ON THE JACOBIAN IDEAL OF THE MODULE OF DIFFERENTIALS[1]

JOSEPH LIPMAN

Let A be a reduced algebra of finite type over a field k, and let J be the Jacobian ideal of A/k (i.e. J is the smallest nonzero Fitting ideal of the module of k-differentials of A). In some geometric investigations it has been found useful to consider the scheme V obtained by blowing up J. (Loosely speaking, the geometric points of V over a point Q of Spec (A) correspond to the various "limiting positions" of the tangent spaces at points P approaching Q along arcs which are simple on Spec (A).) It has even been proposed that one try to resolve singularities by successively blowing up Jacobian ideals. Implicit in this proposal is the assumption that *if J is invertible, then* Spec (A) *is nonsingular*. Our purpose is to show, among other things, that this is indeed so when k has characteristic zero.

Before stating the theorem we recall some definitions. Let M be a (unitary) module of finite type over a ring S (commutative, with identity) generated, say, by $e_1, e_2, \cdots, e_n$. Let (a_{ij}) (i in some indexing set; $j=1, 2, \cdots, n$) be a matrix with entries in S, such that for each i, $\sum_{j=1}^n a_{ij}e_j=0$, and such that any row-vector $(a_1, a_2, \cdots, a_n)$ for which $\sum a_je_j=0$, is a linear combination with coefficients in S of the rows of (a_{ij}); in other words the rows of (a_{ij}) generate the "module of relations" of $e_1, e_2, \cdots, e_n$. For an integer $p \geqq 0$, the $(n-p)\times(n-p)$ minors of (a_{ij}) generate an ideal $I_p(M)$; by convention $I_p(M)=S$ for $p\geqq n$. The ideal $I_p(M)$ is known to depend neither on the choice of the basis $\{e_1, e_2, \cdots, e_n\}$ nor on the choice of the "relation matrix" (a_{ij}); $I_p(M)$ is the pth *Fitting ideal* of M (cf. [2]). We have

$$I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = I_{n+1} = \cdots = S.$$

If I is the annihilator of M, then $I^n \subseteq I_0 \subseteq I$.

The *torsion submodule* M^t of M consists of all elements m in M such that m is annihilated by a *regular element* (i.e. a nonzerodivisor) in S. We denote by dim proj(M) the *projective dimension of M*, i.e. the smallest length (possibly infinite) of a projective resolution of M.

THEOREM. *Let k be a field and let R be a reduced local ring of the form*

Received by the editors July 25, 1968.

[1] Research supported by NSF-GP-6388 at Purdue University.

$A_{\mathfrak{p}}$, where A is a finitely generated k-algebra and $\mathfrak{p}$ is a prime ideal in A. Assume that for each minimal prime $\mathfrak{q}$ of R, the field $R_{\mathfrak{q}}$ is separable over k. Let $D=\Omega^1_{R/k}$ be the module of k-differentials of R, with torsion submodule D^t, and let J be the smallest nonzero Fitting ideal of D. Then the following conditions are equivalent:

(1) *J is principal, generated by a regular element in R (i.e. J is invertible).*

(2) *R is a complete intersection, and D/D^t is free.*

Moreover, if k has characteristic zero, then D/D^t is free if and only if R is regular.

PROOF. R being as above, Ferrand has shown (cf. [1]) that R is a complete intersection if and only if $\dim \operatorname{proj}(D) \leqq 1$. The equivalence of (1) and (2) results therefore from:

LEMMA 1. *Let S be a local ring (not necessarily noetherian) and let M be a finitely generated S-module, with torsion submodule M^t. Let r be a nonnegative integer. The following conditions are equivalent:*

(i) *The smallest nonzero Fitting ideal of M is $I_r(M)$, and $I_r(M)$ is generated by a single regular element of S.*

(ii) *M is of finite presentation,* $\dim \operatorname{proj}(M) \leqq 1$, *and M/M^t is free of rank r.*

PROOF. (ii)$\Rightarrow$(i). Assuming (ii) we have an isomorphism of S-modules $M \cong M^t \oplus S^r$ whence M^t is of finite presentation, and for $p \geqq 0$

$$I_p(M) = I_{p-r}(M^t) \qquad (=(0) \text{ if } p < r).$$

Since M^t is a torsion module, $I_0(M^t)$ *contains a regular element of S*; thus $I_r(M)$ is the smallest nonzero Fitting ideal of M. Since M^t is a direct summand of M, we have also

$$\dim \operatorname{proj}(M^t) \leqq \dim \operatorname{proj}(M) \leqq 1$$

so there is an exact sequence

$$0 \to F_1 \to F_2 \to M^t \to 0$$

with free modules F_1, F_2 of finite ranks, say r_1, r_2 respectively. $r_1 \leqq r_2$ since r_2 linear homogeneous equations in more than r_2 unknowns, with coefficients in S, always have a nontrivial solution in S. (Cf. [3, Proposition 6.1].) On the other hand, since $I_0(M^t) \neq (0)$, $r_1 \geqq r_2$. Thus $r_1 = r_2$, and therefore $I_0(M^t)$ is a principal ideal, whose generator is a *regular* element (since $I_0(M^t)$ *contains* a regular element).

(i)$\Rightarrow$(ii). Let $e_1, e_2, \cdots, e_n$ generate M. Since S is local, there are relations

$$\sum_{j=1}^{n} a_{ij}e_j = 0 \qquad (i = 1, 2, \cdots, n-r)$$

with $a_{ij} \in S$, such that $I_r(M)$ is generated by the determinant D of the $(n-r)\times(n-r)$ matrix

$$A = (a_{ij}), \qquad 1 \leqq i, j \leqq n-r.$$

Since some multiple of D is regular, so is D.

For $1 \leqq h \leqq n-r$, we can multiply the above relations by the cofactors of the hth column in A and add to obtain

$$De_h + \sum_{j=n-r+1}^{n} D_je_j = 0$$

where the D_j are certain determinants which are in $I_r(M)$ and hence are divisible by D; so we have the element

$$e_h + \sum_{j=n-r+1}^{n} (D_j/D)e_j$$

which is annihilated by D, and hence lies in M^t. It follows that the canonical images $\bar{e}_{n-r+1}, \bar{e}_{n-r+2}, \cdots, \bar{e}_n$ of $e_{n-r+1}, e_{n-r+2}, \cdots, e_n$ in M/M^t generate M/M^t over S.

If K is the total ring of fractions of S, then

$$M_{(K)} = M \otimes_S K \cong (M/M^t) \otimes_S K$$

since M^t is the kernel of the canonical map $M \to M_{(K)}$. Identifying M/M^t with an S-submodule of $M_{(K)}$ we see that the r elements $\bar{e}_{n-r+1}, \cdots, \bar{e}_n$ generate $M_{(K)}$ over K. The Fitting ideals of $M_{(K)}$ over K are obtained from those of M by localization. So if

$$\sum_{j=n-r+1}^{n} c_j\bar{e}_j = 0$$

with c_j in $S \subseteq K$, then the ideal $(c_{n-r+1}, \cdots, c_n)K$ is contained in $I_{r-1}(M_{(K)}) = (0)$, i.e. $c_{n-r+1} = \cdots = c_n = 0$. Thus $\bar{e}_{n-r+1}, \cdots, \bar{e}_n$ form a *free basis* of M/M^t.

For the remaining assertions, note first that the rows of the $(n-r)\times n$ matrix (a_{ij}) are linearly independent; for if

$$\sum_{i=1}^{n-r} x_ia_{ij} = 0 \qquad (j = 1, 2, \cdots, n)$$

with elements $x_i \in S$, then $Dx_i = 0$ for all i, i.e. $x_i = 0$. Now suppose that

$$\sum_{j=1}^{n} a_j e_j = 0$$

with $a_j \in S$. Since $I_{r-1}(M) = (0)$, we have for any $h = 1, 2, \cdots, n$

$$\det \left(\begin{array}{c|c} a_{1h} & \\ a_{2h} & \\ \vdots & A \\ a_{n-r,h} & \\ \hline a_h & a_1 \, a_2 \cdots a_{n-r} \end{array} \right) = 0$$

whence

$$Da_h + \sum_{i=1}^{n-r} D_i' a_{ih} = 0$$

where, for $1 \leqq i \leqq n-r$, D_i' is the cofactor of a_{ih} in the above matrix. Note that D_i' does not depend on h. Since D is regular and D divides each D_i' (because $D_i' \in I_r(M)$), we see then that $(a_1, a_2, \cdots, a_n)$ is a linear combination, with coefficients in S, of the rows of the $(n-r) \times n$ matrix (a_{ij}). We have already remarked that these rows are linearly independent; thus M is the quotient of a free module of rank n by a free submodule of rank $n-r$, so that M is of finite presentation and dim proj$(M) \leqq 1$. This completes the proof of Lemma 1.

If k has characteristic zero, then R is regular if and only if J is the unit ideal (Jacobian criterion of regularity); hence if R is regular then D/D^t is free. (In fact it is practically trivial that J is the unit ideal if and only if D itself is free.)

For the converse, we use:

Lemma 2. *Let S be a ring (commutative, with identity) with total ring of fractions K, and let M be a finitely generated S-module with torsion submodule M^t. For any S-module N let N^* be the dual module* $\mathrm{Hom}_S(N, S)$. *The following are equivalent:*

(i) *M/M^t is a free S-module.*

(ii) *$M \otimes_S K$ is a free K-module, M^* is free of finite rank over S, and the canonical map $f: M \to M^{**}$ is surjective.*

Proof. There are canonical isomorphisms

$$M \otimes_S K \xrightarrow{\approx} (M/M^t) \otimes_S K, \qquad (M/M^t)^* \xrightarrow{\approx} M^*$$

and a canonical commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & M^{**} \\ \downarrow & & \downarrow \wr \\ M/M' & \xrightarrow{g} & (M/M')^{**} \end{array}$$

If M/M' is free then g is an isomorphism and so f is surjective. We see also from the above isomorphisms that then $M \otimes_S K$ is free over K and M^* is free of finite rank over S. Thus (i)$\Rightarrow$(ii).

Conversely if (ii) holds then M^{**} is free of finite rank; g is surjective since f is, and since $M \otimes_S K$ is free and $M/M' \subseteq M \otimes_S K$, it is easily seen that g is also injective. Thus M/M' is free. Q.E.D.

Applying Lemma 2 to the case $S=R$, $M=D$, we have that if D/D' is free then D^* is free and $D \to D^{**}$ is surjective. But if k has characteristic zero, this implies that R is regular (cf. [3, §§3, 4]).

This completes the proof of the theorem.

Remark. In Lemma 2, if S is noetherian and every prime ideal in S has depth $\leqq 1$, then "M^* free of finite rank" *implies* that $f: M \to M^{**}$ is surjective (cf. [3, §6]). This gives the main Theorem 3 in [4].

References

1. D. Ferrand, *Suite régulière et intersection complète*, C. R. Acad. Sci. Paris 264 (1967), 427–428.

2. H. Fitting, *Die Determinantenideale eines Moduls*, Jber. Deutsch. Math.-Verein. 46 (1936), 195–228.

3. J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. 87 (1965). 874–898.

4. T. Matsuoka, *On the torsion submodule of a module of type* (F_1), J. Sci. Hiroshima Univ. Ser. A-I Math. 31 (1967), 151–160.

Purdue University