Appendix to Chapter II

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In this appendix we will reconsider, from a different point of view, the main ideas of the foregoing Chapter II (referred to as "the text").

1. Let f be a non-singular irreducible projective surface over an algebraically closed field k, with function field K/k. The (closed) points of f are in one-one correspondence with their (two-dimensional) local rings on f; accordingly we will simply call these local rings "points on f". In fact we will refer to *any* two-dimensional regular local ring with fraction field K as a "point". A point O is said to be "infinitely near" to f if O contains some point on f. This terminology is justified by the following fact (ABHYANKAR, 2, theorem 3, p. 343): *if O is infinitely near to f then there exists a (unique) sequence of points*

$$O_1 < O_2 < \cdots < O_r = O$$

such that O_1 is a point on f, and for each $i = 1, 2, \ldots, r-1$, O_{i+1} is a quadratic transform of O_i (i. e. O_{i+1} is a point on the surface obtained from f by blowing up successively $O_1, O_2, \ldots, O_i$).

By associating to each point O, with maximal ideal $m(O)$, the "order valuation" ord_O (determined by the condition that for non-zero $x \in O$, $\text{ord}_O(x) = \max\{t \mid x \in m(O)^t\}$) we obtain a one-one correspondence between infinitely near points and prime divisors of the second kind with respect to f (i. e. valuations of K/k centered at a point on f, and with residue field transcendental over k) (cf. ABHYANKAR, 2, proposition 3, p. 336).

The text deals, in essence, with the free abelian group Δ on the set of *all* prime divisors, of first and second kind, on f. After identifying prime divisors of the first kind with integral curves on f (namely, their respective centers) and prime divisors of second kind with infinitely near points, we can represent any element W of Δ uniquely in the form $W = C + H$, where C is a divisor on f (= finite formal sum of integral curves) and $H = \sum_{i=1}^{r} s_i O_i$ with infinitely near points O_i and integers s_i; s_i is called the "virtual multiplicity" of W at O_i. To conform with the text, we denote such a W by C_H and think of it as the divisor C together with the "base divisor" H.

Now let f' be a non-singular projective surface birationally equivalent to f, and let Δ' be the free abelian group generated by the prime divisors on f'. We will define an isomorphism $\Theta_{f',f}:\Delta\to\Delta'$; $\Theta_{f',f}(C_H)=C'_{H'}$ will be called the "transform" of C_H on f'. Suppose first that f' is obtained from f by a quadratic transformation, i. e. by blowing up a closed point O on f. Let $T:f'\to f$ be the domination map, and let $L'=T^{-1}(O)$ be the exceptional curve. Then, if i is the virtual multiplicity of C_H at O, we define $C'_{H'}$ by $C'=T^{-1}(C)-i\,L'$, $H'=H-i\,O$ where $T^{-1}(C)$ is the so-called total transform of C. Next assume only that the birational map from f' to f is defined everywhere on f' (i. e. f' dominates f). Then the Factorization Theorem of ZARISKI (cf. ZARISKI, 24, § II.1) asserts that *f' is obtained from f by a sequence of quadratic transformations*. Hence by repeated application of the preceding transformation process we obtain the transform of C_H on f'. (One checks that this definition does not depend on the choice of the sequence of quadratic transformations leading from f to f'.) Finally, in the most general case, we can always find a non-singular surface f'' dominating both f and f', and set $\Theta_{f',f}=(\Theta_{f'',f'})^{-1}\circ(\Theta_{f'',f})$. (This is easily seen to be independent of the choice of f''.)

One studies then those properties of C_H which are *invariant* under $\Theta_{f',f}$ for all f'. It is evident that in order to check that a property is invariant, it is enough to consider only the case when f' is obtained from f by a quadratic transformation. What is more, it can be shown that for given C_H there is an f' on which the transform $C'_{H'}$ of C_H is an ordinary divisor on f', i. e. $H'=0$. Thus, *with each invariant property of C_H there is associated a property of ordinary divisors which is invariant under quadratic transformations, and conversely.* For example, if C and D are divisors on a surface f, if f' is obtained from f by a quadratic transformation, and if C' and D' are the total inverse images of C and D on f', then $(C'\cdot D')=(C\cdot D)$ and $\pi(C')=\pi(C)$ (where π is the virtual arithmetic genus). Therefore it is possible to define *invariantly* (and in just one way) the virtual arithmetic genus of a C_H, and the intersection number of two such objects. Explicitly, if $H=\Sigma\,\mu_i\,O_i$, $G=\Sigma\,\nu_i\,O_i$, then

$$\pi(C_H)=\pi(C)-\tfrac{1}{2}\Sigma\,\mu_i(\mu_i-1)\,,$$
$$(C_H\cdot D_G)=(C\cdot D)-\Sigma\,\mu_i\,\nu_i\,.$$

2. There is another interpretation of the theory which may make it seem more natural. For this purpose, one makes use of the ZARISKI-RIEMANN space Z of K/k, which is the set of valuation rings of K/k topologized by taking as basic open sets those of the form U_S, where S is a finite subset of K and

$$U_S=\{z\in Z\mid S\subseteq z\}$$

(cf. ZARISKI-SAMUEL, 1, Ch. VI, § 17). Z is also a ringed space with the structure sheaf $\mathcal{O}_Z$ whose ring of sections over any open subset U of Z is the intersection of all members of U. Thus it makes sense to speak of (locally principal) divisors on Z. For each surface f' as above, and each f'' dominating f', the domination map $f'' \to f'$ (resp. $Z \to f'$) induces the "inverse image" map from the group of divisors $\mathrm{Div}(f')$ to the group of divisors $\mathrm{Div}(f'')$ (resp. $\mathrm{Div}(Z)$). It is easily shown that

$$(*) \qquad \mathrm{Div}(Z) = \varinjlim_{f'} \mathrm{Div}(f') \, .$$

A key fact is that *there is a family of isomorphisms*

$$\Theta_{f'} : \Delta' \to \mathrm{Div}(Z)$$

(f', Δ', as before) *such that, for any projective non-singular surface f'' birationally equivalent to f, we have*

$$\Theta_{f', f''} = (\Theta_{f'})^{-1} \circ \Theta_{f''} \, .$$

Thus a member of Δ' may be viewed as the representative on f' of a divisor on Z, and then its transform on f'' is the representative on f'' of the same divisor. In other words, *the "invariant" theory outlined above is nothing but the theory of divisors on Z.*

The existence of the isomorphisms $\Theta_{f'}$ follows at once from $(*)$ and the fact that any $C'_{H'}$ on f' transforms into an ordinary divisor C'' on a suitable dominating surface: $\Theta_{f'}(C'_{H'})$ is then the inverse image of C'' on Z. We can obtain more explicit descriptions of Θ_f and Θ_f^{-1} in the following way:

For any divisor D on Z and any valuation v of K/k, there exists $x \in K$ such that D is equal to the divisor of x in some neighborhood on Z of the valuation ring of v (D is locally principal). If x' is another such element then $v(x) = v(x')$, and hence we can define $v(D)$ to be $v(x)$. For example, if H and $\Theta_f(H)$ are as above and O is infinitely near to f, with "quadratic sequence"

$$O_1 < O_2 < \cdots < O_r = O$$

then one finds, with

$$m_i = \text{maximal ideal of } O_i \, ,$$
$$\mathrm{ord}_O(m_i) = \min_{y \in m_i}(\mathrm{ord}_O(y)) \, ,$$
$$s_i = \text{virtual multiplicity of } H \text{ at } O_i$$

that

$$(**) \qquad \mathrm{ord}_O(\Theta_f(H)) = -\sum_{i=1}^{r} s_i \, \mathrm{ord}_O(m_i) \, .$$

This formula determines $\Theta_f(H)$ since a divisor on Z is easily seen to be determined by its values at all the prime divisors on f (and $v(\Theta_f(H)) = 0$ for all v of first kind). We may remark here that in the text the integer $\mathrm{ord}_O(m_i)$ is called "the multiplicity of O_i on a branch of lowest order passing through $O_1, O_2, \ldots, O_{r-1}, O$".

With the divisor D we can associate a divisor C on f, namely

$$C = C(D) = \sum_L \operatorname{ord}_L(D) \cdot L$$

where L runs through all integral curves on f and ord_L is the discrete valuation corresponding to L. The divisor

$$\begin{aligned} D_f &= D - (\text{inverse image of } C \text{ on } Z) \\ &= D - (\Theta_f(C)) \end{aligned}$$

may be called the "base part of D, with respect to f".

For any point O on a surface f', we define the *virtual multiplicity of D at O*, $s_O(D)$ in symbol, to be the integer $-\operatorname{ord}_O(D_{f'})$. (For a given point O, this integer does not depend on the choice of f'.) Then one shows that for almost all points O infinitely near to f, $s_O(D)$ vanishes, and, with

$$H = H(D) = \sum_O s_O(D) \cdot O \quad (O \text{ infinitely near to } f)$$

we have

$$\Theta_f(H) = D_f$$

i. e.

$$\Theta_f^{-1}(D) = C_H \qquad (C, H, \text{ as above}).$$

3. While we have dealt only with *divisors and base divisors*, the emphasis in the text is on *linear systems, with base conditions*. To relate the two, we first remark that any non-zero x in K defines a divisor (x) on Z whose corresponding object $\Theta_f^{-1}((x))$ on the surface f is just the usual divisor $\operatorname{div}_f(x)$ of x on f (with zero base divisor). It follows at once that two divisors D_1 and D_2 on Z are linearly equivalent (i. e. $D_1 - D_2 = (x)$ for some x) if and only if, for the corresponding $C_{i,H_i} = \Theta_f^{-1}(D_i)$ $(i = 1, 2)$ we have that C_1 is linearly equivalent to C_2 (in fact $C_1 - C_2 = \operatorname{div}_f(x)$) and $H_1 = H_2$.

We say naturally that $C_H \geqq 0$ if $\Theta_f(C_H) \geqq 0$, i. e. if

(a) $v(C) \geqq 0$ for all prime divisors v of first kind on f; i. e. C is a positive divisor on f, and

(b) $\operatorname{ord}_O(\Theta_f(C)) + \operatorname{ord}_O(\Theta_f(H)) \geqq 0$ for all O infinitely near to f.

In view of (**), (b) simply says that the divisor C "satisfies the base conditions imposed by H".

Thus, corresponding to $|D|$, the set of all positive divisors on Z linearly equivalent to D, there is (with $C_H = \Theta_f^{-1}(D)$) the set of all positive divisors on f which are linearly equivalent to C and satisfy the base conditions imposed by H; this is the set denoted by $|C|_H$ in the text. Our description of $|C|_H$, being in terms of divisors on Z, *is automatically invariant*. In other words, if $C'_{H'}$ is the transform of C_H on a birationally equivalent surface f', then $|C'|_{H'}$ consists precisely of all the transforms of members of $|C|_H$. So we have the notion of "transform of a linear system with base conditions".

4. Next we discuss "effective base divisors" and "proximity inequalities". For any C_H, the set of all x (including $x = 0$) such that $\text{div}_f(x) + C_H \geqq 0$ forms a finite-dimensional vector space V over k. We assume that $V \neq (0)$, i. e. $|C|_H$ is not empty. Then the sheaf $V\mathcal{O}_Z$ is an invertible $\mathcal{O}_Z$-module, so that $V\mathcal{O}_Z = \mathcal{O}_Z(D)$ for some divisor D on Z. Let $C^*_{H^*} = \Theta_f^{-1}(D)$ be the corresponding object on f. The difference $C_H - C^*_{H^*}$ is then $\geqq 0$ and it depends only on $|C|_H$; it is called the "fixed part" (or "unassigned base") of $|C|_H$. $|C|_H$ is "reduced" if it has zero fixed part. For example, the linear system $|C^*|_{H^*}$, whose members are obtained from those of $|C|_H$ by subtracting off the fixed part, is a reduced linear system.

We say that H is an "effective" base divisor on f if there is a divisor C on f such that $|C|_H$ is a reduced linear system.

An effective base divisor is "simple" if it is not a sum of two other non-zero effective base divisors. It can be shown that there is a one-one correspondence $O \leftrightarrow H_O$ between points infinitely near to f and simple base divisors: if $O_1 < O_2 < \cdots < O_r = O$ is as usual, then the virtual multiplicity of H_O at O_i is $\text{ord}_O(m_i)$ ($m_i = m(O_i)$, $i = 1, 2, \ldots, r$), and at all other infinitely near points is zero. Since H_O has virtual multiplicity one at O, the simple base divisors form a *free basis* for the group of all base divisors.

To gain more information about this situation we use the notion of proximity. For any two *distinct* points $O \subsetneq P$ we say that P is "proximate" to O if the valuation ord_O is non-negative on P. If O is infinitely near to f, and H is a base divisor on f, we set

$$e_O(H) = s_O(H) - \sum_P s_P(H)$$

where P runs through all points proximate to O, and $s_O(H)$, $s_P(H)$ are the virtual multiplicities of H at O, P respectively.

Theorem. *For any base divisor H and any point O infinitely near to f let H_O, $e_O(H)$ be as above. Then $e_O(H) = 0$ for almost all O, and*

$$H = \sum_{\text{all } O} e_O(H) \cdot H_O .$$

Moreover, H is an effective base divisor if and only if $e_O(H) \geqq 0$ for all O (i. e. the virtual multiplicities of H satisfy the "proximity inequalities").

5. We mention, in closing, yet another approach, due to Zariski (2; also Zariski-Samuel, 1, Appendix 5), in terms of *complete ideals*. If $\mathcal{I}$ is any coherent sheaf of ideals on f, then $\mathcal{I}\mathcal{O}_Z$ is an invertible $\mathcal{O}_Z$-module i. e. $\mathcal{I}\mathcal{O}_Z = \mathcal{O}_Z(-D)$, where D is a divisor on Z which we denote $\text{div}_Z(\mathcal{I})$. In this way we map the monoid of coherent ideals homomorphically into the group of divisors on Z. It is a fact that every divisor on Z can be represented in the form $\text{div}_Z(\mathcal{I}) - \text{div}_Z(\mathcal{J})$ for suitable $\mathcal{I}$, $\mathcal{J}$. The divisors on Z which correspond to *effective* base divisors on f are precisely those of the form $-\operatorname{div}_Z(\mathcal{I})$ with $\mathcal{I}$ such that $\mathcal{O}_f : \mathcal{I} = \mathcal{O}_f$.

For given $\mathcal{I}$, there is a largest (in the sense of inclusion) coherent sheaf $\mathcal{I}'$ among those $\mathcal{J}$ such that $\operatorname{div}_Z(\mathcal{I}) = \operatorname{div}_Z(\mathcal{J})$. Such an $\mathcal{I}'$ is said to be "complete". The complete coherent ideals form a monoid with product $\mathcal{I}' * \mathcal{J}' = (\mathcal{I}'\mathcal{J}')'$. (Actually, f being non-singular, ZARISKI shows that $\mathcal{I}'\mathcal{J}' = (\mathcal{I}'\mathcal{J}')'$, i. e. *the product of complete ideals is complete.*) This monoid maps *injectively* into the group $\operatorname{Div}(Z)$, and its image generates $\operatorname{Div}(Z)$. Thus, *divisors on Z can be thought of as formal differences of complete ideals on f.*

The preceding theorem is a geometric counterpart of ZARISKI's theorem that *every complete ideal on f is in a* unique *way the product of simple complete ideals.*

Further remarks on § 1. ZARISKI and SCHILLING (1) prove by valuation-theoretic methods that on any surface F, an irrational pencil can have base points only at singular points of F, slightly generalizing the result stated in the text and extending it to char p. ZARISKI (5), (9), and (24) studied the 2 BERTINI theorems algebraically, and considered their extension to char p. It turns out that the 1st one is false in char p, except for instance for very simple linear systems like the system of hyperplane sections of a non-singular surface; but that the 2nd is true in the slightly weakened form — a reducible linear system is either the set of divisors $p^n D$, where D moves in an irreducible linear system, or else it is composed of the curves of a pencil.