FREE DERIVATION MODULES ON ALGEBRAIC VARIETIES.

By Joseph Lipman.[1,2]

Introduction. The Jacobian criterion for simple points may be formulated in the following way [5; § 3]:

Let P be a point on an algebraic variety V/k over a ground field k, which we assume, for simplicity, to be perfect. Let R be the local ring of P on V, and let $D = D_k(R)$ be the module of k-differentials of R. Then in order that P be a simple point of V, it is necessary and sufficient that D be a free R-module.

$D^* = \mathrm{Hom}_R(D, R)$, the dual module of D, may be identified with the module of k-derivations of R into itself. If P is simple on V, so that D is free, then of course D^* is free. It is tempting to ask for the converse: *If D^* is free, is P simple?*

The answer is in the negative when k has characteristic $p \neq 0$, even under the additional assumption that P is normal, a counterexample being given by the origin on the surface $Z^p = XY$ (cf. § 7).

In characteristic zero, however, the question remains open, even when V is a surface in 3-space. By way of encouragement we have an affirmative answer in some special situations, for example when P is the vertex of a cone, or when P is the origin on a surface whose equation is of the form $Z^n = f(X, Y)$ with $f(0,0) = 0$. (cf. § 7).

Our purpose will be to study, for its own sake, the condition that D^* be free. Although we cannot answer the above question, we can still develop some results which may prove useful toward that end. Assuming that D^* is free, we show in § 3 that when k has characteristic zero, P is a normal point. (Thus if V is a curve, P is simple). In § 5 we give some upper bounds on the codimension of the singular locus in the neighborhood of P. We give a technical criterion for determining whether D^* is free in § 6, and apply this criterion in § 7 to a number of specific examples. In an appendix (§ 8)

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we give a simple characterization, in terms of the local codimension of the singular locus, of those points on a complete intersection whose module of differentials is torsion free or reflexive. Some of the techniques used in the proofs have independent interest (cf. § 4, § 6).

In view of the fact that the module of differentials of R has been defined on certain occasions to be D^{**}, the dual of the module of derivations, one might also ask under what conditions D^{**} is free. However, as long as R is a *reduced* ring, it can be seen that $D^* \cong D^{***}$, from which it follows that D^{**} is free if and only if D^* is free.

We make the convention now that the word "ring" shall mean "non-null commutative ring with identity," and that all modules shall be unitary. We use the terms "finitely generated" and "of finite type" interchangeably. The phrase "V/k is an affine variety" shall mean "k is a field, and $V = \operatorname{Spec} S$ where S is a reduced k-algebra of finite type."

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1. Generalities.

Let V/k be an affine variety over a *perfect* ground field k; thus $V = \operatorname{Spec} S$, where S is a finitely generated k-algebra without nonzero nilpotent elements.

For any k-algebra A, let $D(A)$ be the module of k-differentials of A and let $D^*(A) = \operatorname{Hom}_A(D(A), A)$ be the module of k-derivations of A into itself. (For the definition and properties of differential modules see [**5**; § 1]). If $\mathfrak{p}$ is a prime ideal in S, then $D(S_\mathfrak{p})$ may be identified with the localization $[D(S)]_\mathfrak{p} = D(S) \otimes_S S_\mathfrak{p}$. We recall that if A is any ring, if M and N are two A-modules, and if B is a *flat* A-algebra, then the canonical homomorphism

$$\operatorname{Hom}_A(M, N) \otimes_A B \to \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B)$$

is a monomorphism if M is of finite type, and an isomorphism if M is of finite presentation [**3**; p. 39]. $D(S)$ is of finite presentation and $S_\mathfrak{p}$ is flat over S; we may therefore identify $[D^*(S)]_\mathfrak{p}$ with $D^*(S_\mathfrak{p})$.

If P is a point of V, and $\mathfrak{p}$ is the corresponding prime ideal in S, then we say "D^* is free at P" when $D^*(S_\mathfrak{p})$ is a free $S_\mathfrak{p}$-module.

PROPOSITION 1.1. *The points of V at which D^* is free form a dense open subset of V.*

Proof. The set of points where D^* is free contains the generic point of any irreducible component of V, since $S_{\mathfrak{q}}$ is a field when $\mathfrak{q}$ is a minimal prime ideal of S. Thus the proposition is a consequence of the general fact that for any ring A, and any A-module N of finite presentation, the prime ideals $\mathfrak{p}$ such that $N_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module form an open subset of $\operatorname{Spec} A$ (cf. [**3**; p. 137]). q. e. d.

From now on, we will be concerned only with the purely local properties of P which follow from the assumption that D^* is free at P.

PROPOSITION 1.2. *If D^* is free of rank r at P, then every component of V through P has dimension r.*

Proof. Let R $(=S_{\mathfrak{p}})$ be the local ring of P on V. If $\mathfrak{q}$ is a minimal prime ideal of R then $R_{\mathfrak{q}}$ is the function field of a component of V through P, and all such components are accounted for in this way. Moreover $D^*(R_{\mathfrak{q}}) = D^*(R) \otimes_R R_{\mathfrak{q}}$ is a vector space of dimension r over $R_{\mathfrak{q}}$. Since $R_{\mathfrak{q}}$ is a *separable* extension of k (k is perfect), $R_{\mathfrak{q}}$ has transcendence degree r over k. q. e. d.

Again let R be the local ring of P on V. Whether $D^*(R)$ is free or not depends only on the completion of R:

PROPOSITION 1.3. *Let R' be the completion of R. Then $D^*(R')$ is the completion of $D^*(R)$, and consequently $D^*(R)$ is a free R-module if and only if $D^*(R')$ is a free R'-module.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R'. Let $D' = D(R')/\bigcap_n \mathfrak{m}^n D(R')$. It is known that D' is the completion of $D(R)$, i. e. $D' = D(R) \otimes_R R'$, and that $D^*(R') \cong \operatorname{Hom}_{R'}(D', R')$ [**2**; §§ 2, 3]. (Note: these observations depend on the fact that $D(R)$ is a finitely generated R-module.)

Since R' is a flat R-module, and $D(R)$ is of finite presentation, we have, as above, the isomorphism

$$D^*(R) \otimes_R R' = \operatorname{Hom}_R(D(R), R) \otimes_R R' \cong \operatorname{Hom}_{R'}(D(R) \otimes_R R', R') \cong D^*(R')$$

and the first assertion is proved. Since R and R' are local rings, and R' is a faithfully flat R-algebra, the second assertion is a special case of the following proposition [**3**; p. 53]:

Let A be a ring, and let B be a faithfully flat A-algebra. An A-module F is projective of finite type if and only if the B-module $F \otimes_A B$ is projective of finite type.

(We recall that for modules over local rings, "projective of finite type" means "free of finite type." [**3**; p. 107].)

2. Depth and the conductor; grade and duality. We now give some preliminary results, to be applied in the next section.

Let A be a noetherian ring. A sequence $\{a_1, a_2, \cdots, a_t\}$ of elements in A is a *prime sequence* (of length t) if a_1 is not a zerodivisor in A, and if, for each $i = 2, 3, \cdots, t$, a_i is not a zerodivisor in $A/Aa_1 + Aa_2 + \cdots + Aa_{i-1}$. The *depth* (or *grade*) of a proper ideal $I < A$ is the length of a maximal prime sequence consisting of elements in I, the lengths of any two such sequences being equal [7; Thm. 1.3]. For convenience, the ideal A is said to have depth ∞.

Let $\bar{A}$ be the integral closure of A in its total quotient ring. Let $\mathfrak{C}$ be the *conductor* of A in $\bar{A}$, i.e. the annihilator ideal of the A-module $\bar{A}/A$.

PROPOSITION 2.1. *Let I be an ideal in A such that $Aa : I = Aa$ for every nonzerodivisor a in A. Then $\mathfrak{C} : I = \mathfrak{C}$, and consequently if $\mathfrak{C} \neq A$ then every associated prime ideal of $\mathfrak{C}$ has depth $\leqq 1$.*

Proof. Let $c \in A$ be such that $cI \subseteq \mathfrak{C}$. If $x \in \bar{A}$, then $x = b/a$ ($b, a \in A$, and a is a nonzerodivisor) and $cI \cdot b/a \subseteq A$, i.e. $cb \cdot I \subseteq Aa$. Hence

$$cb \in Aa : I = Aa$$

and $c \cdot b/a \in A$. Thus $c\bar{A} \subseteq A$, i.e. $c \in \mathfrak{C}$ and the first assertion is proved.

If I is an associated prime ideal of $\mathfrak{C}$, then $\mathfrak{C} : I \neq \mathfrak{C}$, so that for some nonzerodivisor a, $Aa : I \neq Aa$, i.e. I is contained in an associated prime ideal $\mathfrak{p}$ of Aa. Since depth $\mathfrak{p} \leqq 1$, the same is true of I. q.e.d.

There is an interesting geometric consequence:

COROLLARY. *Let P be a point of an affine variety V/k, and let A be the local ring of P on V. Then P is a normal point of V (i.e. A is an integrally closed domain) if and only if the singular locus of V is of depth $\geqq 2$ locally at P (i.e. every prime ideal $\mathfrak{p}$ in A such that $A_\mathfrak{p}$ is not regular has depth $\geqq 2$).*

Proof. If A is an integrally closed domain, then every prime ideal $\mathfrak{p}$ of depth 1 is of height 1, and $A_\mathfrak{p}$ is regular for every such prime ideal.

Conversely, if $\mathfrak{p}$ is a prime ideal in A containing $\mathfrak{C}$, then $A_\mathfrak{p}$ is not integrally closed in its total quotient ring [1; p. 506], and so $A_\mathfrak{p}$ is not regular. Therefore, if the singular locus is locally of depth $\geqq 2$, then every prime ideal containing $\mathfrak{C}$ has depth $\geqq 2$, whence, by the proposition, $\mathfrak{C} = A$, so that A is integrally closed in its total quotient ring. Since A is a *reduced* local ring, A is an integral domain. (Otherwise, there is an idempotent e in the total quotient ring of A, $e \neq 0$, $e \neq 1$, and e is necessarily in A since e satisfies a relation of integral dependence: $e^2 - e = 0$. Since e and $e-1$ are zero-divisors in A, they are both nonunits in A, which is impossible). q. e. d.

In particular, if A is a Macaulay ring (for example if V is a complete intersection locally at P), then P is normal if and only if no singular subvariety of V of codimension 1 passes through P. For, A being a Macaulay local ring, the depth of a prime ideal $\mathfrak{p}$ in A is the same as the height of $\mathfrak{p}$, i. e. as the codimension of the subvariety of V corresponding to $\mathfrak{p}$.

* * *

Let A be a ring, M an A-module, and let "$*$" denote the functor "dual" so that $M^* = \mathrm{Hom}_A(M, A)$. Let $M^{**} = (M^*)^*$ be the *bidual* of M.

The canonical bilinear map of $M^* \times M$ into A taking the pair (ω, x) into $\omega(x)$ defines a natural homomorphism $f: M \to M^{**}$. The "naturality" of f means, explicitly, that for any homomorphism of modules $h: M \to N$, the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f_M} & M^{**} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^{**}} \\ N & \xrightarrow{f_N} & N^{**} \end{array}$$

If f is an isomorphism, we say that M is *reflexive*. Any finitely generated free module is reflexive; a direct summand of a reflexive module is reflexive; hence, any finitely generated projective module is reflexive.

The kernel of f consists of those elements of M which are annihilated by every A-homomorphism of M into A. It follows that if M is a submodule of a free module, in particular if M is a projective module, then f is a monomorphism. It also follows that the dual map of $M \to f(M)$ is an isomorphism of $[f(M)]^*$ onto M^*, so that these two modules may be identified.

We denote the cokernel of f by M^{**}/M, even if f is not injective. Thus we have an exact sequence

$$0 \to f(M) \to M^{**} \to M^{**}/M \to 0$$

Applying "$*$" and taking note of the preceding identification we get the exact sequence

$$\begin{aligned} 0 \to (M^{**}/M)^* \to (M^{**})^* \xrightarrow{f^*} M^* \to \\ \to \mathrm{Ext}^1(M^{**}/M, A) \xrightarrow{h} \mathrm{Ext}^1(M^{**}, A) \end{aligned} \qquad (1)$$

which we will use in a moment.

Suppose now that A is noetherian and that M is finitely generated. If I is the annihilator of M, then the depth of I is called the *grade* of M. We check that grade $M = \min\{\text{depth}\,\mathfrak{p} \mid \mathfrak{p} \in \text{Supp}\, M\}$. (Recall that Supp M is the set of prime ideals $\mathfrak{p}$ in A such that $M_\mathfrak{p} \neq 0$, or, equivalently, the set of prime ideals in A which contain the annihilator of M). It is known that the grade of M is the least integer i such that $\text{Ext}^i(M, A) \neq 0$ (unless $M = 0$, in which case grade $M = \infty$) (cf. [7]).

PROPOSITION 2.2. *Let A be a ring, and let M be an A-module such that M^* is free of finite type. Then* $\text{Ext}^0(M^{**}/M, A) = \text{Ext}^1(M^{**}/M, A) = 0$. *Thus, if A is noetherian* $\text{grade}(M^{**}/M) \geqq 2$.

Proof. Since M^* is free of finite type, so is M^{**}, whence $\text{Ext}^1(M^{**}, A) = 0$. Moreover, M^* is reflexive, i.e. the canonical map $g: M^* \to (M^*)^{**} = (M^{**})^*$ is an isomorphism. f^* being as in the above sequence (1), we verify easily that $(f^* \circ g)$ is the identity map of M^*, so that f^* is the inverse of g.

Thus f^* is an isomorphism, and the exactness of (1) shows then that

$$\text{Ext}^0(M^{**}/M, A) = (M^{**}/M)^* = 0$$

and that h is injective, so that

$$\text{Ext}^1(M^{**}/M, A) \subseteq \text{Ext}^1(M^{**}, A) = 0.$$

3. Free D^* and normality.

THEOREM 1. *Let V/k be an affine variety over a field k of characteristic zero, and let P be a point of V. If D^* is free at P, then P is a normal point.*

Remark. The theorem is false when k has characteristic $p \neq 0$, a counter-example being the origin on the plane curve $X^p = Y^{p+1}$ (cf. § 7).

We shall prove Theorem 1 by showing that if R is the local ring of P on V, then *the singular locus of R is* $\text{Supp}(D^{**}(R)/D(R))$. In other words, for any prime ideal $\mathfrak{p}$ in R, $R_\mathfrak{p}$ is not regular if and only if $\mathfrak{p}$ contains the annihilator of $D^{**}(R)/D(R)$. Then Proposition 2.2 shows that the singular locus of R has depth $\geqq 2$, and this being so, the corollary of Proposition 2.1 shows that P is normal.

We write "D" in the place of "$D(R)$." Let $f: D \to D^{**}$ be the canonical map. We have noted in § 1 that we may identify the modules $(D^*)_\mathfrak{p}$ and $(D_\mathfrak{p})^*$, and similarly we may identify $(D^{**})_\mathfrak{p}$ with $(D_\mathfrak{p})^{**}$. When this is done, the "extension" $f_\mathfrak{p}: D_\mathfrak{p} \to (D^{**})_\mathfrak{p}$ of f to $D_\mathfrak{p}$ is easily seen to be the canonical map of $D_{\mathfrak{p}}$ into its bidual $(D_{\mathfrak{p}})^{**}$. To say that $\mathfrak{p} \notin \mathrm{Supp}(D^{**}/D)$ is to say that $(D^{**}/D)_{\mathfrak{p}} = 0$, i. e. that $D_{\mathfrak{p}}^{**}/D_{\mathfrak{p}} = 0$, i. e., that $f_{\mathfrak{p}}$ is surjective. If $R_{\mathfrak{p}}$ is regular, then by the Jacobian criterion, $D_{\mathfrak{p}} = D(R_{\mathfrak{p}})$ is free, and $f_{\mathfrak{p}}$ is certainly surjective.

Conversely, let $dx_1, dx_2, \cdots, dx_t$ generate $D_{\mathfrak{p}}$ ($x_1, x_2, \cdots, x_t \in R_{\mathfrak{p}}$; d is the universal derivation of $R_{\mathfrak{p}}$ into $D(R_{\mathfrak{p}}) = D_{\mathfrak{p}}$). Since D^* is free, by assumption, both $D_{\mathfrak{p}}^*$ and $D_{\mathfrak{p}}^{**}$ are free (over $R_{\mathfrak{p}}$). Since we are dealing with modules over a local ring, $f_{\mathfrak{p}}$ is surjective only if there is a free basis of $D_{\mathfrak{p}}^{**}$ among the elements $f_{\mathfrak{p}}(dx_i)$, $i = 1, 2, \cdots, t$. If $d_1, d_2, \cdots, d_r$ are derivations forming a free basis of $D_{\mathfrak{p}}^*$, then $\alpha_1, \alpha_2, \cdots, \alpha_\nu$ form a free basis of $D_{\mathfrak{p}}^{**}$ only if $\nu = r$ and the matrix $(\alpha_j \cdot d_i)$ is invertible (over $R_{\mathfrak{p}}$). Noting that $f_{\mathfrak{p}}(dx_j) \cdot d_i = d_i x_j$, we see finally that $f_{\mathfrak{p}}$ *is surjective only if there exist elements* $x_1, x_2, \cdots, x_r$ *in* $R_{\mathfrak{p}}$, *and derivations* $d_1, d_2, \cdots, d_r$ *of* $R_{\mathfrak{p}}$ *into itself such that the matrix* $(d_i x_j)$ *is invertible over* $R_{\mathfrak{p}}$. (Here $r = \mathrm{rank}\, D^*$, which, by Proposition 1.2, is the dimension of any component of V through P).

Thus, it will be sufficient to show that the existence of such elements and derivations implies that $R_{\mathfrak{p}}$ is regular. This has been done by Nagata in the case when $R_{\mathfrak{p}}$ is an integral domain [**6**; p. 796]. Presumably, a slight modificacation of his reasoning would take care of the possibility that $R_{\mathfrak{p}}$ has zero-divisors and thereby finish the proof of Theorem 1.

However, for completeness, and for its own interest, we shall present in the next section a somewhat generalized version, due essentially to Zariski, [**9**; p. 526], of Nagata's theorem. The regularity of $R_{\mathfrak{p}}$ will be a special case of that version, in view of the following observation (which is included in Nagata's proof):

If $f_{\mathfrak{p}}$ *is surjective, and* $R_{\mathfrak{p}}$ *has Krull dimension s, then there are* nonunits $y_1, y_2, \cdots, y_s$ *in* $R_{\mathfrak{p}}$, *and derivations* $d_1', d_2', \cdots, d_s'$ *of* $R_{\mathfrak{p}}$ *into itself such that* $(d_i' y_j)$ *is a unit matrix.*

Proof. Supposing $f_{\mathfrak{p}}$ to be surjective, let $(d_i x_j)$ be as above, and let (b_{ij}) be the inverse matrix. Replacing d_i by $\sum_w b_{iw} d_w$ we may assume that $d_i x_j = \delta_{ij}$ (Kronecker δ). Then it is easily seen that $x_1, x_2, \cdots, x_r$ are algebraically independent over k.

Let $\mathfrak{q} = \mathfrak{p} R_{\mathfrak{p}} \cap k[x_1, x_2, \cdots, x_r]$. Then $R_{\mathfrak{p}}$ dominates the regular local ring $T = k[x_1, x_2, \cdots, x_r]_{\mathfrak{q}}$. $D^*(T)$ is generated by the restrictions of the d_i, so that any derivation of T into T extends to one of $R_{\mathfrak{p}}$ into $R_{\mathfrak{p}}$. The above remark about r shows that

$$r = s + \mathrm{tr.\,deg}_k R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \geqq s + \mathrm{tr.\,deg}_k T/\mathfrak{q}T$$

Since $r = \dim T + \text{tr.}\deg_k T/\mathfrak{q}T$, we have $s \leqq \dim T$. If $y_1, y_2, \cdots, y_s$ form part of a regular system of parameters of T then there are derivations $d_1', d_2', \cdots, d_s'$ of T into T such that $(d_i' y_j)$ is the unit $s \times s$ matrix. Extending the d_i' to derivations of $R_\mathfrak{p}$ into $R_\mathfrak{p}$, we complete the proof.

4. Derivations with invertible determinants.

THEOREM 2. *Let A be a ring containing a field of characteristic zero, and let $\mathfrak{m}$ be an ideal in A such that A is a complete Hausdorff space in its $\mathfrak{m}$-adic topology. Suppose there exist derivations $d_1, d_2, \cdots, d_s$ of A into A and elements $x_1, x_2, \cdots, x_s$ in $\mathfrak{m}$ such that the matrix $(d_i x_j)$ is invertible.*

Then there is a subring B of A such that

1) $x_1, x_2, \cdots, x_s$ are analytically independent over B.

2) A is equal to the power series ring $B[[x_1, x_2, \cdots, x_s]]$.

3) B contains the subring A_d of A on which all the d_i vanish (i.e. $A_d = \{z \in A \mid d_i z = 0 \text{ for all } i\}$).

Proof. Let (b_{ij}) be the inverse matrix of $(d_i x_j)$; replacing d_i by $\bar{d}_i = \sum_w b_{iw} d_w$ affects neither the hypotheses nor the conclusions of the theorem, and $\bar{d}_i x_j = \delta_{ij}$; we may therefore assume from the outset that $(d_i x_j)$ is the unit matrix.

As a first step, let B_1 be the subring of A on which d_1 vanishes, and set $d = d_1$, $x = x_1$. Thus $dx = 1$ and $x_2, x_3, \cdots, x_s \in B_1$. For any element y in A set

$$y^\# = y - x dy + (x^2/2!) d^2 y - (x^3/3!) d^3 y + \cdots.$$

Then $y \to y^\#$ is a ring homomorphism: the identity $(y+z)^\# = y^\# + z^\#$ is obvious, and the identity $(yz)^\# = y^\# z^\#$ is a direct consequence of Leibnitz' rule for repeated differentiation of a product.

If $y^\# = 0$, then $y \in Ax$; moreover $x^\# = 0$; thus the kernel of # is the principal ideal Ax. We also note that $d(y^\#) = 0$, so that $y^\# \in B_1$, and that # restricts to the identity on B_1; hence B_1 is the image of #.

Since $(y - y^\#) \in Ax$, and since $x \in \mathfrak{m}$, we see that any element z in A can be written in the form

$$z = z_0 + z_1 x + z_2 x^2 + \cdots \qquad \text{with } z_i \in B_1$$

(for, $z = z^\# + w_1 x = z^\# + (w_1^\# + w_2 x) x = z^\# + w_1^\# x + w_2 x^2$
$= z^\# + w_1^\# x + w_2^\# x^2 + w_3 x^3$ etc. etc.)

Applying d^n, we find that $(d^n z - n!z_n) \in Ax$. Hence

$$0 = (d^n z - n!z_n)^{\#} = (d^n z)^{\#} - (n!z_n)^{\#} = (d^n z)^{\#} - n!z_n$$

so that $z_n = (1/n!)(d^n z)^{\#}$ (cf. Taylor's theorem!).

In particular, if $z = 0$, we have $z_0 = z_1 = z_2 = \cdots = 0$, i.e. x is analytically independent over B_1. Thus $A = B_1[[x]]$.

Let $\mathfrak{m}^{\#}$ be the image of $\mathfrak{m}$ under $\#$. Then $\mathfrak{m}^{\#} \subseteq \mathfrak{m} + Ax + \mathfrak{m}$. Thus $\mathfrak{m}^{\#} = \mathfrak{m} \cap B_1$.

Since $B_1 = A/Ax$ contains a field of characteristic zero, and since B_1 is a complete Hausdorff space in its $\mathfrak{m}^{\#} = \mathfrak{m} \cap B_1$ topology, we can easily complete the proof by induction, after noting that the restrictoin of $(\# \circ d_i)$ to B_1 is a derivation of B_1 into B_1 $(i = 2, 3, \cdots, s)$; that $x_j \in \mathfrak{m} \cap B_1$ $(j = 2, 3, \cdots, s)$; that the matrix $((\# \circ d_i)x_j)$ is the unit $(s-1) \times (s-1)$ matrix; and finally that if $d_i z = 0$ $(i = 1, 2, \cdots, s)$ then z is in B_1 and $(\# \circ d_i)z = 0$ $(i = 2, 3, \cdots, s)$. q.e.d.

Let R be a noetherian local ring with residue field of characteristic zero, let R' be the completion of R, and let k be a subring of R. Suppose there exist k-derivations $d_1, d_2, \cdots, d_s$ of R into R and nonunits $x_1, x_2, \cdots, x_s$ in R such that the matrix $(d_i x_j)$ is invertible. Let

$$\bar{R} = R/Rx_1 + Rx_2 + \cdots + Rx_s,$$

and let $\bar{R}'$ be the completion of $\bar{R}$, so that there is a canonical diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow h \\ \bar{R} & \longrightarrow & \bar{R}' \end{array}$$

where the horizontal arrows represent inclusion maps.

Corollary 1. *Under the preceding circumstances there is a map $g: \bar{R}' \to R'$ such that hg is the identity map of $\bar{R}'$ (hence g is injective), and such that when $\bar{R}'$ is identified with $g(\bar{R}')$, we have*

1) $k \subseteq \bar{R}$.

2) *$x_1, x_2, \cdots, x_s$ are analytically independent over $\bar{R}'$.*

3) *R' is the power series ring $\bar{R}'[[x_1, x_2, \cdots, x_s]]$.*

Proof. The d_i, being uniformly continuous, may be extended to derivations of R' into R'. Thus, by the theorem, $R' = B[[x_1, x_2, \cdots, x_s]]$, with $k \subseteq B$. If π is the projection of R' onto B, then $g = \pi h^{-1}$ is easily seen to be a well-defined isomorphism of $\bar{R}'$ onto B having the required properties (as a map of $\bar{R}'$ into R'). q. e. d.

R and B being as above, we note that if R has Krull dimension s, and if R is analytically unramified, then $B = R'/R'x_1 + R'x_2 + \cdots + R'x_s$ is a reduced local ring of dimension zero, i. e. a field. Hence R is a regular local ring and $x_1, x_2, \cdots, x_s$ are regular parameters. This remark applies in particular to the ring $R_{\mathfrak{p}}$ of § 3; thus *the proof of Theorem* 1 *is complete.*

Geometrically, Corollary 1 may be interpreted as follows. If R is the local ring of a point P on an affine variety V/k over a field k of characteristic zero, then R is analytically unramified, and therefore, by the corollary, $\bar{R}$ is analytically unramified. Hence $\bar{R}$ is the local ring of P on an affine subvariety $\bar{V}$ of V and the corollary states that some neighborhood of P on V is analytically the direct product of a neighborhood of P on $\bar{V}$ with an open subset of the affine space k^s. In particular, the singularity of P on V is completely determined by that of P on $\bar{V}$. For further developments in this direction we refer the reader to a paper of Zariski on the subject of "equisingularity" [**9**].

With regard to the study of free D^*, Corollary 1 can be supplemented by a *reduction principle*:

PROPOSITION 4.1. *In the situation described by Corollary* 1, *assume further that the R-module $D(R) = D_k(R)$ of k-differentials of R is finitely generated. Then the $\bar{R}$-module $D(\bar{R}) = D_k(\bar{R})$ is finitely generated, and $D^*(\bar{R})$ is free if and only if $D^*(R)$ is free.*

Proof. If π is chosen as in the proof of the corollary, and $\bar{R}$ is identified with $g(\bar{R})$, then the restriction of π to R is the canonical map of R onto $\bar{R}$, and π is the identity on k. Thus $D(\bar{R})$ is a homomorphic image of $D(R)$ ($D(\bar{R})$ being thought of as an R-module via the map $R \rightarrow \bar{R}$), and it follows that $D(\bar{R})$ is a finitely generated $\bar{R}$-module.

The proof of Proposition 1.3 shows then that $D^*(\bar{R})$ is free if and only if $D^*(\bar{R}')$ is free, and similarly that $D^*(R)$ is free if and only if $D^*(R')$ is free. Also $D^*(\bar{R}')$, $D^*(R')$, are modules of finite type (over $\bar{R}'$, R', respectively). Thus the proposition is a consequence of the following lemma:

LEMMA. *Let B be a ring, let k be a subring of B, and let A be the power series ring $B[[X_1, X_2, \cdots, X_s]]$. Let "$D^*$" denote "module of k-derivations." Then $D^*(B)$ is a projective B-module of finite type if and only if $D^*(A)$ is a projective A-module of finite type.*

Proof. It is sufficient to deal with the case $s=1$. We set $X_1 = X$, so that $A = B[[X]]$. A k-derivation of A into A is uniquely determined by its restriction to B, and by its value at X, both of which can be assigned arbitrarily. More precisely, we can check that $D^*(A) \cong D^*(B, A) \oplus A$ (the first summand being the module of k-derivations of B into A). Thus we may replace $D^*(A)$ by $D^*(B, A)$ in the statement of the lemma.

The projection of A onto B induces an A-homomorphism of $D^*(B, A)$ into $D^*(B)$, and we see easily that this mapping is a surjection with kernel $XD^*(B, A)$. Thus $D^*(B)$ is isomorphic as a B-module to

$$D^*(B, A)/XD^*(B, A) = D^*(B, A) \otimes_A B.$$

Hence if $D^*(B, A)$ is projective of finite type, then so is $D^*(B)$.

Conversely, if $D^*(B)$ is a projective B-module of finite type, then $D^*[X] = D^*(B) \otimes_B B[X]$ is a projective $B[X]$-module of finite type. In its X-adic topology, $B[X]$ is a Hausdorff space with completion A. Therefore $D^*(B, A)$ is a topological $B[X]$-module, and it is not hard to see that $D^*[X]$, with its X-adic topology, is a dense topological submodule of $D^*(B, A)$ (we identify $D^*(B, A)$ with the additive group of "power-series in X with coefficients in $D^*(B)$", and similarly we identify $D^*[X]$ with the additive group of "polynomials in X with coefficients in $D^*(B)$," and then $D^*[X]$ is a dense $B[X]$-submodule of $D^*(B, A)$ in an obvious way . . .). Thus $D^*(B, A)$ is the completion of $D^*[X]$. Since $D^*[X]$ is a direct summand of a free $B[X]$-module of finite type, and since completion "commutes" with finite direct sums, $D^*(B, A)$ is a direct summand of a free A-module of finite type. q. e. d.

(The preceding situation can be described succinctly: $D^*(B, A)$ is the *complete tensor product* $D^*(B) \otimes_B A$, when B and $D^*(B)$ have discrete topologies, and A has the X-adic topology.)

When we are dealing with free D^* at a point P on an affine variety V/k, with k of characteristic 0, the local ring of P being R, with maximal ideal $\mathfrak{m}$, Proposition 4.1 allows us to assume that $d'\mathfrak{m} \subseteq \mathfrak{m}$ for every k-derivation d' of R into R (otherwise we can pass to a subvariety $\bar{V}$ of V through P, with $\dim \bar{V} < \dim V$).

For example, if P is a (closed) point on a surface S, and if $d'x$ is a unit for some nonunit x in R, then we may pass to R/xR, which is the local ring of P on some curve C through P. If $D^*(R)$ is free, then by Proposition 4.1, $D^*(R/xR)$ is free, whence (Theorem 1) P is a normal point of C, so that R/xR is regular. It follows then from Corollary 1 that R is regular; hence P is a simple point of S.

We close this section with a complement (which we do not require elsewhere) to Theorem 2.

Complement. In the notation of Theorem 2, assume that $d_i x_j = \delta_{ij}$. A necessary and sufficient condition for B to equal A_d is that the d_i commute with each other.

Proof. Necessity is clear. Conversely, let $b \in B$. Set

$$b^{\#} = \sum_{(\alpha_1, \alpha_2, \cdots, \alpha_s)} ((-x_1)^{\alpha_1} \cdots (-x_s)^{\alpha_s} / \alpha_1! \cdots \alpha_s!) d_1^{\alpha_1} \cdots d_s^{\alpha_s}(b)$$

where $(\alpha_1, \alpha_2, \cdots, \alpha_s)$ runs through all s-tuples of nonnegative integers.

Then, assuming that the d_i commute with each other, we verify that $d_j(b^{\#}) = 0$ $(j = 1, 2, \cdots, s)$. Hence $b^{\#} \in A_d \subseteq B$, so that all terms of $b^{\#}$ vanish except that for which $(\alpha_1, \alpha_2, \cdots, \alpha_s) = (0, 0, \cdots, 0)$; i.e. $b^{\#} = b$. Thus $B \subseteq A_d$. q. e. d.

The condition that the d_i commute with each other is always satisfied in the geometric case (cf. remarks following Corollary 1). Thus, in this case, $\bar{R}'$ may be identified with the subring of R' on which all the d_i vanish.

5. Free D^* and the codimension of the singular locus. In [**4**; p. 212] Buchsbaum and Rim have obtained a generalization of Krull's principal ideal theorem:

Let R be a noetherian ring, and let $g: R^t \to R^r$ with $t \geqq r$ be a homomorphism of R-modules. Then $\dim R_{\mathfrak{p}} \leqq t - r + 1$ *for all minimal primes $\mathfrak{p}$ in* Supp(cokernel of g).

If R is the local ring of a point P on an affine variety V/k, and if $D^*(R)$ is free of rank r, then $D^{**}(R)$ is free of rank r, and the above result implies that each irreducible component of $\mathrm{Supp}(D^{**}(R)/D(R))$ is of codimension $\leqq t - r + 1$, t being the least number of generators of $D(R)$ (localizing at a minimal prime of R, we see easily that $t \geqq r$). If k has characteristic zero, then, as we have seen (§ 3), $\mathrm{Supp}(D^{**}(R)/D(R))$ is the singular locus of R. Thus if P is singular (which we hope it is not) we get a bound on the codimension of the singular locus in the neighborhood of P.

It is known [**5**; p. 174] that the number t is characterized by the fact that the t-th *Fitting ideal* of $D(R)$ is the unit ideal in R, while the $(t-1)$-th Fitting ideal is not. If V/k is immersed in affine n-space, i.e. if V is defined by an ideal $I = (f_1, f_2, \cdots, f_s)$ in the polynomial ring

$k[X_1, X_2, \cdots, X_n]$, with $I = \sqrt{I}$, then we have an $s \times n$ Jacobian matrix $(\bar{f}_{ij})$, where $\bar{f}_{ij}$ is the I-residue of the partial derivative $\partial f_i / \partial X_j$, and for any integer q the q-th Fitting ideal of $D(R)$ is generated by the images in R of the $(n-q) \times (n-q)$ subdeterminants of the Jacobian matrix (by convention, such a subdeterminant vanishes if $q < n - s$, and is the identity if $q \geqq n$). Thus $(n-t)$ is the *rank of the Jacobian at* P.

If r is the dimension of V at P, we can associate with P the nonnegative integer $\delta_P = (n-r) - (\text{rank of the Jacobian at } P)$. $\delta_P = t - r$ depends only on R, and not on the particular immersion of V. (The definition of δ_P has nothing to do with the assumption that $D^*(R)$ is free. However, if $D^*(R)$ is free of rank r, then r is indeed the dimension of V at P (Proposition 1.2). In summary:

PROPOSITION 5.1. *Let P be a point of an affine variety V/k over a field of characteristic zero. If D^* is free at P, then every irreducible component of the singular locus which passes through P has codimension $\leqq 1 + \delta_P$ on V.*

If V is a hypersurface in affine n-space, so that $r = \dim V = n - 1$, then $\delta_P \leqq 1$ and Proposition 5.1 shows that the components of the singular locus have codimension $\leqq 2$ at P. Since P is normal (Theorem 1) each component actually is of codimension 2 at P. A more general result has been proved by S. Lichtenbaum and M. Schlessinger for complete intersections:

PROPOSITION 5.2. *Let P be a point of an affine variety V/k over a perfect ground field k. Assume further that V/k is a complete intersection locally at P. If D^* is free at P then each component of the singular locus passing through P has codimension $\leqq 2$ on V.*

Proof. Let R be the local ring of P on V. Since V is locally a complete intersection, $D(R)$ has homological dimension $\leqq 1$. If $\mathfrak{p}$ is a prime ideal in R, then $R_\mathfrak{p}$ is regular if and only if $D(R_\mathfrak{p})$ is a free $R_\mathfrak{p}$-module; but $D(R_\mathfrak{p})$ also has homological dimension $\leqq 1$; thus $D(R_\mathfrak{p})$ is free if and only if

$$0 = \operatorname{Ext}_{R_\mathfrak{p}}{}^1(D(R_\mathfrak{p}), R_\mathfrak{p}) = [\operatorname{Ext}_R{}^1(D(R), R)]_\mathfrak{p}.$$

In other words the singular locus of R is $\operatorname{Supp}(\operatorname{Ext}^1(D(R), R))$.

Again, $D(R)$ has homological dimension $\leqq 1$; thus there is an exact sequence

$$0 \to F \to G \to D(R) \to 0$$

with F, G, free R-modules of finite type. Applying *, we get the exact sequence

$$0 \to D^*(R) \to G^* \to F^* \to \operatorname{Ext}^1(D(R), R) \to 0$$

By assumption, $D^*(R)$ is free; hence $\operatorname{Ext}^1(D(R),R)$ has homological dimension $\leqq 2$.

Now, Theorem 1.1 of [7] implies that if M is a module of finite type over a noetherian ring, and if $\mathfrak{p}$ is an associated prime ideal of M, then depth $\mathfrak{p} \leqq$ homological dimension of M. Applying this result to the minimal prime ideals of $\operatorname{Supp}(\operatorname{Ext}^1(D(R),R))$, we see that each component of the singular locus of R has depth $\leqq 2$. Since R is a Macaulay ring, each such component has codimension $\leqq 2$. q. e. d.

Remark 1. An alternative description of δ_P (cf. Proposition 5.1) is obtained as follows. If $\mathfrak{m}$ is the maximal ideal of R, then it is well-known [2; § 1] that the universal derivation $d: R \to D(R)$ gives rise to an exact sequence of vector spaces over $R/\mathfrak{m}$:

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to D(R)/\mathfrak{m}D(R) \to D(R/\mathfrak{m}) \to 0$$

The dimension of the vector space $D(R)/\mathfrak{m}D(R)$ is the number t, while the dimension of the space $D(R/\mathfrak{m})$ is the k-dimension of the point P. If V/k is assumed to have dimension r at P, then it follows that

$$r = \dim_{R/\mathfrak{m}} D(R/\mathfrak{m}) + \dim R$$

and that

$$t - r = \delta_P = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 - \dim R.$$

In particular, Proposition 5.1 shows that if P is an isolated singular point of V, and if D^* is free at P, then $2 \dim R \leqq 1 + \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

Remark 2. Let $D = D(R)$ and assume that D^* is free of rank r. The canonical map $D \to D^{**}$ gives rise to a canonical map of symmetric algebras $g: S(D) \to S(D^{**})$. We note that $S(D^{**})$ is a polynomial ring in r variables over R.

For each n, let S_n denote the n-th homogeneous component of the symmetric algebra, so that g induces $g_n: S_n(D) \to S_n(D^{**})$. If P is an *isolated* singular point of V, and if the ground field k has characteristic zero, then $\operatorname{Supp}(D^{**}/D)$ contains only the maximal ideal of R, and it follows that the cokernel of g_n has finite length $L(n)$ for each n. It is shown in [**4**; § 3] that $L(n)$ is a polynomial in n for large n, the degree of the polynomial being $r - 1 + \dim R$.

Proposition 5.2 implies therefore that if V is a complete intersection locally at P, then the polynomial $L(n)$ has degree $r + 1$ (since P is normal). A more direct proof of this fact might yield information about $L(n)$ in the general case (when V is not necessarily a complete intersection), and so lead to an improvement of the estimate in Proposition 5.1.

6. Linear equations and free duals. The following proposition on linear equations with solutions in a module will lead to a characterization of free dual modules (Theorem 3) and, in particular, to a useful criterion for free D^* (Proposition 6.2).

PROPOSITION 6.1. *Let A be a ring, let M be an A-module, and let (a_{ij}) be an $n\times r$ matrix with entries in A. Let I be the ideal generated by the $r\times r$ subdeterminants of (a_{ij}) (if $r>n$, an $r\times r$ subdeterminant is defined to be zero). Then the system of homogeneous linear equations*

$$(\mathrm{S})\qquad \sum_j a_{ij}X_j=0 \qquad (i=1,2,\cdots,n;\, j=1,2,\cdots,r)$$

has a nontrivial solution in M if and only if I annihilates some nonzero element of M.

Proof. We can always reduce the case where $r>n$ to the case where $r\leqq n$: enlarging the matrix by setting $a_{ij}=0$ for $i=n+1,n+2,\cdots,r$ affects neither I nor the set of solutions of (S). We assume therefore that $r\leqq n$. Then Cramer's rule implies the necessity of the given condition.

For fixed n, we prove sufficiency by induction on r. If $r=1$, there is nothing to prove. If $r>1$, let m be a nonzero element in M which is annihilated by all $r\times r$ subdeterminants of (a_{ij}) (i.e. which is annihilated by I). If m is annihilated by all $(r-1)\times(r-1)$ subdeterminants of (a_{ij}), then by the inductive hypothesis, the truncated system $\sum_j a_{ij}X_j=0$ $(i=1,2,\cdots,n;j=1,2,\cdots,r-1)$ has a nontrivial solution; hence, *a fortiori*, (S) has a nontrivial solution.

We may therefore assume that $b_1m\neq 0$, $b_1,b_2,\cdots,b_r$ being the cofactors of some row in some $r\times r$ submatrix of (a_{ij}). But $\sum_j a_{ij}b_j=\pm c_i$, where for each i, c_i either is an $r\times r$ subdeterminant of (a_{ij}), or is zero. Hence

$$\sum_j a_{ij}(b_jm)=\pm c_im=0$$

and we have a nontrivial solution of (S). q.e.d.

We mention, without proof, the following corollary (which we will not use elsewhere):

Let A be a ring, let N be an A-module of finite presentation, with annihilator I', and let M be any A-module. Then

a) $\operatorname{Hom}_A(N, M) = 0$ iff $\operatorname{Hom}_A(A/I', M) = 0$.

b) $N \otimes_A M = 0$ iff $A/I' \otimes_A M = 0$.

[(b) follows directly from (a) via the identity

$$\operatorname{Hom}(N \otimes M, M/I'M) = \operatorname{Hom}(N, \operatorname{Hom}(M, M/I'M))].$$

We proceed to the characterization of free duals.

With any $r \times n$ matrix (a_{ij}), the a_{ij} being elements of a ring A, we associate an A-module of finite presentation, viz. the module with generators $e_1, e_2, \cdots, e_n$ subject to the relations $\sum_j a_{ij} e_j = 0$ $(i = 1, 2, \cdots, r)$. If N is any A-module, then we say that N is *torsion free* if every element of A which is a zerodivisor in N is a zerodivisor in A (cf. § 8).

Lemma. *Let (a_{ij}) be an $r \times n$ matrix with entries in a ring A, and let N be the associated module. Let I be the ideal generated by the $r \times r$ subdeterminants of (a_{ij}) $(I = (0)$ if $r > n)$. Then*

1) *The rows of (a_{ij}) are linearly independent over A if and only if $(0) : I = (0)$ in A, and if this is so, then*

2) *N is torsion free if and only if $(a) : I = (a)$ for any nonzerodivisor a in A.*

If A is noetherian, then these two conditions on I mean precisely that depth $I \geqq 2$.

Proof. To say that the rows of (a_{ij}) are linearly independent is to say that the system of equations $\sum_j a_{ij} X_i = 0$ has no nontrivial solution in A. By Proposition 6.1 this is equivalent to $(0) : I = (0)$.

If the rows are linearly independent, then "N is torsion free" means: if a is a nonzerodivisor in A, and $(b_1, b_2, \cdots, b_n)$ is a vector with entries in A, and if $a(b_1, b_2, \cdots, b_n)$ is a linear combination with coefficients $x_i \in A$ of the rows of (a_{ij}), then $(b_1, b_2, \cdots, b_n)$ is already such a linear combination; i.e. a divides each x_i.

This condition may be restated as: the system of equations $\sum a_{ij} X_i = 0$ has no nontrivial solution in the A-module $A/(a)$. By Proposition 6.1, this means that $(a) : I = (a)$.

If A is noetherian, then $(0) : I = (0)$ implies that I contains a nonzerodivisor, say a, and then $(a) : I = (a)$ shows that I contains a nonzerodivisor modulo (a). Hence depth $I \geqq 2$. Conversely, if depth $I \geqq 2$, then $(a) : I = (a)$ for any nonzerodivisor a: this is obvious if a is a unit in A, and otherwise any associated prime ideal of (a) has depth 1, and therefore does not contain I. Similarly $(0):I=(0)$.

COROLLARY. *Let M be a free A-module of finite rank r, let $f_1, f_2, \cdots, f_r$ be linearly independent elements of M, and let Q be the submodule of M generated by the f's. Then M/Q is annihilated by a nonzerodivisor in A.*

Proof. The module M/Q is associated with an $r \times r$ matrix with linearly independent rows. If I is the determinant of this matrix, then $(0):I=(0)$ and I annihilates M/Q (Cramer's rule).

(Conversely, it can be shown that if $f_1, f_2, \cdots, f_r$ are *arbitrary* elements of M, and if M/Q is annihilated by a nonzerodivisor, then $(0):I=(0)$, whence $f_1, f_2, \cdots, f_r$ are linearly independent).

Let A be a ring with total quotient ring K, and let A^0 be the subset of A consisting of the zero element along with all the nonzerodivisors in A. Let M be an A-module and let $M^* = \mathrm{Hom}_A(M, A)$. If $f_1, f_2, \cdots, f_r \in M^*$, and $x_1, x_2, \cdots, x_n \in M$, let $I[f_i x_j]$ be the ideal generated by the $r \times r$ subdeterminants of the matrix $(f_i x_j)$.

THEOREM 3. *With the above notation, assume that M is finitely generated, and let $f_1, f_2, \cdots, f_r \in M^*$. The following statements are equivalent:*

1) *M^* is free and $f_1, f_2, \cdots, f_r$ form a free basis.*
2) *$M^* \otimes_A K$ is a free K-module of rank r, and if $x_1, x_2, \cdots, x_n$ are elements in M which generate M, then $(a):I[f_i x_j]=(a)$ for any a in A^0.*
3) *$M^* \otimes_A K$ is a free K-module of rank r, and there exist elements $x_1, x_2, \cdots, x_n$ in M such that $(a):I[f_i x_j]=(a)$ for any a in A^0.*

Proof. 2) obviously implies 3).

Assume that 3) holds. By enlarging the set $\{x_1, x_2, \cdots, x_n\}$ if necessary, we may assume that $x_1, x_2, \cdots, x_n$ generate M. Let A^n be a free A-module with basis $e_1, e_2, \cdots, e_n$, and let g be the map of A^n onto M such that $g(e_i)=x_i$ $(i=1,2,\cdots,n)$. Thus we have an exact sequence

$$F \to A^n \xrightarrow{g} M \to 0$$

where F is a free A-module. Applying *, we get the exact sequence

$$0 \to M^* \xrightarrow{g^*} (A^n)^* \to F^*$$

so that M^* may be identified with a submodule of $(A^n)^*$. Note that $(A^n)^*/M^*$ is isomorphic to a submodule of F^*; since F^* is a direct product of copies of A, F^* is torsion free, and therefore $(A^n)^*/M^*$ is torsion free.

We identify any element ω of $(A^n)^*$ with the vector $(\omega e_1, \omega e_2, \cdots, \omega e_n)$. Then $f_i = g^*(f_i) = (f_i x_1, f_i x_2, \cdots, f_i x_n)$ for $i = 1, 2, \cdots, r$. Thus if Q is the submodule of M^* generated by $f_1, f_2, \cdots, f_r$, then $(A^n)^*/Q$ is the module associated with the matrix $(f_i x_j)$. According to the lemma, then, the condition "$(a) : I[f_i x_j] = (a)$ for any a in A^0" means precisely "$f_1, f_2, \cdots, f_r$ are linearly independent (over A), and $(A^n)^*/Q$ is torsion free."

To establish 1), it will be sufficient therefore to show that $Q = M^*$, i.e. $M^*/Q = 0$. Since $M^*/Q \subseteq (A^n)^*/Q$ which is torsion free, it is even enough to show that each element of M^*/Q is annihilated by a nonzerodivisor of A, i.e. that $(M^*/Q) \otimes_A K = 0$, i.e. that $(M^* \otimes_A K / Q \otimes_A K) = 0$.

It is clear that M^* is torsion free, and it follows that the canonical map $M^* \to M^* \otimes_A K$ is injective. Hence the images of $f_1, f_2, \cdots, f_r$ in $M^* \otimes_A K$, which generate the submodule $Q \otimes_A K$, are linearly independent over K. Since $M^* \otimes_A K$ is, by assumption, a free K-module of rank r, and since every nonzerodivisor in K is a unit in K, the corollary of the lemma shows that indeed $(M^* \otimes_A K / Q \otimes_A K) = 0$. Thus 3) implies 1).

Forgetting 3), assume now that 1) holds, and apply the preceding considerations to any set of elements $x_1, x_2, \cdots, x_n$ which generate M. In this case, $Q = M^*$, and, having remarked that $(A^n)^*/M^*$ is torsion free, we see that 2) holds by referring to the equivalent conditions set in quotation marks three paragraphs back. This completes the proof.

As an immediate corollary of Theorem 3 and the last assertion of the lemma, we have:

PROPOSITION 6.2. *Let R be the local ring of a point P on an affine variety V/k over a perfect ground field k. Assume that V/k is locally, at P, equidimensional of dimension r. Then the k-derivations $d_1, d_2, \cdots, d_r$ of R into R form a free basis of $D^*(R)$ if and only if there exist elements $x_1, x_2, \cdots, x_n$ in R such that the ideal generated by the $r \times r$ subdeterminants of the matrix $(d_i x_j)$ has depth $\geqq 2$.*

Moreover, if $d_1, d_2, \cdots, d_r$ do form a free basis for $D^*(R)$, then the ideal generated by the $r \times r$ subdeterminants of the matrix $(d_i x_j)$ has depth $\geqq 2$ whenever $x_1, x_2, \cdots, x_n$ are such that the k-differentials $dx_1, dx_2, \cdots, dx_n$ generate $D(R)$.

Remark 6.3. Clearly Proposition 6.2 also holds if R is the *coordinate* *ring* of the affine variety V/k, provided that all irreducible components of V have dimension r.

7. **Examples.** a) If V is an affine curve over a field of characteristic zero, then, by Theorem 1, D^* is free at a point P of V if and only if P is simple. On the other hand, consider the irreducible plane curve C defined over a perfect ground field k of characteristic $p \neq 0$ by the equation $f(X,Y) = X^p - Y^{p+1} = 0$. We have $f_X = 0$, $f_Y = Y^p$. Thus if $R = k[x,y]$ is the local ring of the origin, then there is a derivation $\bar{d}$ of R into R with $\bar{d}x = 1$, $\bar{d}y = 0$. It follows from Remark 6.3 (or it may be checked directly) that $D^*(R)$ is free with generator $\bar{d}$. Thus D^* is free at every point of C. Nevertheless, the origin is not a normal point of C.

An example where D^* is not globally free as above is provided by the curve $Y^2 + X^p + X^{p+1} = 0$ over a perfect field of characteristic $p > 2$. If R is the local ring of the origin, then there os a derivation $\bar{d}$ of R into R with $\bar{d}x = 2 + 2x$, $\bar{d}y = y$. By Proposition 6.2 (or directly) $D^*(R)$ is a free module with generator $\bar{d}$, whereas the origin is a singular point. (The fact that D^* is not globally free is seen by remark 6.3 and by consideration of the points $(0,0)$, $(-1,0)$ on the curve).

b) Consider the surface defined over a perfect ground field k of characteristic $p \neq 0$ by the equation $f(X,Y,Z) = XY - Z^p = 0$. We have $f_X = Y$, $f_Y = X$, $f_Z = 0$. The origin is the only singular point, and by the corollary to Proposition 2.1, the origin is a normal point. Thus, the coordinate ring $R = k[x,y,z]$ is integrally closed, and the ideal (x,y) in R has depth 2. By Remark 6.3, the two derivations d_1, d_2 such that: $d_1x = 0$, $d_1y = 0$, $d_1z = 1$; $d_2x = -x$, $d_2y = y$, $d_2z = 0$; form a free basis of $D^*(R)$. So we can have free D^* in the presence of singular points, even under the assumption of normality.

From now on, we restrict ourselves to a fixed ground field k of *characteristic zero.*

c) Let P be the origin on a cone K in affine 3-space (over k), K being given by $f(X,Y,Z) = 0$ where f is a homogeneous polynomial without multiple factors. Zariski has shown (unpublished notes) that D^* is free at P only if P is a simple point of K (i.e. only if K is a *plane*). The proof is given here with his permission.

Assume that P is a singular point of K. If D^* is free at P, P is normal (Theorem 1), and it follows that P is the only singular point of K (otherwise the line joining P to a singular point would be a multiple curve through

P). If f_X, f_Y, f_Z all vanish at some point P' in the affine space then, by Euler's theorem on homogeneous polynomials, f also vanishes at P', and P' is a singular point of K; hence $P'=P$. In other words, (X,Y,Z) is the only associated prime ideal (in $k[X,Y,Z]$) of the ideal (f_X,f_Y,f_Z). Hence no associated prime ideal of (f_Z,f_Y) is a minimal prime ideal, and it follows that $(f_Z):(f_Y)=(f_Z)$. It also follows, by Macaulay's theorem, that (f_Z,f_Y) is pure one-dimensional so that $(f_Z,f_Y):(f_X)=(f_Z,f_Y)$. We shall make use of these observations below.

Let R be the local ring of P on K, and let x, y, z be the traces of X, Y, Z on K, so that $k[x,y,z]$ is the coordinate ring of K. $k[x,y,z]$ is a subring of R since every irreducible component of a cone contains the vertex. We identify any derivation $\bar{d}$ of R into R with the vector $(\bar{d}x, \bar{d}y, \bar{d}z)$. Thus the derivations of R into R are the vectors (a,b,c) with a, b, c in R such that

$$af_x+bf_y+cf_z=0 \qquad f_x=f_X(x,y,z) \text{ etc.}$$

The derivations (a,b,c) with a, b, c in $k[x,y,z]$ span the R-module of all derivations of R into R. We seek, therefore, the polynomial solutions (A,B,C) of

$$Af_X+Bf_Y+Cf_Z\equiv 0 \pmod{f}$$

These solutions form a $k[X,Y,Z]$-module M which is spanned by the *homogeneous* solutions (since f is homogeneous). So let A, B, C, E be homogeneous polynomials such that

$$Af_X+Bf_Y+Cf_Z=Ef$$

Setting $A_1=A-XE/n$, $B_1=B-YE/n$, $C_1=C-ZE/n$, where n $=$ degree of f, we get

$$A_1f_X+B_1f_Y+C_1f_Z=0 \tag{2}$$

Hence (X,Y,Z) and the homogeneous solutions of (2) span M. Since $(f_Z,f_Y):(f_X)=(f_Z,f_Y)$, (2) implies that $A_1=Gf_Y+Hf_Z$ (G, H homogeneous polynomials). Subtracting from (A_1,B_1,C_1) the vector $G(f_Y,-f_X,0)$ $+H(f_Z,0,-f_X)$ (which is in M) we may assume $A_1=0$. But then, since $(f_Z):(f_Y)=(f_Z)$, (2) implies that $(0,B_1,C_1)=L(0,f_Z,-f_Y)$ (L a polynomial).

Hence M is spanned by the vectors

$$(X,Y,Z),\ (f_Y,-f_X,0),\ (-f_Z,0,f_X),\ (0,f_Z,-f_Y)$$

and therefore $D^*(R)$ is spanned by the vectors

$$(x,y,z),\ (f_y,-f_x,0),\ (-f_z,0,f_x),\ (0,f_z,-f_y).$$

Thus there is a free basis of $D^*(R)$ among these four vectors. Applying Proposition 6.2 to all the possible pairs among these vectors, we conclude that one of f_x, f_y, f_z is a unit in R; this is impossible since P is not simple.

d) We will treat in some detail the following situation: P is the origin on a surface V defined over the field k (of characteristic zero) by an equation of the form $Z^n = f(X, Y)$ where $f(0,0) = 0$, and $n > 1$. If D^* is free at P, then P is normal (Theorem 1); by the corollary of Proposition 2.1, P is normal if and only if, writing $f(X,Y) = g(X,Y)h(X,Y)$, where $h(X,Y)$ is the product of all those factors of $f(X,Y)$ which do not vanish at $(0,0)$, we find that $g(X,Y)$ has no multiple factors.

We will show that if D^* is free at P, then D^* is free at the origin on the curve defined by $g(X,Y) = 0$. By Theorem 1, the origin is a simple point of this curve, whence P is a simple point of V.

(i) Let $k[x,y,z]$ be the coordinate ring of V, and let S be the local ring of P on V. Let R be the local ring of the origin on the (x,y) plane, i.e. let $R = k[x,y]_{(x,y)}$. Then $R[z] \subseteq S$. On the other hand, we can check that $R[z]$ is a local ring, cf. [8; p. 318], and it follows easily that $R[z] = S$.

(ii) We study the derivations of $S = R[z]$ into itself. Any such derivation is uniquely determined by its restriction to R; thus our problem is to study the derivations of R into $R[z]$ which can be extended to derivations of $R[z]$ into $R[z]$.

Let $\bar{d}$ be a derivation of R into $R[z]$. Then clearly

$$\bar{d} = d_0 + zd_1 + z^2d_2 + \cdots + z^{n-1}d_{n-1}$$

where $d_0, d_1, \cdots, d_{n-1}$ are uniquely determined derivations of R into R. We claim that *$\bar{d}$ can be extended to a derivation of $R[z]$ into $R[z]$ if and only if $d_0, d_1, \cdots, d_{n-2}$ can be extended.* (Note that the derivation $z^{n-1}d_{n-1}$ can always be extended.)

If this is so, then, denoting extensions by upper "e," we will have

$$\bar{d}^e = (d_0)^e + z(d_1)^e + z^2(d_2)^e + \cdots + (z^{n-1}d_{n-1})^e$$

since the derivation on the right is obviously an extension of $\bar{d}$, and since $\bar{d}$ has at most one extension. It follows that *$D^*(S)$ is generated by derivations of the form $(\bar{d}_0)^e$, or $(z^{n-1}\bar{d}_0)^e$*, where $\bar{d}_0$ is a derivation of R into R.

The proof of the claim is straightforward. $\bar{d}$ can be extended if and only if there is an s in S such that $nz^{n-1}s = \bar{d}f$, or equivalently,

$$nfs = z(\bar{d}f) = z(d_0f) + z^2(d_1f) + \cdots + z^{n-1}(d_{n-2}f) + f(d_{n-1}f).$$

Thus s exists if and only if f divides $d_0f, d_1f, \cdots, d_{n-2}f$.

Setting $d_1 = d_2 = \cdots = d_{n-1} = 0$, we see that d_0 can be extended if and only if f divides d_0f; similarly d_i can be extended if and only if f divides d_if $(i = 1, 2, \cdots, n-2)$. Thus, the preceding statement becomes "s exists if and only if $d_0, d_1, \cdots, d_{n-2}$ can be extended." q. e. d.

(iii) Assume now that $D^*(S)$ is a free S-module. Since S is a local ring, any set of generators of $D^*(S)$ contains a free basis $\{d', d''\}$ of $D^*(S)$. By the above results, we may assume that d' is either of the form

$$(\#) \qquad (d_0')^e \colon d_0' \in D^*(R) \text{ and } f \text{ divides } d_0'f$$

or of the form

$$(\#\#) \qquad (z^{n-1}d_0')^e \colon d_0' \in D^*(R)$$

and similarly for d''.

We identify d' with the vector $(d'x, d'y, d'z)$ and d'' with the vector $(d''x, d''y, d''z)$. Then

$$d' = (\alpha_1, \alpha_2, z(d_0'f)/nf) \qquad \alpha_1, \alpha_2 \in R$$

or

$$d' = (z^{n-1}\alpha_1, z^{n-1}\alpha_2, z^{n-1} \cdot z(d_0'f)/nf) \qquad \alpha_1, \alpha_2 \in R$$

according as d' is of the form $(\#)$ or $(\#\#)$. Similar remarks apply to d'', with "β" in place of "α."

Thus we are led to the "derivation matrix"

$$J = (z^{n-1})^t \cdot (z/nf) \begin{pmatrix} \alpha_1 & \alpha_2 & d_0'f \\ \beta_1 & \beta_2 & d_0''f \end{pmatrix}$$

where t $(= 0, 1, \text{or } 2)$ is the number of derivations among d', d'' having the form $(z^{n-1} \cdot \cdot)^e$.

Proposition 6.2 states that the 2×2 subdeterminants of J must generate an ideal of depth $\geqq 2$. If $t = 0$, then f divides both $d_0'f$ and $d_0''f$ so that z divides J. If $t = 2$, then $(z^{n-1})^t \cdot (z/nf) = z^{n-1}/n$ and once again z divides J. Hence $t = 1$, and $(z^{n-1})^t \cdot (z/nf)$ is a unit; moreover, we may assume that d' is of the form $(\#)$, so that f divides $d_0'f$.

Now $d_0'f = f_x\alpha_1 + f_y\alpha_2$ and $d_0''f = f_x\beta_1 + f_y\beta_2$, so that the last column of J is a linear combination of the first two. Thus *Proposition* 6.2 *will be satisfied only if* $\alpha_1\beta_2 - \alpha_2\beta_1$ *is a unit in* S (and therefore in R). Hence either $\alpha_1 = d_0'x$ or $\alpha_2 = d_0'y$ is a unit in R. Moreover, since f divides $d_0'f$, d_0' induces a derivation in $R/(f)$, i.e. in $R/(g)$ which is the local ring of the origin on the plane curve $g(X, Y) = 0$. Since, say, α_1 (mod. g) is a unit in $R/(g)$, we see, by Proposition 6.2, that $D^*(R/(g))$ is a free module with generator d_0' (mod. g). This is what we set out to prove.

e) The method of d) can be extended to more complicated situations. For example, the origin on a 3-fold in 5-space given by equations of the form

$$U^n = f(W,X,Y,Z) \qquad f(0,0,0,0)=0$$
$$W^m = g(X,Y,Z) \qquad g(0,0,0)=0$$

has a free D^* if and only if it is a simple point.

f) The purpose of the examples in this section has been to illustrate the conjecture that P is simple if D^* is free at P. We wish to point out that in attempting to prove this conjecture (assuming that k has characteristic zero), we may assume that k is algebraically closed, and that P is a rational point. For, in the first place, Proposition 1.1 shows that if D^* is free at P, then D^* is free at almost every algebraic specialization (over k) of P, so that we may assume that P is algebraic (over k).

Secondly, let $\bar{k}$ be the algebraic closure of k; then we have the canonical projection $V \times_k \bar{k} \to V$. If $\bar{P}$ is any point of $V \times_k \bar{k}$ lying over P, then the local ring $\bar{R}$ of $\bar{P}$ is the localization at one of the maximal ideals of the semi-local ring $R \otimes_k \bar{k}$, R being the local ring of P on V. $\bar{R}$ is a faithfully flat R-algebra.

It is not hard to see that any k-derivation of R into an $\bar{R}$-module M has a unique extension to a $\bar{k}$-derivation of $\bar{R}$ into M. It follows easily that $\bar{D}(\bar{R}) = D(R) \otimes_R \bar{R}$ where "$\bar{D}$" denotes "module of $\bar{k}$-differentials." Hence (cf. proof of Proposition 1.3) $\bar{D}^*(\bar{R}) = D^*(R) \otimes_R \bar{R}$. Since $\bar{R}$ is faithfully flat over R, $\bar{D}(\bar{R})$ (respectively $\bar{D}^*(\bar{R})$) is free if and only if $D(R)$ (respectively $D^*(R)$) is free (cf. proof of Proposition 1.3). This shows that, for the purposes of the conjecture, we may assume $k = \bar{k}$.

8. Appendix: Torsion free and reflexive differential modules. We will indicate a proof of the following facts:

PROPOSITION 8.1. *Let R be the local ring of a point P on an affine variety V/k over a perfect ground field k. Assume that V is locally, at P, a complete intersection. Let $D(R)$ be the R-module of k-differentials of R. Then*

1) *$D(R)$ is torsion free if and only if V is nonsingular in codimension 1 at P, i.e. if and only if P is normal.*

2) *$D(R)$ is reflexive if and only if V is nonsingular in codimension 2 at P.*

Given a ring A with total quotient ring K, and an A-module M, we call the kernel of the canonical map $i: M \to M_{(K)} = M \otimes_A K$ the *torsion submodule* of M. Thus the torsion submodule consists of all elements of M which are annihilated by a nonzerodivisor in A. M is torsion free (cf. § 6) if and only if its torsion submodule is (0). One checks that the torsion submodule of M is contained in the kernel of the canonical map $f: M \to M^{**}$. Conversely, if K is semisimple (equivalently: if the ideal (0) is a finite intersection of prime ideals in A) then the "naturality" of f gives a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & M^{**} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i^{**}} \\ M_{(K)} & \xrightarrow{\quad g \quad} & M_{(K)}{}^{**} \end{array}$$

in which g is injective, since $M_{(K)}$ is a projective K-module (cf. § 2). Hence, in this case, $\ker f \subseteq \ker i$, so that the torsion submodule *is* the kernel of f.

The next lemma gives further information about the kernel and cokernel of $f: M \to M^{**}$.

LEMMA. *Let A be any ring, and let $F_0 \to F_1 \to M \to 0$ be an exact sequence of A-modules, where F_0 and F_1 are projective and of finite type. Let N be the cokernel of the dual map $F_1{}^* \to F_0{}^*$. Then there is an exact sequence*

$$0 \to \mathrm{Ext}^1(N, A) \to M \xrightarrow{\quad f \quad} M^{**} \to \mathrm{Ext}^2(N, A) \to 0$$

Proof. One checks that for any zero-sequence $G_0 \to G_1 \to G_2 \to G_3$ there is an exact sequence

$$\begin{aligned} 0 &\to (\text{homology at } G_1) \to (\text{cokernel of } G_0 \to G_1) \to \\ &\to (\text{kernel of } G_2 \to G_3) \to (\text{homology at } G_2) \to 0 \end{aligned} \qquad (3)$$

We are given the exact sequence

$$0 \to M^* \to F_1{}^* \to F_0{}^* \to N \to 0. \qquad (4)$$

Building an exact sequence

$$F_3 \to F_2 \to M^* \to 0 \qquad (5)$$

with F_3 and F_2 projective modules, combining (4) and (5), and dualizing, we get a zero-sequence

$$0 \to N^* \to F_0{}^{**} \to F_1{}^{**} \to F_2{}^* \to F_3{}^*.$$

Since F_0 and F_1 are reflexive the cokernel of $F_0^{**} \to F_1^{**}$ can be identified with M; also the kernel of $F_2^* \to F_3^*$ is M^{**}. One checks then that (3) gives rise to the desired sequence. q. e. d.

To prove the proposition, we apply the preceding considerations to the case $A = R$, $M = D(R)$. Since $D(R)$ has homological dimension $\leqq 1$, we may assume that $N = \mathrm{Ext}^1(D(R), R)$. K is now semisimple and it follows that $N \otimes_A K = 0$, whence $N^* = \mathrm{Ext}^0(N, R) = 0$.

By the lemma (and the remarks preceding the lemma), $D(R)$ is torsion free iff $\mathrm{Ext}^1(N, R) = 0$, i. e. iff grade $N \geqq 1$, i. e. iff Supp N has depth $\geqq 1$ (cf. § 2); similarly $D(R)$ is reflexive iff $\mathrm{Ext}^1(N, R) = \mathrm{Ext}^2(N, R) = 0$, i. e. iff Supp N has depth $\geqq 2$. However, we have seen, in proving Proposition 5.2, that Supp N is the singular locus of R. Also, since R is a Macaulay ring, depth and codimension coincide. Thus, in view of the corollary to Proposition 2.1, all our assertions are proved.

We remark that Proposition 8.1, applied to the generic point of a component of the singular locus of V, yields an alternative proof of Proposition 5.2.

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