1. Let $A = \begin{bmatrix} 0 & 4 & 8 & 4 \\ 1 & 1 & -5 & 2 \\ 1 & 2 & -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & -2 & 4 \\ 2 & 2 & 0 & 2 \\ 3 & 1 & 6 & 1 \end{bmatrix}$. Let $C = B^T A$ and c_{ij} denote the entries of C.
   What is c_{21}?

   Answer: $c_{21} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 3.$

2. (5 pts) Given the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 2 & 2 \\ 3 & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

   By **Cramer's rule**, $x_2 = \frac{|A_2|}{|A|}$ if $|A| \neq 0$. What is A_2?

   Answer: $A_2 = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$.

3. (20 pts) Given four vectors $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$,
   is $\mathbf{u}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$? If yes, find all possible coefficients.

   Answer: If $\mathbf{u}$ a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, then there are numbers x_1, x_2 and x_3 such that $\mathbf{u} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$. So the answer will be affirmative if the following linear system has at least one solution:

$$\begin{bmatrix} -1 & -1 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

   Solve the linear system above by Gaussian elimination:

$$\left[\begin{array}{ccc|c} -1 & -1 & 2 & 2 \\ 0 & -1 & 1 & 1 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{-r_1 \to r_1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & -1 & 1 & 1 \\ 2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{-2r_1 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 3 & 4 \end{array}\right]$$

$$\xrightarrow{-r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 3 & 4 \end{array}\right] \xrightarrow{2r_2 + r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

By back substitution, we get $x_3 = 2$, $x_2 = 1$ and $x_1 = 1$.

4. (20 pts)

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -3 \end{bmatrix}.$$

(a) Evaluate $|A|$.

(b) Find the inverse of the 3×3 submatrix obtained after deleting the first column and the second row.

Answer:

(a) To compute the det, we do row operations to create more zeros then use cofactor expansion:

$$\begin{aligned}
(\text{row operation} \quad r_3 - r_2 \to r_3) \qquad |A| &= \begin{vmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \end{vmatrix} \\
(\text{cofactor expansion using the first column}) &= (-1) * (-1)^{2+1} \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{vmatrix} \\
(\text{Compute the det directly}) &= 1 * (-1) * (-3) - 1 * (-1) * 1 = 4.
\end{aligned}$$

(b) The 3×3 submatrix obtained after deleting the first column and the second row is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$. To find the inverse:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-r_1 + r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & -1 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{4}r_3 \to r_3]{\frac{1}{2}r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \end{array}\right]$$

$$\xrightarrow{r_1 - r_3 \to r_1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \end{array}\right].$$

So the inverse is $\begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix}$.

5. (20 pts) Consider $A = \begin{bmatrix} 3 & 1 & 2 \\ x & -1 & 2 \\ 0 & 1 & x-2 \end{bmatrix}$.

(a) Find all possible values of x such that A is singular.

(b) Let b_{ij} denote the entries of the adjoint $\text{adj}(A)$. Namely,

$$\text{adj}A = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Find all possible values of x such that $b_{21} = 1$.

Answer:

(a) A being Singular means $|A|$ should be zero. Calculate the det

$|A| = 3 * (-1) * (x-2) + x * 1 * 2 - 3 * 1 * 2 - 1 * x * (x-2) = -x^2 + x$. So $|A| = 0$ implies $-x^2 + x = 0$ thus $x = 0 \quad \text{or} \quad 1$.

(b) By the definition of the adjoint, b_{21} is the cofactor of a_{12}, which is equal to

$$(-1)^{1+2}\begin{vmatrix} x & 2 \\ 0 & x-2 \end{vmatrix} = -x(x-2).$$ So $-x(x-2) = 1$ implies $x^2 - 2x + 1 = 0$ thus $x = 1$.

6. (30 pts) Answer only TRUE or FALSE in the table for the following statements:

a. $\begin{bmatrix} 1 & 4 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is in row echelon form but not reduced row echelon form.

False: it is in reduced row echelon form.

b. By cofactor expansion, $\begin{vmatrix} \cos x & \sin^2 x & \cos^4 x & -\sin^4 x \\ -\cos^2 x & \sin^4 x & -\sin^3 x & 0 \\ \sin^3 x & \cos^3 x & 0 & 0 \\ \cos^4 x & 0 & 0 & 0 \end{vmatrix} = -\sin^7 x \cos^7 x.$

False: it should be $\sin^7 x \cos^7 x$.

c. Let A be a 4×7 matrix , then $A\mathbf{x} = \mathbf{0}$ has either a unique solution or infinitely many solutions. It is impossible for $A\mathbf{x} = \mathbf{0}$ to have three and only three solutions.

True: if the homogeneous system has more than one solution, then it has a nonzero solution $\mathbf{x_0}$. For any real number c, $c\mathbf{x_0}$ is also a solution. Thus the homogeneous system has more than one solution, it has infinitely many solutions.

d. Let A be a 20×17 matrix. Then AA^T is well defined and it is a symmetric matrix.

True: by socks-shoes rule, $(AA^T)^T = (A^T)^T A^T = AA^T$.

e. Let A be a nonsingular matrix, then $A^T A$ is also nonsingular and it has a positive determinant.

True: if $|A| \neq 0$, then $|A^T A| = |A^T||A| = |A|^2 > 0$.

f. If A and B are row equivalent, then they must have the same null spaces.

True: row eqwuivalence implies $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions.

g. For two generic $n \times n$ invertible matrices A and B, $\det(ABA^{-1})$ is not equal to $\det(B)$.

False: $|ABA^{-1}| = |A||B||A^{-1}| = |B||A||A^{-1}| = |B|$.

h. The set of all 3-vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $2a - b + c = 0$ is a subspace of $\mathbb{R}^3$.

True.

i. For a nonsingular matrix A, $\text{adj}(A)$ may be singular.

False: if A is nonsingular, then $A^{-1} = \frac{1}{|A|}\text{adj}(A)$ thus $\text{adj}(A) = |A|A^{-1}$ is invertible, and its inverse is $\frac{1}{|A|}A$.

j. If A is a singular matrix, then the nonhomogeneous system $\mathbf{Ax} = \mathbf{b}$ has infinitely many solutions.

False: if A is a singular matrix, the homogeneous system has infinitely many solutions but the nonhomogeneous system may have no solutions.

7. (**Bonus Problem**, 15 pts) Prove the following.

(a) Let A be a 101×101 matrix. If $A^{101} = -A$, then A must be singular.

Proof: $A^{101} = -A \Rightarrow |A^{101}| = |-A|$. First, $|A^{101}| = |A|^{101}$. Second, $|-A| = |-IA| = |-I||A| = (-1)^{101}|A| = -|A|$. Therefore, $|A|^{101} = -|A| \Rightarrow |A|^{101} + |A| = 0$ thus $|A|(|A|^{100} + 1) = 0$. So $|A| = 0$.

(b) If an upper triangular matrix A is invertible, then its inverse is also upper triangular.

Proof: First of all, the determinant of an upper triangular matrix A is the product of diagonal entries. Thus $|A| \neq 0$ implies all diagonal entries are nonzero. Then to obtain A^{-1}, we do row operations to $[A|I]$. Since A is upper triangular and diagonal entries are nonzero, all we need to do in the row operations is to normalize each row so that the diagonal entries are 1 then zero the entries above the diagonal. Such row operations will not change entries of I below the diagonal, thus the result will always be upper triangular.

One may also use the formula $A^{-1} = \text{adj}(A)/|A|$ and show that $\text{adj}(A)$ is upper triangular.