

Stochastic differential equations

Samy Tindel

Purdue University

Probability Theory 2 - MA 539



Outline

- 1 Introduction and examples
- 2 Existence and uniqueness
- 3 Fractional Brownian motion
- 4 Young integration
- 5 Young differential equations

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Aim

Coefficients: We consider

- $\alpha \in \mathbb{R}^n$ and $b, \sigma^1, \dots, \sigma^d : \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- We denote: $\sigma = (\sigma^1, \dots, \sigma^d) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$
- W , d -dimensional Brownian motion.

Equation: We wish to solve

$$dX_s = b(X_s) ds + \sum_{j=1}^d \sigma^j(X_s) dW_s^j.$$

Integral form: With Itô's integral,

$$X_t = \alpha + \int_0^t b(X_s) ds + \sum_{j=1}^d \int_0^t \sigma^j(X_s) dW_s^j. \quad (1)$$

Infinitesimal drift and covariance

Proposition 1.

Let b, σ bounded, X solution of (1). Then:

$$\partial_t \mathbf{E}[X_t | \mathcal{F}_s]_{|t=s} = b(X_s), \quad \partial_t \mathbf{Cov}(X_t | \mathcal{F}_s)_{|t=s} = a(X_s),$$

with $a = \sigma \sigma^* : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$.

Interpretation:

- $b(X_s) \equiv$ Infinitesimal drift.
- $a(X_s) \equiv$ Infinitesimal covariance.

Itô process

Definition 2.

Let:

- $X : [0, \tau] \rightarrow \mathbb{R}^n$ process in L_a^2
- $\alpha \in \mathbb{R}^n$ initial condition.
- b bounded and adapted process in $\mathbb{R}^n$.
- $\{\sigma^k; k = 1, \dots, d\}$ process of $L_a^2([0, \tau]; \mathbb{R}^n)$.

We say that X is an **Itô process** if it admits a decomposition:

$$X_t = \alpha + \int_0^t b_s ds + \sum_{k=1}^d \int_0^t \sigma_s^k dW_s^k.$$

Remark:

A solution of (1) is an Itô process

$\hookrightarrow$ With $b_s = b(X_s)$ and $\sigma_s = \sigma(X_s)$.

Itô's formula for a solution of (1)

Proposition 3.

Let:

- X Itô process, defined by α, b, σ .
- $f \in \mathcal{C}_b^2$.

Then $f(X_t)$ can be decomposed as:

$$\begin{aligned} f(X_t) &= f(\alpha) + \sum_{j=1}^n \int_0^t \partial_{x_j} f(X_r) b_r^j dr \\ &\quad + \sum_{j=1}^n \sum_{k=1}^d \int_0^t \partial_{x_j} f(X_r) \sigma_r^{jk} dW_r^k \\ &\quad + \frac{1}{2} \sum_{j_1, j_2=1}^n \sum_{k=1}^d \int_0^t \partial_{x_{j_1} x_{j_2}}^2 f(X_r) \sigma_r^{j_1 k} \sigma_r^{j_2 k} dr. \end{aligned}$$

Proof of Proposition 1

Simplification: This will be shown for $s = 0$

$\hookrightarrow$ No conditional expectation to consider.

Drift term: Start from equation

$$X_t = \alpha + \int_0^t b(X_s) ds + \sum_{j=1}^d \int_0^t \sigma^j(X_s) dW_s^j.$$

The terms $\int_0^t \sigma^j(X_s) dW_s^j$ are centered. Therefore:

$$\mathbf{E}[X_t] = \alpha + \int_0^t \mathbf{E}[b(X_s)] ds.$$

and

$$\partial_t \mathbf{E}[X_t]_{|t=0} = \mathbf{E}[b(X_t)]_{|t=0} = b(\alpha).$$

Proof of Proposition 1 (2)

Coordinate product: Let

- $l, m \in \{1, \dots, n\}$.
- $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by the product $f(x) = x^l x^m$.
- $a_s = \sigma_s \sigma_s^*$.

According to Itô's formula we have:

$$\begin{aligned} X_t^l X_t^m &= \alpha^l \alpha^m + \int_0^t \left(X_r^l b_r^m + b_r^l X_r^m \right) dr \\ &\quad + \sum_{k=1}^d \int_0^t \left(X_r^l \sigma_r^{mk} + \sigma_r^{lk} X_r^m \right) dW_r^k + \sum_{k=1}^d \int_0^t a_r^{lm} dr. \end{aligned} \tag{2}$$

Proof of Proposition 1 (3)

Expected value for the product:

Taking expectation in (2) we get:

$$\partial_t \mathbf{E} [X_t^l X_t^m] = \mathbf{E} [X_t^l b_t^m + b_t^l X_t^m] + \mathbf{E} [a_t^{lm}].$$

Therefore:

$$\partial_t \mathbf{E} [X_t^l X_t^m]_{|t=0} = \alpha^l b^m(\alpha) + b^l(\alpha) \alpha^m + a^{lm}(\alpha).$$

Proof of Proposition 1 (4)

Product of expected values: We have

$$\mathbf{E} [X_t^j] = \alpha^j + \int_0^t \mathbf{E} [b_s^j] ds.$$

Therefore

$$\mathbf{E} [X_t^l] \mathbf{E} [X_t^m] = \alpha^l \alpha^m + \int_0^t \mathbf{E} [b_s^l] \mathbf{E} [X_s^m] ds + \int_0^t \mathbf{E} [b_s^m] \mathbf{E} [X_s^l] ds,$$

and differentiating:

$$\partial_t \left(\mathbf{E} [X_t^l] \mathbf{E} [X_t^m] \right) \Big|_{t=0} = \alpha^l b^m(\alpha) + b^l(\alpha) \alpha^m.$$

Proof of Proposition 1 (5)

Infinitesimal covariance: With two previous identities,

$$\begin{aligned}\partial_t \mathbf{Cov} (X_t^l, X_t^m) \Big|_{t=0} &= \partial_t \left(\mathbf{E} [X_t^l X_t^m] - \mathbf{E} [X_t^l] \mathbf{E} [X_t^m] \right) \Big|_{t=0} \\ &= \partial_t \mathbf{E} [X_t^l X_t^m] \Big|_{t=0} - \partial_t \left(\mathbf{E} [X_t^l] \mathbf{E} [X_t^m] \right) \Big|_{t=0} \\ &= a^{lm}(\alpha).\end{aligned}$$

Therefore:

$$\partial_t \mathbf{Cov} (X_t) \Big|_{t=0} = a(\alpha).$$

Geometrical Brownian motion

Proposition 4.

Let:

- W 1-dimensional Brownian motion .
- $\mu \in \mathbb{R}$ and $\sigma > 0$

We set

$$X_t = \alpha \exp(\mu t + \sigma W_t).$$

Then X is solution of (1) with $n = d = 1$ and:

$$b(x) = \left(\mu + \frac{\sigma^2}{2} \right) x, \quad \text{and} \quad \sigma(x) = \sigma x.$$

Geometrical Brownian motion (2)

Remarks:

- In order to show that X is solution of the equation
 $\hookrightarrow$ Apply Itô's formula.
- Exponential Brownian motion is very useful in finance (asset price):
 - ① $X_t \geq 0$.
 - ② Linear trend (infinitesimal drift): $\left(\mu + \frac{\sigma^2}{2}\right) X_t$.
 - ③ Fluctuations (infinitesimal standard deviation) proportional to X_t .

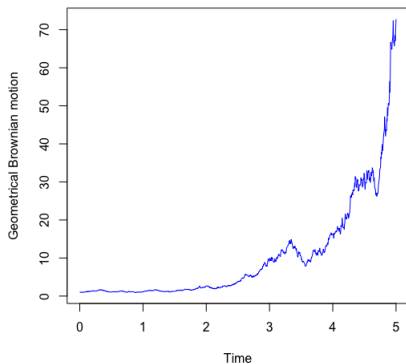
Summarized in Black and Scholes model.

Geometrical Brownian motion: illustration

Equation: Take $\mu = 1$, $\sigma = \frac{1}{2}$, $X_0 = 1$ and

$$dX_t = \left(\mu + \frac{\sigma^2}{2} \right) X_t dt + \sigma X_t dW_t$$

Simulation:



Ornstein-Uhlenbeck process

Situation:

Velocity of a Brownian particle with friction α .

Equation:

$$dX_t = -\alpha X_t dt + dW_t, \quad X_0 = a \in \mathbb{R}$$

Explicit solution:

$$X_t = e^{-\alpha t} \left(a + \sigma \int_0^t e^{\alpha s} dW_s \right)$$

Distribution: For $t > 0$ we have

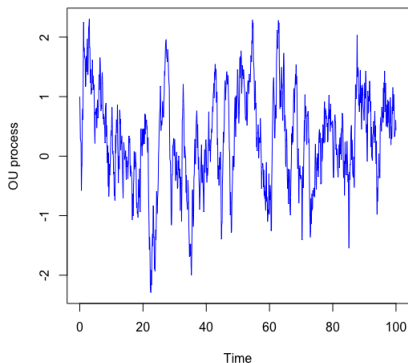
$$X_t \sim \mathcal{N} \left(a e^{-\alpha t}, \sigma_t^2 \right), \quad \text{with} \quad \sigma_t^2 = \frac{\sigma^2 (1 - e^{-2\alpha t})}{2\alpha}$$

Ornstein-Uhlenbeck: illustration

Equation: Take $\alpha = 1$ and

$$dX_t = -\alpha X_t dt + dW_t, \quad X_0 = 1$$

Simulation:



Galton-Watson process

Model: We start from $n\alpha$ persons for generation $m = 0$, then

- At each generation m , individuals have i.i.d offspring
- Common law for offspring: random variable Q .
- For $n \geq 1$, sequence of number of persons at generation m :

$$\{Z_m^n; m \geq 0\}.$$

Assumptions on Q : We suppose

- 1 $\mathbf{E}[Q] = 1 + \frac{\beta}{n}$ with $\beta > 0$.
- 2 $\mathbf{Var}(Q) = \sigma^2$ with $\sigma^2 > 0$.
- 3 For all $\delta > 0$, we have $\lim_{n \rightarrow \infty} \mathbf{E}[Q^2 \mathbf{1}_{(Q > \delta n)}] = 0$

Feller diffusion

Expectation and variance computations: Thanks to offspring $\perp\!\!\!\perp$, we have

$$\mathbf{E}[Z_1^n | Z_0^n = n\alpha] = n\alpha \left(1 + \frac{\beta}{n}\right), \quad \mathbf{Var}(Z_1^n | Z_0^n = n\alpha) = n\alpha\sigma^2.$$

Scaling: We set $X_t^n = \frac{1}{n}Z_{[nt]}^n$. Then:

$$\mathbf{E}[X_{1/n}^n - \alpha | X_0^n = \alpha] = \alpha\beta \frac{1}{n}, \quad \mathbf{Var}(X_{1/n}^n | X_0^n = \alpha) = \alpha\sigma^2 \frac{1}{n}.$$

Limiting equation: Computing limiting drift and variance, we get

$$dX_t = \beta X_t dt + \sigma \sqrt{X_t} dW_t, \quad X_0 = \alpha.$$

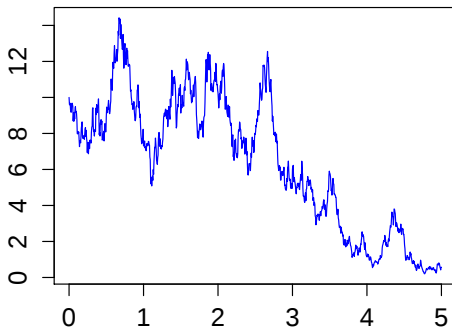
One can show that $\lim_{n \rightarrow \infty} X^n = X$ in law.

Feller diffusion: illustration

Equation:

$$dX_t = .02X_t dt + 2\sqrt{X_t} dW_t$$

Simulation:



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Definition of solution

Definition 5.

We say that $(X, W, (\mathcal{F}_t)_{t \geq 0})$ is solution of (1) if:

- 1 W is a $\mathcal{F}_t$ -Brownian motion.
- 2 X satisfies $X_t = \alpha + \int_0^t b(X_s) ds + \sum_{j=1}^d \int_0^t \sigma^j(X_s) dW_s^j$ for $t \geq 0$.

Definition 6.

We say that (1) admits a **strong solution** if:

$\hookrightarrow$ One can take $\mathcal{F}_t = \mathcal{F}_t^W$ in Definition 5.

Pathwise uniqueness

Definition 7.

Pathwise uniqueness: If X^1, X^2 are two solutions of (1) with:

- ① $X_0^1 = X_0^2 = \alpha$.
- ② Same Brownian motion W .

Then:

$$\mathbf{P} \left(X_t^1 = X_t^2 \text{ for all } t \geq 0 \right) = 1.$$

Remark: Absence of strong solution and non pathwise uniqueness
 $\hookrightarrow$ For very irregular coefficients b, σ .

Existence and uniqueness

Theorem 8.

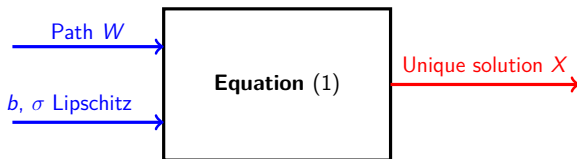
We assume that b and σ are Lipschitz functions:

There exist $c_\sigma, c_b > 0$ such that for all $x, y \in \mathbb{R}^n$ we have

$$|\sigma^j(x) - \sigma^j(y)| \leq c_\sigma |x - y|, \quad |b(x) - b(y)| \leq c_b |x - y|.$$

Then on every interval $[0, \tau]$, equation (1) admits:

- 1 A strong solution in $L^2(\Omega; \mathcal{C}([0, \tau]))$.
- 2 Pathwise uniqueness in $L^2(\Omega; \mathcal{C}([0, \tau]))$.



Proof: strategy

Key application: We define $\Gamma : L_a^2([0, \tau]) \rightarrow L_a^2([0, \tau])$ as:

$$\Gamma(Y) \equiv \tilde{Y}, \quad \tilde{Y}_t = \alpha + \int_0^t b(Y_s) ds + \sum_{j=1}^d \int_0^t \sigma^j(Y_s) dW_s^j.$$

Picard iterations: We set $X^0 \equiv \alpha$ and $X^{n+1} = \Gamma(X^n)$.

Aim: Show that:

- 1 X^n converges to X , where X is a strong solution.
- 2 Pathwise uniqueness.

Simplification in proofs:

We suppose $n = d = 1$ and $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$.

Bounds on application Γ

Lemma 9.

Let:

- $Y, Z \in L_a^2([0, \tau])$.
- $\tilde{Y} = \Gamma(Y)$, $\tilde{Z} = \Gamma(Z)$.

We assume that $c_\sigma, c_b \leq K$. Then:

$$\mathbf{E} \left[\sup_{t \leq \tau} |\tilde{Y}_t - \tilde{Z}_t|^2 \right] \leq c_{K, \tau, d} \mathbf{E} \left[\int_0^\tau |Y_t - Z_t|^2 dt \right].$$

with:

$$c_{K, \tau, d} = (2\tau d + 8d^2) K^2.$$

Proof

Expression for the difference: We have

$$\begin{aligned}\tilde{Y}_t - \tilde{Z}_t &= \int_0^t [b(Y_s) - b(Z_s)] ds + \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \\ &\equiv J_{t,1} + J_{t,2}.\end{aligned}$$

Therefore

$$|\tilde{Y}_t - \tilde{Z}_t|^2 \leq 2(|J_{t,1}|^2 + |J_{t,2}|^2)$$

and:

$$\sup_{t \leq \tau} |\tilde{Y}_t - \tilde{Z}_t|^2 \leq 2 \left(\sup_{t \leq \tau} |J_{t,1}|^2 + \sup_{t \leq \tau} |J_{t,2}|^2 \right).$$

Bounds for the Lebesgue integral

Application of Jensen: We have

$$\begin{aligned}|J_{t,1}|^2 &= \left| \int_0^t [b(Y_s) - b(Z_s)] ds \right|^2 \\ &\leq t \int_0^t |b(Y_s) - b(Z_s)|^2 ds \\ &\leq \tau \int_0^\tau |b(Y_s) - b(Z_s)|^2 ds.\end{aligned}$$

Lipschitz property for b : We get

$$\begin{aligned}\sup_{t \leq \tau} |J_{t,1}|^2 &\leq \tau K^2 \int_0^\tau |Y_s - Z_s|^2 ds \\ \mathbf{E} \left[\sup_{t \leq \tau} |J_{t,1}|^2 \right] &\leq \tau K^2 \int_0^\tau \mathbf{E} [|Y_s - Z_s|^2] ds\end{aligned}$$

Doob's maximal inequality

Proposition 10.

Let $\tau > 0$ and:

- W standard Brownian motion.
- $u \in L^2_a([0, \tau])$.
- $M_t \equiv \int_0^t u_r dW_r$.

Then we have:

$$\mathbf{E} \left[\sup_{t \in [0, \tau]} |M_t|^2 \right] \leq 4 \mathbf{E} [|M_\tau|^2] = 4 \mathbf{E} \left[\int_0^\tau u_r^2 dr \right].$$

Bounds for the stochastic integral

Recall: $J_{t,2} = \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s$

Application of Proposition 10: We have

$$\begin{aligned} \mathbf{E} \left[\sup_{t \leq \tau} |J_{t,2}|^2 \right] &= \mathbf{E} \left[\sup_{t \leq \tau} \left| \int_0^t [\sigma(Y_s) - \sigma(Z_s)] dW_s \right|^2 \right] \\ &\leq 4 \mathbf{E} \left[\int_0^\tau |\sigma(Y_s) - \sigma(Z_s)|^2 dr \right] \end{aligned}$$

Conclusion: Lemma 9 is shown

↪ Combining inequalities for $J_{t,1}$ and $J_{t,2}$.

Bound for iterations of Γ

Lemma 11.

Let X^n Picard iterations on $[0, \tau]$:

$$X^0 \equiv \alpha \quad \text{and} \quad X^{n+1} = \Gamma(X^n).$$

We set:

$$\Delta_n(t) = \mathbf{E} \left[\sup_{s \leq t} |X_s^n - X_s^{n-1}|^2 \right].$$

Then there exist two constants c_1, c_2 such that:

$$\sup_{t \leq \tau} \Delta_n(t) \leq \frac{c_1 c_2^n}{n!}.$$

Proof

Bound for Δ_1 : We have $X_s^1 - X_s^0 = b(\alpha)s + \sigma(\alpha)W_s$. Therefore:

$$\Delta_1(t) \leq c_1 t.$$

Induction: We have

$$X^n - X^{n-1} = \Gamma(X^{n-1}) - \Gamma(X^{n-2}).$$

According to Lemma 9, we get:

$$\Delta_n(t) \leq c_2 \int_0^t \Delta_{n-1}(s) ds.$$

With value of $\Delta_1(t)$, we get the result.

Convergence of X^n

Lemma 12.

Let X^n Picard iterations on $[0, \tau]$.

Then a.s X^n converges to a process X in $\mathcal{C}([0, \tau])$.

Proof

Reduction to a series convergence: We have

$$X^n = X^0 + \sum_{j=1}^{n-1} (X^{j+1} - X^j).$$

Therefore $\lim_{n \rightarrow \infty} X^n$ exists as long as $\sum_j (X^{j+1} - X^j)$ is convergent.

Series convergence:

Let $A_n = (\sup_{t \leq \tau} |X_t^n - X_t^{n-1}| \geq \frac{1}{2^n})$. We will show:

$$\mathbf{P} \left(\limsup_{n \rightarrow \infty} A_n \right) = \mathbf{P} (A_n \text{ realized } \infty\text{-tly often}) = 0.$$

This entails convergence of $\sum_j (X^{j+1} - X^j)$ in $\|\cdot\|_\infty$.

Proof (2)

Application of Borel-Cantelli: We have

$$\begin{aligned}\mathbf{P}(A_n) &\leq 2^{2n} \mathbf{E} \left[\sup_{t \leq \tau} |X_t^n - X_t^{n-1}|^2 \right] \\ &= 4^n \Delta_n(t) \\ &\leq \frac{4^n c_1 c_2^n}{n!} = \frac{c_1 c_3^n}{n!}\end{aligned}$$

Therefore:

$$\sum_{n \geq 1} \mathbf{P}(A_n) < \infty \quad \implies \quad \mathbf{P} \left(\limsup_{n \rightarrow \infty} A_n \right) = 0.$$

This finishes the proof of Lemma 12.

Convergence of X^n in $L^2(\Omega)$

Lemma 13.

Let:

- X^n Picard iterations on $[0, \tau]$.
- X limit of X^n obtained in Lemma 12.
- $\|f\|_{\infty, \tau} \equiv \sup_{t \leq \tau} |f_t|$.

Then:

$$L^2(\Omega) - \lim_{n \rightarrow \infty} \|X^n - X\|_{\infty, \tau} = 0,$$

and thus:

$$L_a^2([0, \tau]) - \lim_{n \rightarrow \infty} X^n = X.$$

Proof

Notation: We set $\|Z\|_2 = \mathbf{E}^{1/2}[Z^2]$ for a real-valued r.v Z .

Cauchy sequence: We will show that

$$\lim_{m,n \rightarrow \infty} \left\| \|X^n - X^m\|_{\infty, \tau} \right\|_2 = 0. \quad (3)$$

However:

$$\begin{aligned} \left\| \|X^n - X^m\|_{\infty, \tau} \right\|_2 &\leq \left\| \sum_{k=m}^{n-1} \|X^{k+1} - X^k\|_{\infty, \tau} \right\|_2 \\ &\leq \sum_{k=m}^{n-1} \left\| \|X^{k+1} - X^k\|_{\infty, \tau} \right\|_2 \leq \sum_{k=m}^{n-1} \Delta_k^{1/2}(\tau). \end{aligned}$$

This proves (3) and Lemma 13.

Existence of a solution to (1)

Lemma 14.

Let:

- X^n Picard iterations on $[0, \tau]$.
- X limit of X^n obtained by Lemma 12.

Then X is solution of equation (1).

Proof

Strategy: We set $\tilde{X} \equiv \Gamma(X)$. We will show that $\tilde{X} = X$.

Sufficient condition: We will see that:

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[\|\tilde{X} - X^{n+1}\|_{\infty, \tau}^2 \right] = 0.$$

Verification: Recall that

- $X^{n+1} = \Gamma(X^n)$.
- Lemmas 9 and 13.

This yields:

$$\mathbf{E} \left[\|\tilde{X} - X^{n+1}\|_{\infty, \tau}^2 \right] = \mathbf{E} \left[\|\Gamma(X) - \Gamma(X^n)\|_{\infty, \tau}^2 \right] \leq c \|X^n - X\|_{L_a^2}^2 \longrightarrow 0.$$

Gronwall's lemma

Lemma 15.

Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ continuous. We assume:

$$\varphi_t \leq c + d \int_0^t \varphi_s ds,$$

with two constants $c, d > 0$. Then we have:

$$\varphi_t \leq c \exp(dt).$$

Proof

Majorizing function: For $\varepsilon > 0$, we set $\psi_t = (c + \varepsilon) \exp(dt)$.

Comparison between φ and ψ : We assume $\tau < \infty$ with:

$$\tau = \inf \{t \geq 0; \varphi_t \geq \psi_t\}.$$

Then $\tau > 0$ and $\varphi_\tau = \psi_\tau$ because φ, ψ continuous.

Contradiction: We have

$$\begin{aligned}\psi_\tau &= c + \varepsilon + d \int_0^\tau \psi_s ds > c + d \int_0^\tau \varphi_s ds \\ &\geq c + d \int_0^\tau \varphi_s ds \geq \varphi_\tau.\end{aligned}$$

Therefore $\psi_\tau > \varphi_\tau$, contradiction with $\varphi_\tau = \psi_\tau$.

Pathwise uniqueness

Lemma 16.

We consider:

- Lipschitz coefficients b, σ .
- Space of processes $L^2(\Omega; \mathcal{C}([0, \tau]))$, characterized by:

$$\|Z\|_{L^2(\Omega; \mathcal{C}([0, \tau]))}^2 = \mathbf{E} \left[\sup_{t \leq \tau} |Z_t|^2 \right].$$

Then we have pathwise uniqueness for equation (1)
 $\hookrightarrow$ In $L^2(\Omega; \mathcal{C}([0, \tau]))$.

Proof

Aim: Let $(X^1, W, \mathcal{F}_t^1), (X^2, W, \mathcal{F}_t^2)$ two solutions
 $\hookrightarrow$ We wish to show that $X^1 = X^2$.

Filtrations: Let $\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2$

$\hookrightarrow$ We have X^1, X^2 adapted for $\mathcal{F}_t$

$\hookrightarrow$ Estimates for stochastic integrals can be applied to $X^1 - X^2$.

Application of Lemma 9:

We set $\varphi_t = \|X^1 - X^2\|_{L^2(\Omega; \mathcal{C}([0, \tau]))}^2$. Then:

$$\varphi_t = \|\tilde{X}^1 - \tilde{X}^2\|_{L^2(\Omega; \mathcal{C}([0, t]))}^2 \leq d \int_0^t \varphi_s ds,$$

with $d = c_{K, \tau, d}$. Therefore $\varphi \equiv 0$ and $X^1 = X^2$.

More existence and uniqueness results

Extensions:

We have existence and uniqueness for (1) in following situations:

- 1 Coefficients $b(s, x)$ and $\sigma(s, x)$ with uniform Lipschitz conditions:

$$|b(s, x) - b(s, y)| + |\sigma(s, x) - \sigma(s, y)| \leq c |x - y|.$$

- 2 Coefficients b, σ locally Lipschitz with linear growth:

$$\begin{aligned} |b(x) - b(y)| + |\sigma(x) - \sigma(y)| &\leq c_n |x - y|, \text{ for } |x|, |y| \leq n. \\ |b(x)| + |\sigma(x)| &\leq c(1 + |x|). \end{aligned}$$

- 3 Case $d = 1$ and

- ▶ b Lipschitz.
- ▶ σ Hölder-continuous Hölder exponent $\alpha \geq 1/2$.

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Definition of fBm

Complete probability space: $(\Omega, \mathcal{F}, \mathbf{P})$

Definition 17.

A 1-d fBm is a continuous process $B = \{B_t; t \geq 0\}$ such that $B_0 = 0$ and for $H \in (0, 1)$:

- B is a centered Gaussian process
- $\mathbf{E}[B_t B_s] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H})$

d -dimensional fBm: $B = (B^1, \dots, B^d)$, with B^i independent 1-d fBm

Variance of the increments:

$$\mathbf{E}[|\delta B_{st}^j|^2] \equiv \mathbf{E}[|B_t - B_s|^2] = |t - s|^{2H}$$

Kolmogorov criterion

Notation: If $f : [0, \tau] \rightarrow \mathbb{R}^d$ is a function, we shall denote:

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \|f\|_\mu = \sup_{s, t \in [0, \tau]} \frac{|\delta f_{st}|}{|t - s|^\mu}$$

Theorem 18.

Let $X = \{X_t; t \in [0, \tau]\}$ be a process defined on $(\Omega, \mathcal{F}, \mathbf{P})$, such that

$$\mathbf{E} [|\delta X_{st}|^\alpha] \leq c |t - s|^{1+\beta}, \quad \text{for } s, t \in [0, \tau], \quad c, \alpha, \beta > 0$$

Then there exists a modification $\hat{X}$ of X such that almost surely $\hat{X} \in \mathcal{C}_1^\gamma$ for any $\gamma < \beta/\alpha$, i.e. $\mathbf{P}(\omega; \|\hat{X}(\omega)\|_\gamma < \infty) = 1$.

FBm regularity

Proposition 19.

FBm $B \equiv B^H$ is γ -Hölder continuous for all $\gamma < H$, up to modification.

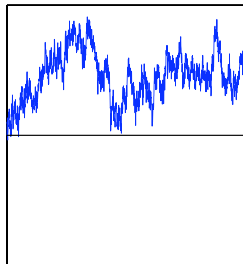
Proof: We have $\delta B_{st} \sim \mathcal{N}(0, |t - s|^{2H})$. Thus for $n \geq 1$,

$$\mathbf{E} \left[|\delta B_{st}|^{2n} \right] = c_n |t - s|^{2Hn} \quad \text{i.e.} \quad \mathbf{E} \left[|\delta B_{st}|^{2n} \right] = c_n |t - s|^{1+(2Hn-1)}$$

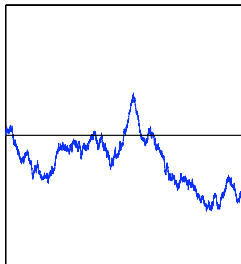
Kolmogorov: B is γ -Hölder for $\gamma < (2Hn - 1)/2n = H - 1/(2n)$.

Proof finished by letting $n \rightarrow \infty$.

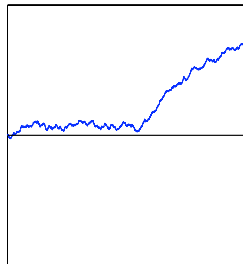
Examples of fBm paths



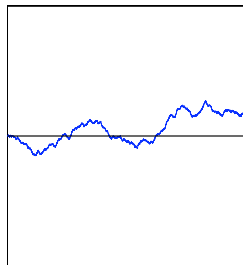
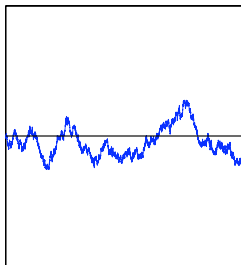
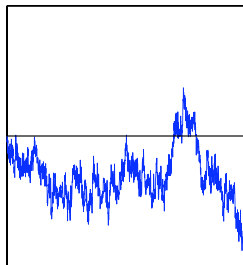
$H = 0.3$



$H = 0.5$



$H = 0.7$



Some properties of fBm

Proposition 20.

Let B be a fBm with parameter H . Then:

- ① $\{a^{-H}B_{at}; t \geq 0\}$ is a fBm (scaling)
- ② $\{B_{t+h} - B_h; t \geq 0\}$ is a fBm (stationarity of increments)
- ③ B is not a semi-martingale unless $H = 1/2$
And B is nowhere differentiable a.s

Proof of claim 3:

If B were a semi-martingale, we would get on $[0, 1]$:

$$\mathbf{P} - \lim_{n \rightarrow \infty} \sum_{i=1}^n (B_{i/n} - B_{(i-1)/n})^2 = \langle B \rangle_1,$$

where $\langle B \rangle$ is the (non trivial) quadratic variation of B .

We will show that $\langle B \rangle$ is trivial (0 or ∞) whenever $H \neq 1/2$.

Proof of claim 3 (2)

Define

$$V_{n,2} = \sum_{i=1}^n |B_{i/n} - B_{(i-1)/n}|^2, \quad \text{and} \quad Y_{n,2} = n^{2H-1} V_{n,2}.$$

By scaling properties, we have:

$$Y_{n,2} \stackrel{(d)}{=} \hat{Y}_{n,2}, \quad \text{with} \quad \hat{Y}_{n,p} = n^{-1} \sum_{i=1}^n |B_i - B_{i-1}|^2.$$

The sequence $\{B_i - B_{i-1}; i \geq 1\}$ is stationary and mixing
 $\Rightarrow \hat{Y}_{n,2}$ converges $\mathbf{P} - a.s$ and in L^1 towards $\mathbf{E}[|B_1 - B_0|^2]$
 $\Rightarrow \mathbf{P} - \lim_{n \rightarrow \infty} Y_{n,2} = E[|B_1|^2]$
 $\Rightarrow \mathbf{P} - \lim_{n \rightarrow \infty} V_{n,2} = 0$ if $2H > 1$, ∞ if $2H < 1$

Outline

- 1 Introduction and examples
- 2 Existence and uniqueness
- 3 Fractional Brownian motion
- 4 Young integration**
- 5 Young differential equations

Strategy for $H > 1/2$

- Generally speaking, take advantage of two aspects of fBm:
 - ▶ Gaussianity
 - ▶ Regularity

For $H > 1/2$, regularity is almost sufficient

- Notation: $\mathcal{C}_1^\gamma = \mathcal{C}_1^\gamma(\mathbb{R}) \equiv \gamma$ -Hölder functions of 1 variable
- If $H > 1/2$, $B \in \mathcal{C}_1^\gamma$ for any $1/2 < \gamma < H$ a.s
- We shall try to solve our equation in a pathwise manner

Pathwise strategy

Aim: Let x be a function in $\mathcal{C}_1^\gamma$ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s) dx_s \quad (4)$$

Steps:

- Define an integral $\int z_s dx_s$ for $z \in \mathcal{C}_1^\kappa$, with $\kappa + \gamma > 1$
- Solve (4) through fixed point argument in $\mathcal{C}_1^\kappa$ with $1/2 < \kappa < \gamma$

Remark: We treat a real case and $b \equiv 0$ for notational sake.

$\hookrightarrow$ Extensions to dimension d by adding indices.

Particular Riemann sums

Aim: Define $\int_0^1 z_s dx_s$ for $z \in \mathcal{C}_1^\kappa, x \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$

Dyadic partition: set $t_i^n = i/2^n$, for $n \geq 0, 0 \leq i \leq 2^n$

Associated Riemann sum:

$$I_n \equiv \sum_{i=0}^{2^n-1} z_{t_i^n} [x_{t_{i+1}^n} - x_{t_i^n}] = \sum_{i=0}^{2^n-1} z_{t_i^n} \delta x_{t_i^n t_{i+1}^n}.$$

Question: Can we define $\mathcal{J}_{01}(z dx) \equiv \lim_{n \rightarrow \infty} I_n$?

Possibility: Control $|I_{n+1} - I_n|$ and write (if the series is convergent):

$$\mathcal{J}_{01}(z dx) = I_0 + \sum_{n=0}^{\infty} (I_{n+1} - I_n).$$

Control of $l_{n+1} - l_n$

We have:

$$l_n = \sum_{i=0}^{2^n-1} z_{t_i^n} \delta x_{t_i^n t_{i+1}^n} = \sum_{i=0}^{2^n-1} z_{t_{2i}^{n+1}} \left[\delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right]$$

$$l_{n+1} = \sum_{i=0}^{2^{n+1}-1} \left[z_{t_{2i}^{n+1}} \delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + z_{t_{2i+1}^{n+1}} \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right]$$

Therefore:

$$\begin{aligned} |l_{n+1} - l_n| &= \left| \sum_{i=0}^{2^n-1} \delta z_{t_{2i}^{n+1} t_{2i+1}^{n+1}} \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}} \right| \\ &\leq \sum_{i=0}^{2^n-1} \|z\|_{\kappa} |t_{2i+1}^{n+1} - t_{2i}^{n+1}|^{\kappa} \|x\|_{\gamma} |t_{2i+2}^{n+1} - t_{2i+1}^{n+1}|^{\gamma} \\ &= \frac{\|z\|_{\kappa} \|x\|_{\gamma}}{2^{\kappa+\gamma} 2^{n(\kappa+\gamma-1)}} \end{aligned}$$

Definition of the integral

We have seen: for $\alpha \equiv \kappa + \gamma - 1 > 0$ and $n \geq 0$:

$$|I_{n+1} - I_n| \leq \frac{C_{x,z}}{2^{\alpha n}}$$

Series convergence:

Obviously, $\sum_{n=0}^{\infty} (I_{n+1} - I_n)$ is a convergent series

$\hookrightarrow$ yields definition of $\mathcal{J}_{01}(z dx)$, and more generally: $\mathcal{J}_{st}(z dx)$

Remark:

One should consider more general partitions π , with $|\pi| \rightarrow 0$

$\hookrightarrow$ C.f Lejay (Séminaire 37)

Young integral

Proposition 21.

Let

- $z \in \mathcal{C}_1^\kappa([0, \tau])$, $x \in \mathcal{C}_1^\gamma([0, \tau])$
- $\kappa + \gamma > 1$, and $0 \leq s < t \leq T$
- $(\pi_n)_{n \geq 0}$ a sequence of partitions of $[s, t]$ such that $\lim_{n \rightarrow \infty} |\pi_n| = 0$
- I_n corresponding Riemann sums

Then:

- 1 I_n converges to an element $\mathcal{I}_{st}(z \, dx)$
- 2 The limit does not depend on the sequence $(\pi^n)_{n \geq 0}$
- 3 Integral linear in z , and coincides with Riemann's integral for smooth z, x

A bound for Young integrals

Theorem 22.

Let $f \in \mathcal{C}_1^\kappa, g \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$. Then:

- 1 If $0 \leq s < u < t \leq T$, we have
$$\mathcal{J}_{st}(z \, dx) = \mathcal{J}_{su}(z \, dx) + \mathcal{J}_{ut}(z \, dx)$$
- 2 Generalized integral $\mathcal{J}(f \, dg)$ satisfies:

$$|\mathcal{J}_{st}(f \, dg)| \leq \|f\|_\infty \|g\|_\gamma |t - s|^\gamma + c_{\gamma, \kappa} \|f\|_\kappa \|g\|_\gamma |t - s|^{\gamma + \kappa}.$$

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Pathwise strategy (repeated)

Aim: Let x be a function in $\mathcal{C}_1^\gamma$ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s) dx_s \quad (5)$$

Steps:

- Define an integral $\int z_s dx_s$ for $z \in \mathcal{C}_1^\kappa$, with $\kappa + \gamma > 1$
- Solve (5) through fixed point argument in $\mathcal{C}_1^\kappa$ with $1/2 < \kappa < \gamma$

Remark: We treat a real case and $b \equiv 0$ for notational sake.

Existence-uniqueness result

Theorem 23.

Consider

- Noise: $x \in \mathcal{C}_1^\gamma \equiv \mathcal{C}_1^\gamma([0, \tau])$, with $\gamma > 1/2$
- Coefficient: $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ a C_b^2 function
- Equation: $\delta y = \mathcal{I}(\sigma(y) dx)$, and $y_0 = a$.

Then:

- 1 Our equation admits a unique solution y in $\mathcal{C}_1^\kappa$ for any $1/2 < \kappa < \gamma$.
- 2 Application $(a, x) \mapsto y$ is continuous from $\mathbb{R} \times \mathcal{C}_1^\gamma$ to $\mathcal{C}_1^\kappa$.

Fixed point: strategy

A map on a small interval:

Consider an interval $[0, \tau]$, with τ to be determined later

Consider κ such that $1/2 < \kappa < \gamma < 1$

In this interval, consider $\Gamma : \mathcal{C}_1^\kappa([0, \tau]) \rightarrow \mathcal{C}_1^\kappa([0, \tau])$ defined by:
 $\Gamma(z) = \hat{z}$, with $\hat{z}_0 = a$, and for $s, t \in [0, \tau]$:

$$\delta \hat{z}_{st} = \int_s^t \sigma(z_r) dx_r = \mathcal{J}_{st}(\sigma(z) dx)$$

Aim: See that for a small enough τ , the map Γ is a contraction
 $\hookrightarrow$ our equation admits a unique solution in $\mathcal{C}_1^\kappa([0, \tau])$