_{a,b}(W) = \infty$ for fixed $a < b$.

We proceed by contradiction. Let:

- $\omega \in \Omega_0$ such that $\mathbf{P}(\Omega_0) = 1$ and $\lim_{n \rightarrow \infty} S_{\pi_n}(\omega) = b - a > 0$.
- Assume $V_{a,b}(W(\omega)) < \infty$.

Bound on increments: Thanks to continuity of W :

- 1 $\sup_{n \geq 1} \sum_{k=0}^{m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| \leq c(\omega)$
- 2 $\lim_{n \rightarrow \infty} \max_{0 \leq k \leq m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| = 0$

Thus

$$S_{\pi_n}(\omega) \leq \max_{0 \leq k \leq m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| \sum_{k=0}^{m_n-1} |\delta W_{t_k t_{k+1}}(\omega)| \longrightarrow 0.$$

Contradiction: with $\lim_{n \rightarrow \infty} S_{\pi_n}(\omega) = b - a > 0$.

Proof (4)

Step 3: Verify $V_{a,b}(W) = \infty$ for all couple $(a, b) \in \mathbb{R}_+^2$

$\hookrightarrow$ It is enough to check $V_{a,b}(W) = \infty$ for all couple $(a, b) \in \mathbb{Q}_+^2$

Recall: For all couple $(a, b) \in \mathbb{Q}_+^2$, we have found:

$\hookrightarrow \exists \Omega_{a,b}$ s.t $\mathbf{P}(\Omega_{a,b}) = 1$ and $V_{a,b}(W(\omega)) = \infty$ for all $\omega \in \Omega_{a,b}$.

Full probability set: Let

$$\Omega_0 = \bigcap_{(a,b) \in \mathbb{Q}_+^2} \Omega_{a,b}.$$

Then:

- $\mathbf{P}(\Omega_0) = 1$.
- If $\omega \in \Omega_0$, for all couple $(a, b) \in \mathbb{Q}_+^2$ we have $V_{a,b}(W(\omega)) = \infty$.

Irregularity of W

Proposition 43.

Let:

- W Wiener process
- $\gamma > 1/2$ and $0 \leq a < b$

Then **almost surely** W does not belong to $\mathcal{C}^\gamma([a, b])$.

Proof

Strategy: Proceed by contradiction. Let:

- $\omega \in \Omega_0$ such that $\mathbf{P}(\Omega_0) = 1$ and $\lim_{n \rightarrow \infty} S_{\pi_n}(\omega) = b - a > 0$.
- Suppose $W \in \mathcal{C}^\gamma$ with $\gamma > 1/2$, i.e. $|\delta W_{st}| \leq L|t - s|^\gamma$
 $\hookrightarrow$ With L random variable.

Bound on quadratic variation: We have:

$$S_{\pi_n}(\omega) \leq L^2 \sum_{k=0}^{m_n-1} |t_{k+1} - t_k|^{2\gamma} \leq L^2 |\pi_n|^{2\gamma-1} (b - a) \longrightarrow 0.$$

Contradiction: with $\lim_{n \rightarrow \infty} S_{\pi_n}(\omega) = b - a > 0$.

Irregularity of W at each point

Theorem 44.

Let:

- W Wiener process
- $\gamma > 1/2$ and $\tau > 0$

Then

- 1 Almost surely the paths of W are not γ -Hölder continuous at each point $s \in [0, \tau]$.
- 2 In particular, W is nowhere differentiable.