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BALANCED FIELD EXTENSIONS

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Let k be a field, and let K be an algebraic extension of k. K is then a purely inseparable extension of a separable extension of k; for reasons of symmetry, one might wonder when K will be a separable extension of a purely inseparable extension of k. (This is not always so: cf. example at the end of this note.) When this does happen, let us say that K/k is "balanced." We wish to set down some simple observations about such extensions.

For basic notions of field theory see O. Zariski and P. Samuel: *Commutative Algebra*, Van Nostrand, Princeton, N. J., 1958, Volume I, chapter 2; also chapter 3, section 15, for the definition and properties of free joins.

PROPOSITIONS: A. *The following are equivalent:*

1. *K/k is balanced.*

2. *There exists a separable algebraic extension of K which is normal over k.*

3. *If L is a field of algebraic functions over k, then the order of inseparability $[(L, K): K]_i$ is the same for all free joins (L, K) of L/k and K/k.*

In geometric language, 3. reads: If V/k is an irreducible algebraic variety, then all the irreducible components of V/K have the same order of inseparability.

B. *Let $\bar{k}$ be an algebraic closure of k, and let $\bar{k}_s$ (respectively $\bar{k}_i$) be the subfield of $\bar{k}$ consisting of all elements which are separable, (resp. purely inseparable) over k. We know that the subfields of $\bar{k}$ (resp. $\bar{k}_s$, $\bar{k}_i$) form a lattice Z (resp. Z_s, Z_i) under the operations of field composition and intersection.*

The balanced extensions of k in $\bar{k}$ form a sublattice of Z isomorphic with the direct product $Z_s \times Z_i$.

Proofs: A. We show that $1 \xrightarrow{a)} 3 \xrightarrow{b)} 2 \xrightarrow{c)} 1$.

a) Let $k \subseteq I \subseteq K$, I being a field such that I/k is purely inseparable, and K/I is separable. Any free join (L, K) contains a free join (L, I), and since K/I is separable, we have $[(L, K): K]_i = [(L, I): I]_i$. Thus we may assume that

$K=I$. But then there is nothing to prove, since all free joins of L/k and I/k are equivalent.

b) Let $x\in K$, and let $L=k(x)$. The free joins of L/k, K/k, are all the fields of the form $K(\bar{x})$, where $\bar{x}$ is k-conjugate to x, in some fixed algebraic closure $\bar{K}$ of K. Since $[K(x):K]_i=[K:K]_i=1$, we have $[K(\bar{x}):K]_i=1$, i.e. *all k-conjugates of x (in $\bar{K}$) are separable over K*. From this it follows immediately that if N is the least extension of K normal over k, then N/K is separable.

c) Let $N\supseteq K$ be such that N/K is separable and N/k is normal. Let $I\subseteq N$ be the field of invariants of all automorphisms of N/k. Then I/k is purely inseparable, and N/I is separable. It will be sufficient to show that $I\subseteq K$. But any x in I is separable over K (since N/K is separable), and purely inseparable over K (since I/k is purely inseparable).

B. Let B be the set of balanced extensions of k in $\bar{k}$. Clearly $K\in B$ iff $K/(K\cap\bar{k}_i)$ is separable, and this latter condition may be expressed as follows:

(1′) If $x\in K$, if $f(X)$ is the minimum monic polynomial of x over k, and if $\bar{f}(X)\in\bar{k}[X]$ is *the* polynomial without multiple roots, of which $f(X)$ is a power, then $\bar{f}(X)\in(K\cap\bar{k}_i)[X]$.

It follows immediately that an arbitrary intersection of members of B is again a member of B.

Again, if $K\in B$, then K is a separable extension of $K\cap\bar{k}_i$; also K is purely inseparable over $K\cap\bar{k}_s$; if K' is the compositum $(K\cap\bar{k}_s, K\cap\bar{k}_i)$, then K/K' is both separable and purely inseparable, i.e., $K=K'$. We see then, that $K\in B$ *iff K is generated by a separable extension of k and a purely inseparable extension of k.*

It follows easily that the field generated by an arbitrary collection of members of B is itself a member of B.

We have shown, therefore, that B is a sublattice of Z (in fact, a complete sublattice). We have also given an order-preserving map F from $Z_s\times Z_i$ *onto* B; if $S\in Z_s$, $I\in Z_i$, then $F(S, I)$ is the composed field (S, I). Now if $x\in(S, I)\cap\bar{k}_i$, then x is separable over I and purely inseparable over I; hence $(S, I)\cap\bar{k}_i=I$. Similarly $(S, I)\cap\bar{k}_s=S$. Thus F is injective, and the proof is complete.

EXAMPLE. For an example of a nonbalanced extension, let L be a field of characteristic two, let Y, Z, be indeterminates over L, let $k=L(Y, Z)$, and let $K=k(x)$, x being a root of

$$f(X) = X^4 + YX^2 + Z = 0.$$

One checks that $f(X)$ is irreducible over k, that $[K:k]_i=2$, and that, in the notation of (1′) above, $\bar{f}(X)=X^2+\sqrt{Y}X+\sqrt{Z}$. According to (1′), K/k cannot be balanced unless $\sqrt{Y}\in K$ and $\sqrt{Z}\in K$. Since $[k(\sqrt{Y}, \sqrt{Z}):k]_i=4$, this is impossible.