Diffusions as Markov processes

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Outline

1. Restarting a diffusion
2. Generator of a diffusion
3. Transition densities and Kolmogorov equations
4. Summary

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Setting

Data:

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\tau]}, \mathbf{P})$ stochastic basis
- W d-dimensional Brownian motion
- $b, \sigma^1, \ldots, \sigma^d : \mathbb{R}^n \to \mathbb{R}^n$ Lipschitz, $\alpha \in \mathbb{R}^n$

Equation: X unique solution in $L^2(\Omega; \mathcal{C}([0,\tau]))$ of

$$X_t = \alpha + \int_0^t b(X_r)\, dr + \sum_{k=1}^{d} \int_0^t \sigma^k(X_r)\, dW_r^k \tag{1}$$

Aim: Markov property for X, as for Brownian motion
$\hookrightarrow$ Future after time s driven by the increments of W after s

Restarted Brownian motion

Notation: For $s \in [0, \tau)$ we set

$$\begin{aligned} W_u^s &= W_{s+u} - W_s, \quad u \in [0, \tau - s] \\ \mathcal{G}^s &= \sigma\{W_u^s;\ u \in [0, \tau - s]\} \end{aligned}$$

$\hookrightarrow$ Restarted filtration: $(\mathcal{F}_{s+u})_{u \in [0, \tau - s]}$

Lemma 1

Let $s \in [0, \tau)$. Then:

- *W^s is a Brownian motion with respect to $(\mathcal{F}_{s+u})_{u \in [0, \tau - s]}$*
- *$\mathcal{G}^s$ and $\mathcal{F}_s$ are independent*

Restarted diffusion

Notation: For $s \in [0, \tau)$ and $x \in \mathbb{R}^n$,
$X^{s,x} = \{X_u^{s,x};\ u \in [0, \tau - s]\}$ solution of

$$X_u^{s,x} = x + \int_0^u b(X_r^{s,x})\, dr + \sum_{k=1}^{d} \int_0^u \sigma^k(X_r^{s,x})\, dW_r^{s,k}$$

Well posedness:

- Same equation as (1), on the restarted basis, driven by W^s
- Existence and uniqueness theorem applies verbatim
- $X^{s,x}$ adapted to $(\mathcal{F}_{s+u})_{u \in [0,\tau-s]}$, and $X^{0,\alpha} = X$

Restarting the Itô integral

Lemma 2

Let

- $v \in L^2_a([0, \tau - s])$ *on the restarted basis*
- $\bar{v}_r = v_{r-s}\, \mathbf{1}_{(s,\tau]}(r)$

Then for $u \in [0, \tau - s]$,

$$\int_0^u v_r \, dW_r^s = \int_s^{s+u} \bar{v}_r \, dW_r$$

Meaning: Integrals against W^s on $[0, u]$

- Are integrals against W on $[s, s + u]$
- Needed to restart equation (1) at time s

Proof

Formal derivation: $dW_r^s = dW_{s+r}$

$\hookrightarrow$ Then change of variable $t = s + r$:

$$\int_0^u v_r \, dW_{s+r} = \int_s^{s+u} v_{t-s} \, dW_t = \int_s^{s+u} \bar{v}_t \, dW_t$$

Rigorous proof: No change of variables inside an Itô integral

- Check the identity for simple processes v
  $\hookrightarrow$ finite sums of increments of W
- Pass to the limit in $L^2(\Omega)$
  $\hookrightarrow$ by density of simple processes in L_a^2

Independence from the past

Lemma 3

Let $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $u_1, \dots, u_m \in [0, \tau - s]$. Then:

$$\left(X_{u_1}^{s,x}, \dots, X_{u_m}^{s,x}\right) \perp\!\!\!\perp \mathcal{F}_s$$

In particular, for $\Phi : \mathbb{R}^n \to \mathbb{R}$ bounded measurable and $u \in [0, \tau - s]$,

$$\mathbf{E}\left[\Phi(X_u^{s,x}) \,|\, \mathcal{F}_s\right] = \mathbf{E}\left[\Phi(X_u^{s,x})\right]$$

Interpretation: Analog of $W_{s+u} - W_s \perp\!\!\!\perp \mathcal{F}_s$ for the diffusion
$\hookrightarrow$ This is why the equation is restarted from a deterministic point x

Proof

Measurability: $X^{s,x}$ is a strong solution on the restarted basis
$\hookrightarrow X_u^{s,x}$ measurable with respect to $\sigma(W_r^s;\, r \leq u) \subset \mathcal{G}^s$

Independence: $\mathcal{G}^s \perp\!\!\!\perp \mathcal{F}_s$, by the lemma on restarted Brownian motion
$\hookrightarrow (X_{u_1}^{s,x}, \ldots, X_{u_m}^{s,x}) \perp\!\!\!\perp \mathcal{F}_s$

Conditional expectation: For $Z \perp\!\!\!\perp \mathcal{F}_s$ integrable,

$$\mathbf{E}\left[Z \,|\, \mathcal{F}_s\right] = \mathbf{E}\left[Z\right]$$

$\hookrightarrow$ Apply to $Z = \Phi(X_u^{s,x})$

Restart property

Proposition 4

Let $s \in [0, \tau)$ and X solution of (1). Then for $u \in [0, \tau - s]$,

$$X_{s+u} = X_s + \int_0^u b(X_{s+r})\, dr + \sum_{k=1}^{d} \int_0^u \sigma^k(X_{s+r})\, dW_r^{s,k}$$

Interpretation: After time s, X solves the restarted equation
$\hookrightarrow$ Started from the random point $X_s \in \mathcal{F}_s$, driven by W^s

Proof

Increment: Subtract (1) at times $s+u$ and s:

$$X_{s+u} - X_s = \int_s^{s+u} b(X_r)\, dr + \sum_{k=1}^{d} \int_s^{s+u} \sigma^k(X_r)\, dW_r^k$$

Lebesgue term: Change of variables $r = s + r'$, path by path

Stochastic term: Restarted Itô integral, with $v_r = \sigma^k(X_{s+r})$ and $\bar{v}_r = \sigma^k(X_r)$ on $(s, \tau]$

$$\int_s^{s+u} \sigma^k(X_r)\, dW_r^k = \int_0^u \sigma^k(X_{s+r})\, dW_r^{s,k}$$

Transition operators

Definition 5

Let:

- $0 \leq s \leq t \leq \tau$
- $\Phi : \mathbb{R}^n \to \mathbb{R}$ bounded measurable

The transition operator $p_{s,t}$ is defined by

$$p_{s,t}\Phi(x) = \mathbf{E}\left[\Phi\left(X_{t-s}^{s,x}\right)\right], \quad x \in \mathbb{R}^n$$

Interpretation:

- Average of Φ over the solution restarted from x at time s
- Generalizes the heat semigroup of Chapter 1

Transition operators: properties

Properties:

- $p_{s,s}\Phi = \Phi$, since $X_0^{s,x} = x$
- $\|p_{s,t}\Phi\|_\infty \leq \|\Phi\|_\infty$
- Φ bounded Lipschitz $\Longrightarrow$ $p_{s,t}\Phi$ bounded Lipschitz
  $\hookrightarrow$ stability of $X^{s,x}$ in x

Brownian case: $b = 0$ and $\sigma = \mathrm{Id}$, so that

- $X_u^{s,x} = x + W_u^s$
- $p_{s,t} = p_{t-s}$, heat semigroup of Chapter 1

Markov property

Theorem 6

Let:

- b, σ *Lipschitz,* X *solution of* (1)
- $0 \leq s \leq t \leq \tau$, *and* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *bounded measurable*

Then $p_{s,t}\Phi$ *is measurable, and*

$$\mathbf{E}\left[\Phi(X_t) \,|\, \mathcal{F}_s\right] = p_{s,t}\Phi(X_s)$$

Interpretation:

Future after s predicted with X_s only

$\hookrightarrow$ Solve (1) restarted from X_s, then average

Markov property: idea of proof

Step 1, X_s with countably many values x_i: Set $A_i = (X_s = x_i) \in \mathcal{F}_s$
$\hookrightarrow$ $Z = \sum_i \mathbf{1}_{A_i} X^{s,x_i}$ solves the restarted equation from X_s
$\hookrightarrow$ $Z = X_{s+\cdot}$ by uniqueness, and independence from the past gives

$$\mathbf{E}\left[\Phi(X_t) \,|\, \mathcal{F}_s\right] = \sum_i \mathbf{1}_{A_i} \mathbf{E}\left[\Phi(X^{s,x_i}_{t-s})\right] = p_{s,t}\Phi(X_s)$$

Step 2, general X_s: Discretize X_s on dyadic cubes
$\hookrightarrow$ Stability with respect to the initial condition, for Φ Lipschitz

Step 3, general Φ: Monotone class theorem, as for Brownian motion

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Diffusion matrix

Definition 7

Let $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ the coefficients of the SDE. The diffusion matrix is

$$a = \sigma\sigma^* : \mathbb{R}^n \to \mathbb{R}^{n\times n}, \quad \text{that is} \quad a^{j_1 j_2}(x) = \sum_{k=1}^{d} \sigma^{j_1 k}(x)\, \sigma^{j_2 k}(x)$$

Properties:

- $a(x)$ symmetric and nonnegative, since $\langle a(x)y, y\rangle = |\sigma(x)^* y|^2$
- a Lipschitz on bounded sets whenever σ is Lipschitz

Generator

Definition 8

Let:

- b, σ coefficients of the SDE, $a = \sigma\sigma^*$ the diffusion matrix
- $f \in \mathcal{C}^2(\mathbb{R}^n)$

The generator $\mathcal{L}$ of the diffusion is defined by

$$\mathcal{L}f(x) = \sum_{j=1}^{n} b^j(x)\,\partial_{x_j} f(x) + \frac{1}{2}\sum_{j_1,j_2=1}^{n} a^{j_1 j_2}(x)\,\partial^2_{x_{j_1}x_{j_2}} f(x)$$

Interpretation: Second order differential operator

$\hookrightarrow$ First order part from the drift, second order part from the noise

Generator: examples

Brownian motion: $b = 0$, $\sigma = \mathrm{Id}$, hence $a = \mathrm{Id}$ and

$$\mathcal{L}f = \frac{1}{2}\sum_{j=1}^{n} \partial^2_{x_j x_j} f = \frac{1}{2}\Delta f$$

$\hookrightarrow$ Operator of the heat equation seen in Chapter 1

Case $n = d = 1$: $b, \sigma : \mathbb{R} \to \mathbb{R}$, hence $a = \sigma^2$ and

$$\mathcal{L}f = b\, f' + \frac{1}{2}\sigma^2\, f''$$

$\hookrightarrow$ Setting of the proof below

Integrated generator identity

Lemma 9

Let b, σ Lipschitz, $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then for $u \in [0, \tau - s]$,

$$p_{s,s+u} f(x) = f(x) + \int_0^u \mathbf{E}\left[\mathcal{L} f(X_r^{s,x})\right] dr$$

Interpretation: Evolution of $\mathbf{E}[f(X_u^{s,x})]$ governed by $\mathcal{L}$
$\hookrightarrow$ Analog of the heat semigroup identity for Brownian motion

Proof: strategy

Two steps:

- $X^{s,x}$ is an Itô process on the restarted basis
- Itô's formula for $f(X_u^{s,x})$, then take expectations

Below: Proof for $n = d = 1$, after recalling the Itô process slides
$\hookrightarrow$ General case: same computation with indices

Recall: Itô process in dimension 1

Definition 10

Let:

- $\alpha \in \mathbb{R}$, and b bounded adapted real process
- σ real process in $L^2_a([0,\tau])$

We say that X is an Itô process if

$$X_t = \alpha + \int_0^t b_r \, dr + \int_0^t \sigma_r \, dW_r$$

Solution of (1) with $n = d = 1$: Itô process with
$\hookrightarrow$ $b_r = b(X_r)$ and $\sigma_r = \sigma(X_r)$
$\hookrightarrow$ $b(X_\cdot)$ only continuous, which is enough for $\int_0^t b_r \, dr$

Recall: Itô's formula in dimension 1

Proposition 11

Let X Itô process defined by α, b, σ, and $f \in \mathcal{C}_b^2(\mathbb{R})$. Then:

$$f(X_t) = f(\alpha) + \int_0^t f'(X_r)\, b_r\, dr + \int_0^t f'(X_r)\, \sigma_r\, dW_r + \frac{1}{2}\int_0^t f''(X_r)\, \sigma_r^2\, dr$$

Remark: $\int_0^t f'(X_r)\, \sigma_r\, dW_r$ is a martingale

$\hookrightarrow$ Since f' bounded and $\sigma \in L_a^2([0, \tau])$

$\hookrightarrow$ Vanishing expectation

Proof of the integrated identity for $n = d = 1$

Itô process: On the restarted basis, with $b_r = b(X_r^{s,x})$ and $\sigma_r = \sigma(X_r^{s,x})$

Itô's formula: Grouping the dr terms with $\mathcal{L}f = b\,f' + \frac{1}{2}\sigma^2 f''$,

$$f(X_u^{s,x}) = f(x) + \int_0^u \mathcal{L}f(X_r^{s,x})\,dr + \int_0^u f'(X_r^{s,x})\,\sigma(X_r^{s,x})\,dW_r^s$$

Expectation:

- Stochastic integral has mean 0, and $\mathbf{E}[f(X_u^{s,x})] = p_{s,s+u}f(x)$
- Fubini for the dr term: $|\mathcal{L}f(z)| \leq c_f(1 + |z|^2)$ and $\mathbf{E}[\sup_r |X_r^{s,x}|^2] < \infty$

Generator of a diffusion

Theorem 12

Let b, σ Lipschitz, $s \in [0, \tau)$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$,

$$\lim_{h \to 0^+} \frac{p_{s,s+h} f(x) - f(x)}{h} = \mathcal{L} f(x)$$

Interpretation: $\mathcal{L}$ is the infinitesimal generator of $\{p_{s,t};\ 0 \leq s \leq t \leq \tau\}$

$\hookrightarrow$ Deterministic operator, coefficients depend on the current position only

Brownian case: $p_{s,s+h} = p_h$ heat semigroup and $\mathcal{L} = \frac{1}{2}\Delta$

$\hookrightarrow$ Generator of Brownian motion, seen in Chapter 2

Proof

Step 1, continuity at $r = 0$: We have

$$\lim_{r \to 0^+} \mathbf{E}\left[\mathcal{L}f(X_r^{s,x})\right] = \mathcal{L}f(x)$$

- $X_r^{s,x} \to x$ a.s. (continuous paths, $X_0^{s,x} = x$), and $\mathcal{L}f$ continuous
- Domination: $|\mathcal{L}f(X_r^{s,x})| \leq c_f(1 + \sup_r |X_r^{s,x}|^2)$, integrable

Step 2, averages: Integrated identity divided by h:

$$\frac{p_{s,s+h}f(x) - f(x)}{h} = \frac{1}{h}\int_0^h \mathbf{E}\left[\mathcal{L}f(X_r^{s,x})\right] dr \xrightarrow[h \to 0^+]{} \mathcal{L}f(x)$$

$\hookrightarrow$ Average of a bounded function, continuous at $r = 0$

Evolution equation for $p_{s,t}f$

Corollary 13

Let b, σ Lipschitz, $s \in [0, \tau)$, $x \in \mathbb{R}^n$ and $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. Then $t \mapsto p_{s,t}f(x)$ is $\mathcal{C}^1$ on $[s, \tau]$, and

$$\partial_t \, p_{s,t}f(x) = \mathbf{E}\left[\mathcal{L}f\left(X_{t-s}^{s,x}\right)\right]$$

Interpretation:

- Right hand side is $p_{s,t}(\mathcal{L}f)(x)$, for $\mathcal{L}f$ with quadratic growth
- At $t = s$, the generator theorem: $\partial_t \, p_{s,t}f(x)|_{t=s} = \mathcal{L}f(x)$

Proof

Continuity: $r \mapsto \mathbf{E}[\mathcal{L}f(X_r^{s,x})]$ continuous on $[0, \tau - s]$
$\hookrightarrow$ Continuous paths, plus the domination of Step 1 above

Integrated identity with $u = t - s$:

$$p_{s,t}f(x) = f(x) + \int_0^{t-s} \mathbf{E}\left[\mathcal{L}f(X_r^{s,x})\right] dr$$

$\hookrightarrow$ Fundamental theorem of calculus in the variable t

Towards a PDE for $p_{s,t}f$

Brownian case: $p_{s,t} = p_{t-s}$ and $\mathcal{L} = \frac{1}{2}\Delta$, so that

$$\partial_t\, p_t f = p_t\left(\tfrac{1}{2}\Delta f\right) = \tfrac{1}{2}\Delta\left(p_t f\right)$$

$\hookrightarrow$ Heat equation of Chapter 1

General case, formally: $u(s,x) = p_{s,t}f(x)$ smooth in (s,x)
$\hookrightarrow$ $p_{s-h,t}f = p_{s-h,s}(p_{s,t}f)$, then generator theorem as $h \to 0^+$

$$\partial_s u + \mathcal{L}u = 0, \quad u(t,\cdot) = f$$

$\hookrightarrow$ Backward equation, proved in the next section in the additive case

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Additive noise

Setting: $d = n$, and the equation with additive noise

$$X_t = X_0 + \int_0^t b(r, X_r)\, dr + \int_0^t \sigma(r)\, dW_r \tag{2}$$

- $b : [0, \tau] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, \tau] \to \mathbb{R}^{n \times n}$
- $X_0 \in \mathcal{F}_0$ random initial condition, with law μ

Comments:

- Well posed path by path (Chapter 3)
- Forward equation of generative models, reversed in Chapter 5

Smoothness assumption

Assumption:

1. b infinitely differentiable, bounded, with bounded derivatives of all orders
2. σ continuously differentiable, and $\sigma(t)$ invertible for all $t \in [0, \tau]$

Remarks:

- Far from minimal, chosen for convenience
- In force for the rest of the deck

Well posedness: b Lipschitz in x, uniformly in t
$\hookrightarrow$ Existence and uniqueness for (2)

Uniform ellipticity

Lemma 14

Let σ as in the smoothness assumption, and $a(t) = \sigma(t)\sigma(t)^*$. Then a is bounded on $[0, \tau]$, and there is $\lambda > 0$ with

$$\langle a(t)\, y, y\rangle = |\sigma(t)^* y|^2 \geq \lambda\, |y|^2, \quad t \in [0, \tau],\ y \in \mathbb{R}^n$$

Interpretation:

- Noise acts in every direction, with strength bounded below
- Hypothesis of the parabolic theory giving smooth densities

Proof

Compactness argument:

- $(t, y) \mapsto |\sigma(t)^* y|^2$ continuous on the compact set $K = [0, \tau] \times \{|y| = 1\}$
- Positive on K, since $\sigma(t)^*$ invertible and $y \neq 0$
- Hence a positive minimum λ on K

Conclusion: For $y \neq 0$, by homogeneity,

$$|\sigma(t)^* y|^2 = |y|^2 \left| \sigma(t)^* \frac{y}{|y|} \right|^2 \geq \lambda |y|^2$$

$\hookrightarrow$ Boundedness of a by continuity of σ on $[0, \tau]$

Restarted process in the additive case

Restarted process: For $s \in [0, \tau)$ and $x \in \mathbb{R}^n$, $Y^{s,x}$ solution of

$$Y_u^{s,x} = x + \int_0^u b(s+r, Y_r^{s,x})\, dr + \int_0^u \sigma(s+r)\, dW_r^s, \quad u \in [0, \tau - s]$$

$\hookrightarrow$ Transition operator $\widetilde{p}_{s,t}\Phi(x) = \mathbf{E}\left[\Phi(Y_{t-s}^{s,x})\right]$

Markov property: For X solution of (2), with random X_0,

$$\mathbf{E}\left[\Phi(X_t) \,|\, \mathcal{F}_s\right] = \widetilde{p}_{s,t}\Phi(X_s)$$

$\hookrightarrow$ Same proof as in the Lipschitz case, restarted equation well posed

Generator in the additive case

Generator: Time dependent operator, for $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$,

$$\mathcal{L}_s\varphi(x) = \sum_{j=1}^{n} b^j(s,x)\,\partial_{x_j}\varphi(x) + \frac{1}{2}\sum_{j_1,j_2=1}^{n} a^{j_1 j_2}(s)\,\partial^2_{x_{j_1}x_{j_2}}\varphi(x)$$

Integrated identity: Same proof as before, for $\varphi \in \mathcal{C}^2_b(\mathbb{R}^n)$

$$\widetilde{p}_{s,s+u}\varphi(x) = \varphi(x) + \int_0^u \mathbf{E}\left[\mathcal{L}_{s+r}\varphi(Y_r^{s,x})\right] dr$$

$\hookrightarrow$ Two time variables: $s + r$ for the operator, r for the process

Adjoint operator

Definition 15

Let:

- b, σ satisfying the smoothness assumption, $a(t) = \sigma(t)\sigma(t)^*$
- $t \in [0, \tau]$ and $\psi \in \mathcal{C}^2(\mathbb{R}^n)$

The adjoint operator $\mathcal{L}_t^*$ is defined by

$$\mathcal{L}_t^* \psi(x) = -\sum_{j=1}^{n} \partial_{x_j} \left[b^j(t,x)\, \psi(x) \right] + \frac{1}{2} \sum_{j_1, j_2 = 1}^{n} a^{j_1 j_2}(t)\, \partial^2_{x_{j_1} x_{j_2}} \psi(x)$$

Interpretation: Formal adjoint of $\mathcal{L}_t$ in $L^2(\mathbb{R}^n)$ (see duality below)

Adjoint operator: remarks

Comparison with $\mathcal{L}_t$:

- Only the first order term changes with respect to $\mathcal{L}_t$
- Second order term unchanged, since $a(t)$ does not depend on x
  $\hookrightarrow$ With $\sigma(t,x)$, one gets $\frac{1}{2}\partial^2_{x_{j_1}x_{j_2}}\left[a^{j_1 j_2}(t,x)\,\psi(x)\right]$ instead

Role: $\mathcal{L}_t^*$ governs the evolution of densities (see below)

Duality

Lemma 16

Let $t \in [0, \tau]$, $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^2(\mathbb{R}^n)$. Then:

$$\int_{\mathbb{R}^n} \mathcal{L}_t \varphi(x)\, \psi(x)\, dx = \int_{\mathbb{R}^n} \varphi(x)\, \mathcal{L}_t^* \psi(x)\, dx$$

Comments:

- Justifies the name adjoint
- Only property of $\mathcal{L}_t^*$ used in the sequel
- Integrals over a ball containing Supp(φ), no convergence issue

Proof

First order term: One integration by parts, no boundary term (φ compactly supported)

$$\int_{\mathbb{R}^n} b^j(t,x)\,\partial_{x_j}\varphi(x)\,\psi(x)\,dx = -\int_{\mathbb{R}^n} \varphi(x)\,\partial_{x_j}\left[b^j(t,x)\,\psi(x)\right]dx$$

Second order term: Two integrations by parts, $a^{j_1 j_2}(t)$ constant in x

$$\int_{\mathbb{R}^n} a^{j_1 j_2}(t)\,\partial^2_{x_{j_1}x_{j_2}}\varphi(x)\,\psi(x)\,dx = \int_{\mathbb{R}^n} a^{j_1 j_2}(t)\,\varphi(x)\,\partial^2_{x_{j_1}x_{j_2}}\psi(x)\,dx$$

Conclusion: Sum over j, then over j_1, j_2
$\hookrightarrow$ Definition of $\mathcal{L}_t^*$ under the integral sign

Chapman–Kolmogorov equation

Theorem 17

Let b, σ satisfying the smoothness assumption, $0 \leq s \leq t \leq t' \leq \tau$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ bounded measurable. Then:

$$\widetilde{p}_{s,t'}\Phi = \widetilde{p}_{s,t}\left(\widetilde{p}_{t,t'}\Phi\right)$$

Interpretation: Predict over $[s, t]$, then over $[t, t']$
$\hookrightarrow$ Same as predicting over $[s, t']$ directly

Remarks:

- Brownian case: semigroup property $p_{t'-s} = p_{t-s} \circ p_{t'-t}$
- Same statement and proof for $p_{s,t}$ in the Lipschitz case

Proof

Markov property on the restarted basis: For $Y^{s,x}$, at times $t - s \leq t' - s$,

$$\mathbf{E}\left[\Phi(Y^{s,x}_{t'-s}) \,|\, \mathcal{F}_t\right] = \widetilde{p}_{t,t'}\Phi(Y^{s,x}_{t-s})$$

$\hookrightarrow$ Restarting $Y^{s,x}$ at time $t - s$ is restarting X at time t
Shifted coefficients $b(t + r, \cdot)$ and $\sigma(t + r)$

Tower property:

$$\widetilde{p}_{s,t'}\Phi(x) = \mathbf{E}\left[\Phi(Y^{s,x}_{t'-s})\right] = \mathbf{E}\left[\mathbf{E}\left[\Phi(Y^{s,x}_{t'-s}) \,|\, \mathcal{F}_t\right]\right] = \mathbf{E}\left[\widetilde{p}_{t,t'}\Phi(Y^{s,x}_{t-s})\right]$$

$\hookrightarrow$ Right hand side is $\widetilde{p}_{s,t}(\widetilde{p}_{t,t'}\Phi)(x)$, by definition of $\widetilde{p}_{s,t}$

Transition density

Theorem 18

Under the smoothness assumption, there is a function $\widetilde{q}_{s,t}(x,y)$, *for* $0 \leq s < t \leq \tau$ *and* $x, y \in \mathbb{R}^n$, *such that:*

- $\widetilde{q}_{s,t}(x,\cdot)$ *is the density of* $Y_{t-s}^{s,x}$, *that is*

$$\widetilde{p}_{s,t}\Phi(x) = \int_{\mathbb{R}^n} \widetilde{q}_{s,t}(x,y)\,\Phi(y)\,dy$$

- $\widetilde{q}$ *is infinitely differentiable in* (s,t,x,y), *and* $\widetilde{q}_{s,t}(x,y) > 0$

Comments:

- Admitted: parabolic PDE theory (Friedman, Durrett Section 7.3)
- Fundamental solution of the Kolmogorov equations (next slide)

Kolmogorov equations

Theorem 19

Under the smoothness assumption, $\widetilde{q}$ satisfies:

- *Backward equation, (t, y) fixed, $\mathcal{L}_s$ acts on x:*

$$-\partial_s \widetilde{q}_{s,t}(x, y) = \mathcal{L}_s \widetilde{q}_{s,t}(\cdot, y)(x)$$

- *Forward (Fokker–Planck) equation, (s, x) fixed, $\mathcal{L}_t^*$ acts on y:*

$$\partial_t \widetilde{q}_{s,t}(x, y) = \mathcal{L}_t^* \widetilde{q}_{s,t}(x, \cdot)(y)$$

Dirac condition: $\widetilde{q}_{s,t}(x, y)\, dy \to \delta_x$ weakly as $t - s \to 0$
$\hookrightarrow$ Continuity of the paths of $Y^{s,x}$

Backward versus forward

Backward equation:

- Starting point (s, x) moves, arrival point (t, y) frozen
- Operator $\mathcal{L}_s$, produced by Itô's formula

Forward equation:

- Arrival point (t, y) moves, starting point (s, x) frozen
- Adjoint operator $\mathcal{L}_t^*$, describes the transport of mass

In the sequel: Forward equation is the relevant one
$\hookrightarrow$ Evolution of the law of X_t, and the reversal of Chapter 5

Chapman–Kolmogorov for densities

Corollary 20

Let $0 \leq s < r < t \leq \tau$ *and* $x \in \mathbb{R}^n$. *Then for a.e.* $y \in \mathbb{R}^n$,

$$\widetilde{q}_{s,t}(x,y) = \int_{\mathbb{R}^n} \widetilde{q}_{s,r}(x,z)\, \widetilde{q}_{r,t}(z,y)\, dz$$

Interpretation: Chapman–Kolmogorov at the level of densities
$\hookrightarrow$ Go from x to y through an intermediate point z at time r

Remark: Holds for every y, by continuity of $\widetilde{q}$ (not needed below)

Proof

Operators: Chapman–Kolmogorov $\widetilde{p}_{s,t}\Phi = \widetilde{p}_{s,r}(\widetilde{p}_{r,t}\Phi)$, written with densities

$$\int_{\mathbb{R}^n} \widetilde{q}_{s,t}(x,y)\,\Phi(y)\,dy = \int_{\mathbb{R}^n} \widetilde{q}_{s,r}(x,z)\left(\int_{\mathbb{R}^n} \widetilde{q}_{r,t}(z,y)\,\Phi(y)\,dy\right)dz$$

Fubini: Exchange dy and dz (absolutely convergent, Φ bounded)

$$\int_{\mathbb{R}^n} \widetilde{q}_{s,t}(x,y)\,\Phi(y)\,dy = \int_{\mathbb{R}^n} \Phi(y)\left(\int_{\mathbb{R}^n} \widetilde{q}_{s,r}(x,z)\,\widetilde{q}_{r,t}(z,y)\,dz\right)dy$$

$\hookrightarrow$ Φ arbitrary, so both integrands in y coincide a.e.

Marginal density

Notation: X solution of (2) with $X_0 \sim \mu$. For $t \in (0, \tau]$,

$$p_t(x) = \int_{\mathbb{R}^n} \widetilde{q}_{0,t}(z, x)\, \mu(dz), \quad x \in \mathbb{R}^n$$

$\hookrightarrow$ μ arbitrary (e.g. empirical measure of a data set)

Proposition 21

Under the smoothness assumption:

- *p_t is the density of X_t*
- *p infinitely differentiable on $(0, \tau] \times \mathbb{R}^n$, and $p_t(x) > 0$*
- *Fokker–Planck equation: $\partial_t p_t(x) = \mathcal{L}_t^* p_t(x)$*

Weak form of the forward equation

Proposition 22

Let b Lipschitz in x only, X solution of (2), and ν_t law of X_t. Then for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $0 \leq s \leq t \leq \tau$,

$$\int_{\mathbb{R}^n} \varphi \, d\nu_t = \int_{\mathbb{R}^n} \varphi \, d\nu_s + \int_s^t \left(\int_{\mathbb{R}^n} \mathcal{L}_r \varphi \, d\nu_r \right) dr$$

Comments:

- Itô's formula alone, no parabolic theory, no density needed
- Fokker–Planck equation in weak form, tested against φ

Proof (1)

Integrated identity: For $z \in \mathbb{R}^n$, with $r = s + u$ and r frozen in $\mathcal{L}_r\varphi$,

$$\widetilde{p}_{s,t}\varphi(z) = \varphi(z) + \int_s^t \mathbf{E}\left[\mathcal{L}_r\varphi(Y_{r-s}^{s,z})\right] dr = \varphi(z) + \int_s^t \widetilde{p}_{s,r}\left(\mathcal{L}_r\varphi\right)(z)\, dr$$

Markov property: For Ψ bounded measurable and $r \in [s, \tau]$,

$$\int_{\mathbb{R}^n} \widetilde{p}_{s,r}\Psi\, d\nu_s = \mathbf{E}\left[\widetilde{p}_{s,r}\Psi(X_s)\right] = \mathbf{E}\left[\Psi(X_r)\right] = \int_{\mathbb{R}^n} \Psi\, d\nu_r$$

Next: Integrate the first identity in $\nu_s(dz)$, then apply the second

Proof (2)

Integration in $\nu_s(dz)$: Then Fubini, since the integrand is bounded

$$\int_{\mathbb{R}^n} \widetilde{p}_{s,t}\varphi \, d\nu_s = \int_{\mathbb{R}^n} \varphi \, d\nu_s + \int_s^t \left(\int_{\mathbb{R}^n} \widetilde{p}_{s,r} \left(\mathcal{L}_r \varphi \right) d\nu_s \right) dr$$

Markov identity twice:

- $\Psi = \varphi$ at time t: left hand side is $\int_{\mathbb{R}^n} \varphi \, d\nu_t$
- $\Psi = \mathcal{L}_r\varphi$ at time r (bounded, φ compactly supported): inner integral is $\int_{\mathbb{R}^n} \mathcal{L}_r\varphi \, d\nu_r$

Conclusion:

$$\int_{\mathbb{R}^n} \varphi \, d\nu_t = \int_{\mathbb{R}^n} \varphi \, d\nu_s + \int_s^t \left(\int_{\mathbb{R}^n} \mathcal{L}_r\varphi \, d\nu_r \right) dr$$

From the weak form to Fokker–Planck

Density of X_t: Markov property at $s = 0$, then Fubini

$$\mathbf{E}\left[\Phi(X_t)\right] = \int_{\mathbb{R}^n} \widetilde{p}_{0,t}\Phi(z)\,\mu(dz) = \int_{\mathbb{R}^n} \Phi(x)\left(\int_{\mathbb{R}^n} \widetilde{q}_{0,t}(z,x)\,\mu(dz)\right)dx$$

$\hookrightarrow \nu_t(dx) = p_t(x)\,dx$, with $p_t \in \mathcal{C}^2$ by the parabolic theory

Duality: The weak form becomes

$$\int_{\mathbb{R}^n} \varphi(x)\left[p_t(x) - p_s(x) - \int_s^t \mathcal{L}_r^* p_r(x)\,dr\right]dx = 0$$

$\hookrightarrow \varphi$ arbitrary, bracket continuous, hence $= 0$. Differentiate in t

Outline

1. Restarting a diffusion
2. Generator of a diffusion
3. Transition densities and Kolmogorov equations
4. Summary

Summary: notation (1)

Restarted process: For $s \in [0, \tau)$ and $x \in \mathbb{R}^n$, $X^{s,x}$ solution of

$$X_u^{s,x} = x + \int_0^u b(s+r, X_r^{s,x})\, dr + \int_0^u \sigma(s+r)\, dW_r^s, \quad u \in [0, \tau - s]$$

Transition operator: For $0 \leq s \leq t \leq \tau$ and Φ bounded measurable

$$\widetilde{p}_{s,t}\Phi(x) = \mathbf{E}\left[\Phi\left(X_{t-s}^{s,x}\right)\right]$$

Markov property: $\mathbf{E}[\Phi(X_t) \,|\, \mathcal{F}_s] = \widetilde{p}_{s,t}\Phi(X_s)$
$\hookrightarrow$ Prediction of X_t given $\mathcal{F}_s$ through the operator $\widetilde{p}_{s,t}$

Summary: notation (2)

Diffusion matrix: $a(t) = \sigma(t)\sigma(t)^*$

Generator and adjoint: For $\varphi, \psi \in \mathcal{C}^2(\mathbb{R}^n)$,

$$\begin{aligned} \mathcal{L}_t\varphi &= \sum_{j=1}^n b^j(t,\cdot)\,\partial_{x_j}\varphi + \frac{1}{2}\sum_{i,j=1}^n a^{ij}(t)\,\partial^2_{x_i x_j}\varphi \\ \mathcal{L}_t^*\psi &= -\sum_{j=1}^n \partial_{x_j}\left[b^j(t,\cdot)\,\psi\right] + \frac{1}{2}\sum_{i,j=1}^n a^{ij}(t)\,\partial^2_{x_i x_j}\psi \end{aligned}$$

Duality: $\int \mathcal{L}_t\varphi\,\psi\,dx = \int \varphi\,\mathcal{L}_t^*\psi\,dx$ for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

Summary: Kolmogorov equations for $\widetilde{p}_{s,t}$

Chapman–Kolmogorov: For $0 \leq s \leq t \leq t' \leq \tau$,

$$\widetilde{p}_{s,t'} = \widetilde{p}_{s,t} \circ \widetilde{p}_{t,t'}$$

$\hookrightarrow$ Brownian case: $\widetilde{p}_{s,t}$ depends on $t - s$ only, semigroup property

Forward and backward equations: For $f \in \mathcal{C}_b^2(\mathbb{R}^n)$

$$\begin{aligned} \partial_t \, \widetilde{p}_{s,t} f &= \widetilde{p}_{s,t} \left(\mathcal{L}_t f \right), \quad \widetilde{p}_{s,s} f = f \\ \partial_s \, \widetilde{p}_{s,t} f &= -\mathcal{L}_s \left(\widetilde{p}_{s,t} f \right), \quad \widetilde{p}_{t,t} f = f \end{aligned}$$

Summary: Kolmogorov equations for $\widetilde{q}$

Transition density: Smoothness assumption
$\hookrightarrow$ $\widetilde{q}_{s,t}(x,\cdot)$ density of $X_{t-s}^{s,x}$, smooth and positive
$\hookrightarrow$ Dirac condition: $\widetilde{q}_{s,t}(x,y)\,dy \to \delta_x$ weakly as $t-s \to 0$

Backward in (s,x), forward in (t,y):

$$\begin{aligned} \partial_s \widetilde{q}_{s,t}(x,y) &= -\mathcal{L}_s \widetilde{q}_{s,t}(\cdot,y)(x) \\ \partial_t \widetilde{q}_{s,t}(x,y) &= \mathcal{L}_t^* \widetilde{q}_{s,t}(x,\cdot)(y) \end{aligned}$$

Chapman–Kolmogorov: $\widetilde{q}_{s,t}(x,y) = \int \widetilde{q}_{s,r}(x,z)\,\widetilde{q}_{r,t}(z,y)\,dz$

Summary: marginal densities

Marginal density: X solution of (2), $X_0 \sim \mu$
$\hookrightarrow$ For $t \in (0, \tau]$, X_t has density

$$p_t(x) = \int_{\mathbb{R}^n} \widetilde{q}_{0,t}(z, x)\, \mu(dz)$$

Fokker–Planck equation:

$$\partial_t p_t(x) = \mathcal{L}_t^* p_t(x), \quad (t, x) \in (0, \tau] \times \mathbb{R}^n$$

Next: Run X backward in time
$\hookrightarrow$ Reversed diffusion, drift corrected by the score $\nabla \log p_t$