STABLE IDEALS AND ARF RINGS.

By Joseph Lipman.*

Contents

Introduction. This paper deals (except in § 4) with one-dimensional commutative rings. It originated with the observation that if A is a complete one-dimensional local domain with an algebraically closed residue field of characteristic zero and if A is *saturated* (cf. § 5) then the embedding dimension of A is equal to the multiplicity of A. The proof was based on the fact that if A is saturated then A is an "Arf ring," that is to say A satisfies a certain condition studied in detail some twenty five years ago by C. Arf [1]. The next observation was that the condition of Arf made sense for arbitrary one-dimensional local rings, and that it still implied the equality of embedding dimension and multiplicity. This naturally raised the question whether *any* one-dimensional saturated local ring satifies Arf's condition, and this question was answered in the affirmative by Zariski (cf. § 5).

The first two sections are an outgrowth of the above observations. We work throughout with a one-dimensional *semi-local* Macaulay ring A. (It is inconvenient in practice to be restricted to *local* rings.) An open ideal I in A is "stable" if the length of A/I^n is given by the characteristic polynomial of I for *all* integers $n > 0$ (cf. Corollary 1.6). In Theorem 1.9 we find, for fixed $n > 0$, that I^n is stable if and only if the length λ_n of I^n/I^{n+1} is equal to the multiplicity μ of the ideal I. (For any $n \geqq 0$, the inequality $\lambda_n \leqq \mu$ holds, and moreover if $\lambda_n = \mu$, then $\lambda_m = \mu$ for all $m \geqq n$.) In particular, if A is local then the maximal ideal of A is stable if and only if the embedding dimension and multiplicity of A are equal (Corollary 1.10). In 1.3 and 1.5 we interpret the characteristic polynomial and the stability

Received December 18, 1970.

* Supported by National Science Foundation grant GP 9567.

of I in terms of the ring A^I obtained by blowing up I; the principal results along these lines (Proposition 1.1 and Theorem 1.5) generalize some parts of Northcott's paper [5]. There are also some more technical, but useful, facts about stability involving "I-transversal" elements (cf. 1.7, 1.8, 1.11), and some remarks concerning the behavior of stability under various types of ring extension (cf. 1.4, 1.12).

In § 2, the main result (Theorem 2.2) is that A is an Arf ring (Definition 2.1) if and only if A satisfies one of the following (equivalent) conditions:

(i) Every integrally closed open ideal in A is stable.

(ii) For every local ring B infinitely near to A, the embedding dimension of B is equal to the multiplicity of B.

(The definitions of "integrally closed ideal" and "infinitely near" are reviewed in § 2.)

§ 3 is a direct generalization of some (but not all) of the theory developed by Arf in [1]. Assume for simplicity that A is a complete one-dimensional local domain. (We assume less in § 3.) Among all Arf rings between A and its integral closure $\bar{A}$ there is a smallest, denoted A' and called the "Arf closure" of A. The basic facts about this closure operation are:

(1) "Arf closure" commutes with "quadratic transform" (i. e. blowing up of the maximal ideal) (Theorem 3.5).

(2) A' is a local ring with the same residue field and the same multiplicity as A (Theorem 3.4).

(3) The embedding dimension of A' is equal to the multiplicity of A'. (This is a part of the above mentioned Theorem 2.2.)

A consequence of (1) and (2) is that A and A' have the same multiplicity sequence (Corollary 3.7; the definition of "multiplicity sequence" precedes Proposition 3.6). Moreover there is an inequality for the length of the A-module $\bar{A}/A$ in terms of the multiplicity sequence of A (Theorem 3.9) which, because of (1), (2), (3), becomes an equality if and only if $A = A'$; and it follows that for any A, A' is the *largest* ring between A and $\bar{A}$ having the same multiplicity sequence as A (Corollary 3.10). (We may remark here that the multiplicity sequence is one of the most important "geometric invariants" associated with A.)

Corollary 3.8 is a characterization of Arf rings by means of the "semigroup of values".

In § 4 we turn to another aspect of the theory. With A and $\bar{A}$ as above, consider the subring A^* of $\bar{A}$ consisting of those elements x such that $x\otimes 1=1\otimes x$ in $\bar{A}\otimes_A\bar{A}$. (Thus $\operatorname{Spec}(A^*)$ is obtained from $\operatorname{Spec}(\bar{A})$ by factoring out the equivalence relation defined by the map $\operatorname{Spec}(\bar{A})\to\operatorname{Spec}(A)$). It had been suggested by Grothendieck that A^* might serve the same purpose as the saturation $\tilde{A}$ of A. Zariski showed that this was not so by giving an example of two local rings A_1 and A_2 belonging to *equivalent* singularities of plane curves such that $A_1{}^*$ and $A_2{}^*$ are not isomorphic (whereas local rings of equivalent singularities have *isomorphic* saturations). Zariski also showed that, always, $A'\subseteq A^*\subseteq\tilde{A}$ where A' is, as above, the Arf closure of A (Propositions 4.5 and 5.1); and he conjectured that $A'=A^*$. The central result of § 4 is that this is so if A contains a field (Corollary 4.8). The question remains open in the mixed characteristic case.

Actually, the construction of A^* (unlike that of A') applies to arbitrary commutative rings A, and some of the results in § 4, notably Theorems 4.2 and 4.11, are not restricted to the one-dimensional case.

In § 5 we review the definitions needed to discuss saturated rings, and then indicate the relation between Arf rings and saturated rings. In addition to the results of Zariski mentioned before, we give an example showing that the Arf closures of local rings belonging to equivalent singularities can vary in continuous families. (This raises some interesting questions about the moduli of Arf rings, which we do not pursue.)

This paper was inspired by discussions with Professor Zariski during his visit to Purdue University in the spring of 1970. I wish to express my appreciation both for the motivation and for the many useful suggestions which he provided.

1. Stable ideals in one-dimensional semi-local rings. *In §§ 1-3 A will be a commutative semi-local noetherian ring of Krull dimension one; we assume also that the radical M* (=intersection of the maximal ideals of A) *contains a regular element* (non-zerodivisor) *of A, i. e. A is a Macaulay ring. $\bar{A}$ will be the integral closure of A in its total ring of fractions.* Note that if B is a ring between A and $\bar{A}$ such that B is a finitely generated A-algebra, then B is also a semi-local one-dimensional Macaulay ring.

The length of an A-module E will be denoted $\lambda(E)$, or, if necessary, $\lambda_A(E)$. An ideal I in A is *open* if I contains a regular element of A, or, equivalently, $M^n\subseteq I$ for some $n>0$, or, equivalently, $\lambda(A/I)$ is finite.

The following proposition describes the ring A^I obtained by *blowing up* an open ideal I in A. First, some more notation. For any two A-submodules E, F of $\bar{A}$ set

$$E:F=\{y\in\bar{A}\mid yF\subseteq E\}.$$

If I is an ideal in A, then $I:I$ is a subring of $\bar{A}$, and for any $n \geqq 0$, $I^n : I^n \subseteq I^{n+1} : I^{n+1}$.

PROPOSITION 1.1. *Let I be an open ideal in A and let A^I be the ring $\bigcup_{n>0} I^n : I^n$. Then:*

(i) *A^I is a finitely generated A-module, and $A^I = I^n : I^n$ for all sufficiently large n.*

(ii) *$IA^I = xA^I$ for some regular element x in A^I.*

(iii) *If B is any ring between A and $\bar{A}$ such that IB is a principal ideal in B, then $A^I \subseteq B$.*

Proof. We first prove (iii). If IB is principal, say $IB = wB$, then w is a regular element in B because I contains an element which is regular in A (hence also in B). If $yI^n \subseteq I^n$ $(y \in \bar{A})$, then $yI^nB \subseteq I^nB$, i.e. $yw^n \in w^nB$, so that $y \in B$ (since w^n is regular in $\bar{A}$). Thus $A^I \subseteq B$.

Next we note that in proving (i), we may replace I by any one of its powers I^s $(s > 0)$. Once (i) has been proved, we know that A^I is semi-local, and so (ii) becomes equivalent to: there exists an A^I-submodule J of the total ring of fractions of A such that $IJ = A^I$ (cf. for example [2, Chap. 2; p. 148, Thm. 4, and p. 143, Prop. 5]). Hence if I^sA^I is principal, i.e. $I^sJ' = A^I$ for some J', then $I(I^{s-1}J') = A^I$, so that IA^I is principal. In other words, after proving (i) we may also replace I by I^s in proving (ii).

Now *there exist an integer $s > 0$ and an element $x \in I^s$ such that $xI^r = I^{r+s}$ for all sufficiently large integers r.* [Since A is one-dimensional, there exists $x \in A$ whose leading form $\bar{x}$ in the graded ring $G = \bigoplus_{n \geqq 0} I^n/MI^n$ is of positive degree, say s, and such that $\bar{x}G$ is an irrelevant ideal in G (i.e. $\sqrt{\bar{x}G}$ contains all homogeneous elements of positive degree); then for all large r, $I^{r+s}/MI^{r+s} \subseteq \bar{x}G$, i.e. $I^{r+s}/MI^{r+s} = \bar{x}(I^r/MI^r)$, i.e. $I^{r+s} = xI^r + MI^{r+s}$, so by Nakayama's lemma $I^{r+s} = xI^r$.] Replacing I by I^s, we may assume that $s = 1$.

Note that x is regular, since for large r, $xA \supseteq xI^r = I^{r+1}$. Any member of A^I, being in $I^n : I^n$ for some n, is of the form z/x^n, $z \in I^n$; and conversely, for any $n > 0$ and any $z \in I^n$ we have, for sufficiently large r,

$$(z/x^n)I^{r+n} = (z/x^n)x^nI^r = zI^r \subseteq I^{r+n}$$

so that $z/x^n \in \bar{A}$ (cf. remarks (a) and (b) near the beginning of § 2), and hence $z/x^n \in I^{r+n} : I^{r+n} \subseteq A^I$. Thus

$$A^I = \{\frac{z}{x^n} \mid n > 0, z \in I^n\}$$

$$\text{i.e.}\quad A^I = A\left[\frac{z_1}{x}, \frac{z_2}{x}, \cdots, \frac{z_t}{x}\right] \qquad (*)$$

where $\{z_1, z_2, \cdots, z_t\}$ is any set of generators of the ideal I in A. So A^I ($\subseteq \bar{A}$) is a finitely generated module over the noetherian ring A, and since

$$I : I \subseteq I^2 : I^2 \subseteq \cdots \subseteq I^n : I^n \subseteq \cdots$$

is an ascending sequence of A-submodules of A^I, (i) is proved. The preceding expression (*) for A^I also makes it evident that $IA^I = xA^I$. Q.E.D.

COROLLARY 1.2. *Let $f: A \to B$ be a ring homomorphism, where B is also a one-dimensional semi-local Macaulay ring. Assume that if x is regular in A then $f(x)$ is regular in B. Then if I is an open ideal in A (so that IB is an open ideal in B) there is a unique extension of f to a homomorphism $f^I: A^I \to B^{IB}$. The corresponding map*

$$f^I{}_{(B)} : B \otimes_A A^I \to B^{IB}$$

is surjective, and $f^I{}_{(B)}$ is an isomorphism if B is flat over A.

Proof. f extends uniquely to a homomorphism $f_T: T_A \to T_B$ where T_A, T_B are the total rings of fractions of A, B respectively. If f^I exists, it *must* be obtained by restricting f_T to A^I; the expression (*) for A^I (and the similar expression for B^{IB}) shows that $f_T(A^I) \subseteq B^{IB}$, so f^I does indeed exist (uniquely). Again (*) shows that $f^I{}_{(B)}$ is surjective. The final assertion follows from the fact that $B \otimes_A T_A$ can be identified via f_T with a subring of T_B. ($B \otimes_A A^I \to B \otimes_A T_A$ is injective if B is flat.) Q.E.D.[1]

Definition 1.3. An open ideal I of A is **stable** (in A) if $A^I = I : I$, or, equivalently, $IA^I = I$.

COROLLARY 1.4. (i) *I^n is stable for some $n > 0$.*

(ii) *If I^n is stable, then I^m is stable for all $m \geqq n$.*

(iii) *Let $f: A \to B$ be as in Corollary 1.2. If I is stable in A, then IB is stable in B; and the converse is true if B is faithfully flat over A.*

(iv) *I is stable in A if and only if IA_P is stable in A_P for all maximal ideals P of A.*

Proof. (i) follows from (i) of Proposition 1.1 since clearly $A^I = A^{I^n}$ for any $n > 0$. Similarly (ii) holds because $I^nA^I = I^n \Rightarrow I^mA^I = I^m$ for

[1] 1.1 and 1.2 could be deduced from the general theory of blowing-up, but it seems more reasonable, since we are working only with one-dimensional rings, to proceed as we have. In any case, these results are more or less "well-known."

$m \geqq n$. The first assertion of (iii) holds because, in view of Corollary 1.2, $IA^I = I \Rightarrow (IB)B^{IB} = IB$. The second assertion of (iii) holds because when B is faithfully flat over A we have (in view of Corollary 1.2) $IB = (IB)B^{IB}$ if and only if the canonical map $I \otimes_A B \to IA^I \otimes_A B$ is an isomorphism, i.e. if and only if the inclusion map $I \to IA^I$ is an isomorphism. Similarly IA_P is stable in A_P for all maximal ideals P if and only if the canonical map $I \otimes_A A_P \to IA^I \otimes_A A_P$ is an isomorphism for all P, i.e. if and only if $I \to IA^I$ is an isomorphism. Q.E.D.

Let I, A^I be as above, and set

$$\mu = \mu(I) = \lambda_A(A^I/IA^I)$$
$$\nu = \nu(I) = \lambda_A(A^I/A).$$

THEOREM 1.5. *With the preceding notation we have, for all $n > 0$,*

$$\lambda(A/I^n) \geqq \mu n - \nu$$

with equality if and only if I^n is stable.

Remark. Since I^n is stable for all large n (Corollary 1.4), Theorem 1.5 shows that the *characteristic polynomial* [7; p. 285] of the ideal I is $\mu n - \nu$; in particular, $\mu(I)$ *is the multiplicity of the ideal* I (a special case of the well-known "projection formula" [4; §(23.4)]).

Proof. Let $IA^I = xA^I$ with x regular in A^I (Proposition 1.1). Then for all $t > 0$, we have

$$I^tA^I/I^{t+1}A^I = x^tA^I/x^{t+1}A^I \cong A^I/xA^I$$

(isomorphic A-modules) and it follows that

$$\lambda(A^I/I^nA^I) = \sum_{t=0}^{n-1} \lambda(I^tA^I/I^{t+1}A^I) = \mu n.$$

Hence we have

$$\lambda(A^I/I^n) = \nu + \lambda(A/I^n) = \mu n + \lambda(I^nA^I/I^n)$$

so that

$$\lambda(A/I^n) - (\mu n - \nu) = \lambda(I^nA^I/I^n) \geqq 0$$

with equality if and only if $I^nA^I = I^n$. Q.E.D.

COROLLARY 1.6. *I^{n_0} is stable if and only if for all $n \geqq n_0$.*

$$\lambda(I^n/I^{n+1}) = \mu(I).$$

Proof. If I^{n_0} is stable and $n \geqq n_0$ then I^n and I^{n+1} are stable, and

1.5 shows that $\lambda(A/I^{n+1})-\lambda(A/I^n)=\mu(I)$ as required. Conversely if $\lambda(I^n/I^{n+1})=\mu$ for all $n\geqq n_0$, and if $\lambda(A/I^{n_0})=\mu n_0-\nu_0$, then the characteristic polynomial of I is $\mu n-\nu_0$, whence $\nu_0=\nu$, and so I^{n_0} is stable.

Q. E. D.

The next Theorem (1.9) improves on Corollary 1.6. It will be convenient first to introduce some more terminology.

Definition 1.7. An element x of the open ideal I in A is **I-transversal** if $xI^n=I^{n+1}$ for some integer $n>0$.[2]

Lemma 1.8(i) below (along with (ii) of Proposition 1.1) gives the existence of I-transversal elements whenever A^I is a *local* ring.

In the general case we see, as in the proof of Proposition 1.1, that x *is I-transversal if and only if $x\in I$ and the image of x in I/MI generates an irrelevant ideal in the graded ring* $\bigoplus_{n\geqq 0} I^n/MI^n$. Thus the argument given in [4; §(22.1)] shows that, *if A/P is infinite for each maximal ideal P in A, then there exist I-transversal elements for every open ideal I in A.*

(Actually it is easy to see that x is I-transversal if and only if: x is a superficial element of I (cf. [4; §22]) and $\sqrt{xA}=\sqrt{I}$.)

LEMMA 1.8. (i) *An element x in I is I-transversal if and only if* $xA^I=IA^I$.

(ii) *If x is I-transversal and $n>0$ then $xI^n=I^{n+1}$ if and only if I^n is stable.*

Proof. If $xI^n=I^{n+1}$, then $xI^nA^I=I^{n+1}A^I$, and since IA^I is generated by a regular element of A^I (Proposition 1.1) it follows that $xA^I=IA^I$. Conversely if $xA^I=IA^I$ and I^n is stable, i.e. $I^nA^I=I^n$, then

$$xI^n=xI^nA^I=I^n(IA^I)=I^{n+1},$$

proving (i) (cf. Corollary 1.4(i)) and the "if" part of (ii). Finally if $xI^n=I^{n+1}$ then by induction $I^{n+r}=x^rI^n$ for all $r>0$, and since x is regular $(xA\supseteq I^{n+1})$, we have

$$A^I=\bigcup_{r>0} I^{n+r}:I^{n+r}=I^n:I^n$$

i.e. I^n is stable. Q. E. D.

THEOREM 1.9. *If I is an open ideal in A, with multiplicity $\mu(I)$, then for all $n\geqq 0$ we have*

[2] Cf. remark (a) following Corollary 1.10 for a clarification of the term "transversal."

$$\lambda(I^n/I^{n+1}) \leqq \mu(I),$$

and for $n > 0$ *equality holds if and only if* I^n *is stable.*

Proof. Let $P_1, P_2, \cdots, P_t$ be the maximal ideals of A. Following a suggestion of M. Nagata, we reduce to the case where A/P_i is infinite (for all i) in the standard way: let X be an indeterminate, let S be the multiplicative system

$$S = A[X] - \bigcup_{i=1}^{t} P_i A[X]$$

and let $A(X)$ be the ring of fractions

$$A(X) = A[X]_S.$$

Then $A(X)$ is a semi-local ring with maximal ideals $P_iA(X)$ $(i = 1, 2, \cdots, t)$, and $A(X)/P_iA(X)$ is isomorphic to the field of rational functions of one variable over A/P_i. Moreover, for any $n \geqq 0$, since $A(X)$ is flat over A, and since the maximal ideals of A generate maximal ideals of $A(X)$, we have

$$\lambda_A(I^n/I^{n+1}) = \lambda_{A(X)}(I^nA(X)/I^{n+1}A(X))$$

i.e. I and $IA(X)$ have the same characteristic function; in particular $\mu(I) = \mu(IA(X))$, and $A(X)$ is one-dimensional. Finally I^n is stable if and only if $[IA(X)]^n$ is stable because of Corollary 1.4(iii) (or Corollary 1.6).

Thus, and in view of the remarks following Definition 1.7, we may assume that there exists an I-transversal element x. x is regular (since xA contains some power of I) and therefore multiplication by x gives an isomorphism of A-modules $A/I^n \xrightarrow{\approx} xA/xI^n$ $(n \geqq 0)$. Hence

$$\begin{aligned}\lambda(A/xA) &= \lambda(A/xA) + \lambda(xA/xI^n) - \lambda(A/I^n)\\ &= \lambda(A/xI^n) - \lambda(A/I^n)\\ &= \lambda(I^n/xI^n)\\ &\geqq \lambda(I^n/I^{n+1}) \qquad \text{(since } xI^n \subseteq I^{n+1}\text{)}\end{aligned}$$

with equality if and only if $xI^n = I^{n+1}$. Since $xI^n = I^{n+1}$ for large n, we have $\lambda(A/xA) = \mu(I)$, and in view of Lemma 1.8(ii), the Theorem is proved. Q. E. D.

If A is a local ring, the *embedding dimension* of A, emdim(A), is $\lambda(M/M^2)$. The *multiplicity* of A, which we can denote without fear of confusion by $\mu(A)$, is, by definition, $\mu(M)$. For $I = M$, $n = 1$ in Theorem 1.9, we have:

COROLLARY 1.10. *If A is* local *then*

$$\operatorname{emdim}(A) \leqq \mu(A)$$

with equality if and only if the maximal ideal M is stable.

Remarks (a). From the last part of the proof of 1.9, we obtain:

An element x in I is I-transversal if and only if x is regular and

$$\mu(I) = \lambda(A/xA) \quad (= \mu(xA)).$$

(b). The following fact will be used repeatedly in the sequel:

If x is a regular element in A and J is a finitely generated A-submodule of $\bar{A}$ containing a regular element of $\bar{A}$, then

$$\lambda(A/xA) = \lambda(J/xJ).$$

(*Proof.* For a suitable y regular in A, we have that $I = yJ$ is an open ideal in A, and clearly $\lambda(J/xJ) = \lambda(I/xI)$; but in proving 1.9 we have seen that $\lambda(A/xA) = \lambda(I/xI)$).

The following criteria, not depending on Theorems 1.5 and 1.9, will be very useful:

LEMMA 1.11. *An open ideal I in A is stable if and only if there exists an element x in I satisfying one of the following equivalent conditions:*

(i) $I^2 = xI$.

(ii) *x is regular and Ix^{-1} is a ring.*

(iii) *x is regular and $Ix^{-1} = A^I$.*

Moreover, if I is stable and x is any *I-transversal element, then* (i), (ii), (iii) *hold.*

Proof. We first check the equivalence of (i), (ii), and (iii). If $I^2 = xI$, then x is regular (since $Ax \supseteq I^2$) and $(Ix^{-1})^2 = Ix^{-1}$, i.e. Ix^{-1} is closed under multiplication, so that (i) $\Rightarrow$ (ii). Conversely if $(Ix^{-1})^2 = Ix^{-1}$, then $I^2 = xI$, i.e. (ii) $\Rightarrow$ (i). Obviously (iii) $\Rightarrow$ (ii). Finally, if (i) holds, then $xI^r = I^{r+1}$ for all $r \geqq 1$, and the proof of Proposition 1.1 shows that A^I is the least subring of $\bar{A}$ containing Ix^{-1}; since (i) $\Rightarrow$ (ii), Ix^{-1} is itself a ring, so $A^I = Ix^{-1}$; thus (i) $\Rightarrow$ (iii).

Now $IA^I = xA^I$ for some x in A^I (Proposition 1.1); and if I is stable then $IA^I = I$ so that $x \in I$; in other words, there exists an I-transversal element (Lemma 1.8(i)). The rest follows from Lemma 1.8(ii) (with $n = 1$). Q.E.D.

We conclude this section with an application of Theorem 1.9.

PROPOSITION 1.12. *Let $p_1, p_2, \cdots, p_h$ be the associated prime ideals of (0) in A; for each $i = 1, 2, \cdots, h$ let $A_i = A/p_i$, and let e_i be the length of the artinian local ring A_{p_i}. Let I be an open ideal in A and for each i let $I_i = IA_i$ (so that I_i is an open ideal in A_i). Then I is stable in A if and only if: I_i is stable in A_i for each i and*

$$\lambda_A(I/I^2) = \sum_{i=1}^{h} \lambda_{A_i}(I_i/I_i^2) \cdot e_i. \tag{**}$$

Proof. If I is stable then so is I_i by Corollary 1.4(iii) (or else by Lemma 1.11(i)). Furthermore we have $\lambda_A(I/I^2) = \mu(I)$ (Corollary 1.6 or Theorem 1.9) and similarly $\lambda_{A_i}(I_i/I_i^2) = \mu(I_i)$ (where $\mu(I_i)$ is the multiplicity *with respect to* A_i of the ideal I_i). But by [4; §(23.5)] we have

$$\mu(I) = \sum_{i=1}^{h} \mu(I_i) \cdot e_i$$

so that (**) holds.

Conversely, if each I_i is stable and (**) holds, then we conclude that $\lambda_A(I/I^2) = \mu(I)$ so that I is stable (Theorem 1.9). Q. E. D.

2. Arf rings. After some technical preliminaries, we recall the definition, suitably generalized, of a class of rings studied by Arf in [1] (cf. also [3] for some geometric interpretations); we will call such rings "Arf rings." The defining condition, while not very exciting in itself, is convenient to work with. Theorem 2.2 gives some more interesting characterizations of Arf rings. We will also see that Arf rings behave well with respect to some standard types of ring extension (cf. 2.5, 2.7, 2.8, 2.9).

We will need some basic properties of integral dependence over *ideals*, which we review briefly before proceeding. Notation remains as in §1. Recall that an element z in A is said to be **integrally dependent on the ideal I** if z satisfies a relation

$$z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0 \qquad (n > 0)$$

with $a_j \in I^j$ for $j = 1, 2, \cdots, n$. It is equivalent to say that in the polynomial ring $A[T]$, the element zT is integral over the graded subring

$$A[IT] = A \oplus IT \oplus I^2T^2 \oplus \cdots.$$

From this latter condition it follows at once (by well-known properties of integral dependence over *rings*) that the set $\bar{I}$ consisting of all elements in A which are integral over I is an ideal in A. $\bar{I}$ is called the **integral closure of I** in A. We have the usual closure properties:

(i) $I \subseteq \bar{I}$ (obvious).

(ii) $I \subseteq J \Rightarrow \bar{I} \subseteq \bar{J}$ (obvious).

(iii) $\bar{\bar{I}} = \bar{I}$ (because every element of $\bar{I}T$, and hence of $A[\bar{I}T]$, is integral over $A[IT]$).

I is said to be **integrally closed** (in A) if $I = \bar{I}$. For any ideal I, $\bar{I}$ is the smallest integrally closed ideal in A containing I.

Remarks. Without fear of confusion, we will continue to denote by $\bar{A}$ the integral clousure of A in its total ring of fractions (*not* the integral closure of the unit ideal in A!).

(a) *If x is regular in A and z is integral over the ideal xA then $z/x \in \bar{A}$.*

(To see this divide an equation of integral dependence of z over xA

$$z^n + b_1 x z^{n-1} + b_2 x^2 z^{n-2} + \cdots + b_n x^n = 0 \qquad (b_j \in A, 1 \leqq j \leqq n)$$

by x^n).

Conversely, *if $z/x \in \bar{A}$ then z is integral over xA.*

(b) *If $z \in A$, J is an ideal of A, and $zI \subseteq JI$, where I is some ideal of A containing a regular element, say x, then z is integral over J.*

(Indeed, for any finite basis $\{x_1, x_2, \cdots, x_t\}$ of I we have

$$zx_i = \sum_{j=1}^{t} c_{ij} x_j \qquad (i = 1, 2, \cdots, t; c_{ij} \in J)$$

i.e.

$$0 = \sum_{j=1}^{t} (c_{ij} - z\delta_{ij}) x_j \qquad (\text{Kronecker } \delta_{ij})$$

whence $dx_j = 0$ $(1 \leqq j \leqq t)$ where d is the determinant

$$d = \det(c_{ij} - z\delta_{ij}).$$

Hence $dx = 0$, i.e. $d = 0$, and this is an equation of integral dependence for z over J).

(c) *Let $J \subseteq I$ be two ideals of A, where I contains a regular element of A. Then $I \subseteq \bar{J}$ if and only if*

$$I^{n+1} = JI^n$$

for some integer $n \geqq 0$.

(For, if $I^{n+1} = JI^n$ and $z \in I$, then

$$zI^n \subseteq I^{n+1} = JI^n$$

whence z is integral over J by remark (b)

Conversely, if $I \subseteq \bar{J}$ and $\{x_1, x_2, \cdots, x_t\}$ is a basis of I, then an equation of integral dependence for x_i over J shows that, for some integer n_i,

$$x_i^{n_i} \in JI^{n_i - 1} \qquad (i = 1, 2, \cdots, t).$$

It follows easily that any monomial $x_1^{m_1} x_2^{m_2} \cdots x_t^{m_t}$ of degree

$$m = m_1 + m_2 + \cdots + m_t > n_1 + n_2 + \cdots + n_t - t$$

lies in JI^{m-1} (since $m_i \geqq n_i$ for at least one i); consequently

$$I^m = JI^{m-1}$$

for any $m > n_1 + n_2 + \cdots + n_t - t$).

(d) As a particular case of (c) we have:

An element x of an open ideal I in A is I-transversal (Definition 1.7) *if and only if every element of I is integrally dependent on xA, i.e.* (remark (a)) : *x is regular and $z \in I \Rightarrow z/x \in \bar{A}$.*

(e) (This will be used only in the proof of Proposition 4.5).

Let A' be an A-algebra such that, if B is any A-algebra and C is a B-algebra in which B is integrally closed, then also $B \otimes_A A'$ is integrally closed in $C \otimes_A A'$. If I is an ideal in A with integral closure $\bar{I}$ in A, then the integral closure of $I' = IA'$ in A' is $\bar{I}A'$.

(The condition on A' is satisfied, for example, if $A' = A_S$ where S is a multiplicative system in A [2, Chap. 5; p. 22, Prop. 16], or if A' is a polynomial ring over A [2, Chap. 5; p. 19, Prop. 13]).

(*Proof.* Let $C = A[T]$, $B_I = A[IT]$. The integral closure of B_I in C is a graded ring $B = A \oplus \bar{I}T \oplus \cdots$ [2, Chap. 5; p. 30, Prop. 20]. It follows that the integral closure of $A'[I'T]$ in $A'[T]$ is $A' \oplus \bar{I}A'T \oplus \cdots$).

Definition 2.1. Let A be as in §1. A is an **Arf ring** if there exists an I-transversal element for every integrally closed open ideal I in A (cf. remarks following Definition 1.7) and if the following condition is satisfied:

(#) whenever x, y, z in A are such that x is regular and both y and z are integral over xA (i.e. $y/x, z/x \in \bar{A}$, cf. remark (a) above), then $yz \in xA$.

The usefulness of this definition will become more apparent as we proceed. There are many examples of Arf rings in [1]. For the moment, we give only a counterexample: it can be checked that if F_2 is the field with two elements, then the complete local ring $A=F_2[[X,Y]]/XY(X+Y)$ satisfies condition (#), but there is no M-transversal element, M being the maximal ideal.

We need also to recall the notion of a local ring **infinitely near** to A. Define the sequence of semilocal rings

$$A=A_0\subseteq A_1\subseteq A_2\subseteq\cdots\subseteq\bar{A}$$

by taking, for each $i\geqq 0$, A_{i+1} to be the ring obtained from A_i by blowing up the radical of A_i. A local ring of the form $(A_i)_P$, where P is a maximal ideal in A_i, is said to be infinitely near to A.

THEOREM 2.2. *The following conditions on A are equivalent:*

(i) *A is an Arf ring.*

(ii) *Every integrally closed open ideal in A is stable.*

(iii) *If B is any local ring infinitely near to A, then the embedding dimension of B is equal to the multiplicity of B.*

Proof. **(i)$\Rightarrow$(ii).** Suppose that A is an Arf ring, let I be an integrally closed open ideal in A, and let x be an I-transversal element. We must show that Ix^{-1} is a ring (Lemma 1.11). But if y/x, z/x are two members of Ix^{-1}, then $y/x, z/x\in\bar{A}$ (remark (d) above) and so $yz\in xA$, i.e.

$$\frac{y}{x}\cdot\frac{z}{x}=\frac{w}{x}\qquad (w\in A).$$

Since $w/x\in\bar{A}$ and I is integrally closed, remark (a) shows that $w\in I$, and so Ix^{-1} is closed under multiplication, as required.

(ii)$\Rightarrow$(i). If (ii) holds, and J is an integrally closed open ideal in A, then $J^2=wJ$ for some w in J (Lemma 1.11) and this w is J-transversal. If x, y, z are now any elements of A, with x regular, and if I is the integral closure of xA, then x is I-transversal (remark (d)), and $I^2=xI$ (Lemma 1.11); hence if both y and z are integral over xA, then $yz\in I^2\subseteq xA$.

(ii)$\Rightarrow$(iii). Because of Corollary 1.10 it is enough to show that if (ii) holds in A, then (ii) holds in every local ring B infinitely near to A. It is even enough to show that (ii) holds in each of the rings A_i appearing in the definition of "infinitely near" (preceding Theorem 2.2); for, B is a localization of such an A_i, and so if J is an integrally closed open ideal in B, then $J=IB$ where I is the inverse image of J in A_i; but I is easily seen to be integrally closed and open in A_i, and if I is stable then so is J, because of Lemma 1.11(i) (or Corollary 1.4(iii)). By induction, we are reduced to showing that: if (ii) holds in A_j then (ii) also holds in A_{j+1}. Now clearly the radical of A_j is integrally closed in A_j, and so the conclusion results from the following:

LEMMA 2.3. *Let A be as usual, let I be a stable integrally closed open ideal in A, and let x be an I-transversal element, so that $A^I = Ix^{-1}$ (Lemma 1.11). Then $J \leftrightarrow Jx^{-1}$ is a one-one correspondence between integrally closed open A-ideals $J \subseteq I$ and integrally closed open A^I-ideals, and such a J is stable in A if and only if Jx^{-1} is stable in A^I. In particular, if condition* (ii) *of Theorem 2.2 holds in A then it also holds in A^I.*

Proof. We first show that $(Ix^{-1})(Jx^{-1}) \subseteq Jx^{-1}$ when J is as in Lemma 2.3; it follows then that Jx^{-1} is an ideal in $A^I = Ix^{-1}$. Clearly $(Ix^{-1})(Jx^{-1}) \subseteq Ix^{-1}$, so $(Ix^{-1})J \subseteq I \subseteq A$. Moreover if $y \in Ix^{-1}$ and $z \in J$, then it is immediate (since $y \in \bar{A}$) that yz is integral over $zA \subseteq J$, and since J is integrally closed in A, $yz \in J$. Thus $(Ix^{-1})J \subseteq J$, i.e. $(Ix^{-1})(Jx^{-1}) \subseteq Jx^{-1}$.

Jx^{-1} is an open ideal in Ix^{-1} since J (and hence Jx^{-1}) contains a regular element. Also, if $y \in Ix^{-1}$ and y is integral over Jx^{-1}, then $yx \in I \subseteq A$ and yx is integral over J, so $yx \in J$ and $y \in Jx^{-1}$; in other words Jx^{-1} is integrally closed in Ix^{-1}.

Conversely, let J' be an integrally closed open ideal in Ix^{-1}. Then obviously $J'x$ is an open ideal in A, $J'x \subseteq I$. If $y \in A$ is integral over $J'x$ then $y \in I$ (since $J'x \subseteq I$ and I is integrally closed) and hence yx^{-1} is an element of Ix^{-1} which is integral over J', i.e. $yx^{-1} \in J'$, i.e. $y \in J'x$; thus $J'x$ is integrally closed.

Finally J is stable in A if and only if $J^2 = wJ$ for some w in J (Lemma 1.11), i.e. if and only if $(Jx^{-1})^2 = (wx^{-1})(Jx^{-1})$, i.e. if and only if Jx^{-1} is stable. This completes the proof of Lemma 2.3.

(iii) $\Rightarrow$ (ii). Let J be an integrally closed open ideal in A; assuming that (iii) holds for A we want to prove that J is stable. We may as well assume that $J \neq A$. Let I be a maximal ideal in A containing J; in view of Corollaries 1.4(iv) and 1.10, condition (iii) of Theorem 2.2 (with $B = A_I$, the localization of A at I) entails that I is stable. If x is an I-transversal element, then, as in Lemma 2.3, Jx^{-1} is an ideal in $A^I = Ix^{-1}$, and if Jx^{-1} is stable in A^I then J is stable in A. Furthermore we have

$$\begin{aligned} \lambda_{A^I}(A^I/Jx^{-1}) &\leqq \lambda_A(Ix^{-1}/Jx^{-1}) \\ &= \lambda_A(I/J) \\ &< \lambda_A(A/J). \end{aligned}$$

Thus if we could show that (iii) holds for A^I, then the desired conclusion would follow by induction on $\lambda_A(A/J)$.

It will therefore suffice to show that *if I is any maximal ideal of A, then every local ring infinitely near to A^I is also infinitely near to A.* This just involves straightforward technical points, as follows. First note that *blowing up commutes with localization*: let J be any open ideal in A, and let S be a multiplicative system in A such that the ring of fractions A_S is one-dimensional and Macaulay (so that JA_S is an open ideal in A_S); then the rings $(A^J)_S$ and $(A_S)^{JA_S}$ are canonically isomorphic (Corollary 1.2, with $I=J$, $B=A_S$). Next, let Q be a maximal ideal in A^I and let $P=Q\cap A$. The preceding remark gives (up to canonical isomorphism)

$$(A_P)^{IA_P}=(A^I)_P \quad (=A^I\otimes_A A_P);$$

hence $(A^I)_Q$, which is a localization of $(A^I)_P$ at one of its maximal ideals is infinitely near to the local ring A_P. The conclusion then follows from:

LEMMA 2.4. *Let A be as usual. Then:*

(i) *A local ring B is infinitely near to A if and only if B is infinitely near to A_P for some maximal ideal P in A.*

(ii) *If B' is a local ring infinitely near to A, and B is a local ring infinitely near to B', then B is infinitely near to A.*

[(i) shows then that if B is infinitely near to A^I, then B is infinitely near to $(A^I)_Q$ for some maximal ideal Q in A^I, and since, as above, $(A^I)_Q$ is infinitely near to A_P, (i) gives that $(A^I)_Q$ is infinitely near to A; finally (ii) (with $B'=(A_I)_Q$) shows that B is infinitely near to A, as desired.]

Proof of Lemma 2.4. Bearing in mind the commutativity of blowing up and localization, and the fact that if S is a multiplicative system in A such that A_S is one-dimensional and Macaulay then MA_S is the radical of A_S (M being the radical of A), one sees that if

$$A=A_0\subseteq A_1\subseteq A_2\subseteq\cdots\subseteq A_n\subseteq\cdots$$

is, as usual, the sequence of rings derived from A by successively blowing up radicals, then (up to canonical isomorphism)

$$A_S=(A_0)_S\subseteq(A_1)_S\subseteq(A_2)_S\subseteq\cdots\subseteq(A_n)_S\subseteq\cdots$$

is the sequence derived similarly from A_S. (i) follows easily. As for (ii), if B' is a localization of some A_i at one of its maximal ideals, then by (i) B is infinitely near to A_i, and hence to A.

This completes the proof of Lemma 2.4 and Theorem 2.2.

COROLLARY 2.5. (i) *A is an Arf ring if and only if A_P is an Arf ring for every maximal ideal P of A.*

(ii) *If A is an Arf ring and B' is a local ring infinitely near to A, then B' is an Arf ring.*

Example. If the multiplicity of M, $\mu(M)$, is $\leqq 2$, then A is an Arf ring.

Proof. If B is a one-dimensional Macaulay local ring with $\mu(B) \leqq 2$, then $\operatorname{emdim}(B) = \mu(B)$, because $\operatorname{emdim}(B) \leqq \mu(B)$ (Corollary 1.10) and $\operatorname{emdim}(B) = 1 \Rightarrow B$ regular $\Rightarrow \mu(B) = 1$. So the assertion follows from the general fact that *if B is infinitely near to A, then $\mu(B) \leqq \mu(M)$*. [As in the proof that (ii) $\Rightarrow$ (iii) in Theorem 2.2, we are reduced to showing that if M_1 is the radical of $A_1 = A^M$, then $\mu_{A_1}(M_1) \leqq \mu_A(M)$. However, since $MA_1 = xA_1$ with x regular in A_1 (Proposition 1.1), remark (b) following Corollary 1.10 (with $J = A_2 = A_1{}^{M_1}$) gives

$$\lambda_A(A_1/MA_1) = \lambda_A(A_2/MA_2) \geqq \lambda_{A_1}(A_2/MA_2) \geqq \lambda_{A_1}(A_2/M_1A_2)$$

i. e. (Theorem 1.5)

$$\mu_A(M) \geqq \mu_{A_1}(M_1).]$$

* * *

In what follows (2.6, 2.7, 2.8), we consider a ring homomorphism $f\colon A \to B$ where B is also a semi-local one-dimensional Macaulay ring. We assume that the map f is *continuous*, i. e. if J is an open ideal in B, then $f^{-1}(J)$ is an open ideal in A. (It suffices to check this for *maximal* ideals J in B, i. e. $f^{-1}(J)$ *should be a maximal ideal in A for each such J.*) We look for conditions under which "A is an Arf ring" implies "B is an Arf ring" (or vice-versa (2.9)).

PROPOSITION 2.6. *$f\colon A \to B$ being as above, suppose that every integrally closed open ideal in B is extended, i. e. of the form IB for some ideal I in A. Then if A is an Arf ring, so is B.*

Proof. Using (ii) of Theorem 2.2, we can argue just as in the middle of the paragraph preceding Lemma 2.3.

COROLLARY 2.7. *If A is an Arf ring then so is B in any one of the following situations (f being the obvious map in each case):*

(i) *$B = A_S$ where S is a multiplicative system in A (such that A_S is a one-dimensional Macaulay ring).*

(ii) *$B = \hat{A}$, the completion of A.*

(iii) *$B = A/J$, where J is an ideal in A none of whose associated prime ideals is maximal.*

PROPOSITION 2.8. *$f: A \to B$ being as above, assume that B is a flat A-module and that MB is the radical of B. (M is, as always, the radical of A). Assume further that if Q is a maximal ideal of B, then B/Q is a separable field extension of $A/(Q \cap A)$. (Our conclusion will also hold if "B" is replaced by "$\bar{A}$" in this last assumption.) Under these conditions, if A is an Arf ring then so is B.*

Proof. Let $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ be as in the definition of "infinitely near" (preceding Theorem 2.2), and let $B = B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$ be similarly defined. Then in view of (iii) of Theorem 2.2, and Corollaries 1.4(iv) and 1.10, A is an Arf ring if and only if the radical M_i of A_i is stable in A_i for all $i \geqq 0$; and similarly for B. By induction on i, we shall show that $B_i = B \otimes_A A_i$ and that $M_i B_i$ is the radical of B_i. In view of Corollary 1.4(iii), this will prove Proposition 2.8.

If $B_i = B \otimes_A A_i$ and $M_i B_i$ is the radical of B_i, then B_i is flat over A_i, and Corollary 1.2 shows that

$$B_{i+1} = B_i^{M_i B_i} = B_i \otimes_{A_i} A_{i+1} = B \otimes_A A_{i+1}.$$

Moreover, every maximal ideal in B_{i+1} contracts to a maximal ideal in B (since B_{i+1} is integral over B), hence in A, hence in A_{i+1}; in other words $M_{i+1} B_{i+1}$ is contained in the radical of B_{i+1}. Also, since M_{i+1} contains a regular element and B_{i+1} is flat over A_{i+1}, the ring $B_{i+1}/M_{i+1}B_{i+1}$ is zero-dimensional, and we have only to prove that it has no nilpotent elements. But

$$B_{i+1}/M_{i+1}B_{i+1} \cong (B/MB) \otimes_{A/M} (A_{i+1}/M_{i+1})$$

and the conclusion follows in a straightforward way from the separability assumptions in Proposition 2.8, and the fact that the tensor product is compatible with direct products of algebras [i.e. if C is a ring and $E_1, E_2, \cdots, E_n$, $F_1, F_2, \cdots, F_m$ are C-algebras, then there is a canonical isomorphism of C-algebras

$$\left(\prod_{i=1}^{n} E_i\right) \otimes_C \left(\prod_{j=1}^{m} F_j\right) \xrightarrow{\sim} \prod_{i,j} E_i \otimes_C F_j]. \qquad \text{Q.E.D.}$$

In connection with Proposition 2.8, cf. also Corollary 4.9.

PROPOSITION 2.9. *Let B be a faithfully flat A-algebra, with B a one-dimensional semi-local Macaulay ring. If B is an Arf ring then A is an Arf ring.*

Proof. Let $A(X)$ be as in the proof of Theorem 1.9, and let $B(X)$

be similarly defined. It is easily seen that $B(X)$ is faithfully flat over $A(X)$. By Proposition 2.8, $B(X)$ is an Arf ring. If we knew that $A(X)$ was an Arf ring, then the proof of Proposition 2.8 would show that A is an Arf ring (use the "converse" part of Corollary 1.4(iii)). We may therefore assume that A/P is infinite for every maximal ideal P in A, so that every open ideal in A has transversal elements.

It remains then to be seen that the condition (#) in Definition 2.1 is satisfied. We may identify A with its image in B. Given x, y, z as in (#), we have (by flatness) that x is regular in B, and clearly y, z, are both integral over xB. Hence $yz \in xB \cap A = xA$.

3. Arf closure. *We assume throughout this section that $\bar{A}$ is a finitely generated A-module* (notation as in § 1).

(It follows that A and its completion $\hat{A}$ are reduced: for if w is a nilpotent element in A and x is a regular element in the radical M, then

$$A(w/x) \subseteq A(w/x^2) \subseteq \cdots \subseteq A(w/x^n) \subseteq A(w/x^{n+1}) \subseteq \cdots$$

is a strictly increasing sequence of A-submodules of $\bar{A}$ unless $w = 0$; hence A is reduced, $\bar{A}$ is a direct product of semilocal Dedekind domains, $(\bar{A})^\wedge$ is a direct product of discrete valuation rings, $(\bar{A})^\wedge$ is reduced, $\hat{A}$ is reduced, and incidentally $(\bar{A})^\wedge = \bar{A} \otimes_A \hat{A}$ is the integral closure of $\hat{A}$ in its total ring of fractions. *Conversely*, if $\hat{A}$ is reduced then $\bar{A}$ is a finitely generated A-module (cf. [4; § (32.2)]).)

PROPOSITION-DEFINITION 3.1. *Among the Arf rings between A and $\bar{A}$ there is one, A', which is contained in all the others. A' will be called the* **Arf closure** *of A.*

Proof. Let $A \subseteq B \subseteq \bar{A}$, B being an Arf ring. Let I be any open ideal in A, let $J = IB$ and let $\bar{J}$ be the integral closure of J in B. Then (with notation as in Proposition 1.1):

$$(*) \qquad IA^I \subseteq JB^J \subseteq \bar{J}B^{\bar{J}} \subseteq B.$$

To see this, note first of all that $IB^J = JB^J$ is principal, so that $A^I \subseteq B^J$ (Proposition 1.1), which gives the first inclusion in (*). Next we have, for some $n > 0$, $J\bar{J}^n = \bar{J}^{n+1}$ (remark (c) near the beginning of § 2), and since $\bar{J}B^{\bar{J}}$ is principal, generated by a regular element of $B^{\bar{J}}$, it follows that $JB^{\bar{J}} = \bar{J}B^{\bar{J}}$; thus, by Proposition 1.1 again, $B^J \subseteq B^{\bar{J}}$, whence the second inclusion. Finally, since B is an Arf ring, $\bar{J}$ is stable in B (Theorem 2.2), i.e. $\bar{J}B^{\bar{J}} = \bar{J} \subseteq B$.

Now let $A_{(1)}$ be the ring generated over A by all the elements in IA^I with I ranging over all open ideals in A. Then, from what we have just seen, $A_{(1)} \subseteq B$. Replacing A by $A_{(1)}$ in the preceding, we can construct a new ring $A_{(2)} = (A_{(1)})_{(1)} \subseteq B$. Repeating the procedure, we obtain a sequence of rings

$$A \subseteq A_{(1)} \subseteq A_{(2)} \subseteq \cdots \subseteq B$$

(with $A_{(i+1)} = (A_{(i)})_{(1)}$ for each $i > 0$). Since B is a finitely generated A-module, we have $A_{(n)} = A_{(n+1)} = A_{(n+2)} = \cdots$ for n sufficiently large. To prove Proposition 3.1, it is enough now to show, for such n, that $A_{(n)}$ is an Arf ring.

If I is an integrally closed open ideal in $A_{(n)}$, then $I(A_{(n)})^I \subseteq A_{(n+1)} = A_{(n)}$. But clearly if $x \in I$ and $y \in (A_{(n)})^I$ $(\subseteq \bar{A})$ then xy is integral over $xA_{(n)} \subseteq I$ so that $xy \in I$; thus $I(A_{(n)})^I \subseteq I$, i.e. I is stable in $A_{(n)}$. By Theorem 2.2, $A_{(n)}$ is an Arf ring. Q.E.D.

Remark. If A/P were infinite for every maximal ideal P, then Proposition 3.1 would be trivial, since only the condition (#) of Definition 2.1 would be involved.

COROLLARY 3.2. *Let B be an Arf ring, and let $f: A \to B$ be a homomorphism such that $f(x)$ is regular in B whenever x is regular in A. Then f extends uniquely to a homomorphism $f': A' \to B$, and $f'(A')$ is an Arf ring.*

Proof. f extends uniquely to a homomorphism $\bar{f}: \bar{A} \to \bar{B}$, and we need only show that $\bar{f}(A') \subseteq B$. If I is any open ideal in A and $J = IB$, then Corollary 1.2 shows that $\bar{f}(A^I) \subseteq B^J$, whence $\bar{f}(IA^I) \subseteq JB^J \subseteq B$, and it follows that $\bar{f}(A_{(1)}) \subseteq B$ (cf. proof of 3.1). Repeating the argument, we get $\bar{f}(A_{(2)}) \subseteq B, \cdots, \bar{f}(A_{(n)}) \subseteq B, \cdots$, and for large n this gives $\bar{f}(A') \subseteq B$. Furthermore Corollary 2.7(iii) shows that $\bar{f}(A')$ is an Arf ring.

COROLLARY 3.3. *"Arf closure" commutes with completion and localization. In other words, if $B = \hat{A}$ (resp. $B = A_S$ for some multiplicative system S in A such that B is one-dimensional and Macaulay), then $B' = (A')^\wedge$ (resp. $B' = (A')_S$).*

Proof. In either case let f be the composed map $A \to B \to B'$, and let $f': A' \to B'$ be as in 3.2. There is a canonical isomorphism $B \otimes_A A' \xrightarrow{\approx} B[f'(A')]$ (since B is flat over A), so that, after identifying, we have

$$B \subseteq B \otimes_A A' \subseteq B'.$$

But $B \otimes_A A' = (A')^\wedge$ (resp. $(A')_S$) is an Arf ring (Corollary 2.7) and hence $B \otimes_A A' = B'$. Q.E.D.

In connection with 3.3, cf. also Corollary 4.9.

THEOREM 3.4. *Let A' be the Arf closure of A and let C be any ring between A and A'. Then:*

(i) *For every maximal ideal P of A there is precisely one maximal ideal Q of C such that $Q \cap A = P$. In particular if A is a local ring then so is C.*

(ii) *For each P, Q as in* (i), *the fields A/P and C/Q are canonically isomorphic.*

(iii) *For each P, Q as in* (i), *the local rings A_P and C_Q have the same multiplicity.*

Proof. Because of Corollary 3.3, we may replace A by A_P, i.e. we may assume that A is local. As usual, M will be the maximal ideal of A, and $\bar{A}$ will be the integral closure of A. It is easily checked that $A + M\bar{A}$ is a local ring with maximal ideal $M\bar{A}$, and that the canonical map $A/M \to (A + M\bar{A})/M\bar{A}$ is an isomorphism. With notation as in the proof of 3.1, we have $IA^I \subseteq A + M\bar{A}$ for every open ideal I in A, whence $A_{(1)} \subseteq A + M\bar{A}$. Hence $A_{(1)}$ is a local ring, with maximal ideal $M_{(1)} = M\bar{A} \cap A_{(1)}$, and

$$A_{(1)} + M_{(1)}\bar{A} = A + M\bar{A}.$$

So we may repeat the argument to obtain

$$A_{(1)} \subseteq A_{(2)} \subseteq \cdots \subseteq A_{(n)} \subseteq \cdots \subseteq A + M\bar{A}.$$

For n large, this gives $A' \subseteq A + M\bar{A}$. (i) and (ii) follow at once.

Now if Q is the (unique) maximal ideal of C, then $Q = M\bar{A} \cap C$, so $Q\bar{A} = M\bar{A}$. The multiplicity of A is $\lambda_A(A^M/MA^M)$ (Theorem 1.5), and this is the same as $\lambda_A(\bar{A}/M\bar{A})$, (cf. remark (b) following Corollary 1.10, with A replaced by A^M and J by $\bar{A}$). Similarly the multiplicity of C is $\lambda_C(\bar{A}/Q\bar{A})$. But since $C/Q = A/M$ is the only C-module of length one, we see that

$$\lambda_C(\bar{A}/Q\bar{A}) = \lambda_A(\bar{A}/Q\bar{A}) = \lambda_A(\bar{A}/M\bar{A}).$$

This proves (iii). Q.E.D.

Remark. Theorem 3.4 also follows from Proposition 4.5 and Theorem 4.2.

THEOREM 3.5. *"Arf closure" commutes with "quadratic transform." In other words, with the usual notations, if M' is the radical of A', then*

$$(A^M)' = (A')^{M'}.$$

Proof. Let $C = A^M$, so that $MC = xC$ for some regular x in C (Proposition 1.1). As in the proof of Proposition 3.1, we have xC $(= MA^M) \subseteq A'$, so that $x \in A'$ and $C \subseteq A' : xA'$. But $A' : xA'$ *is an Arf ring*: for if $y \in A' : xA'$ then $yx \in I$ where I is the integral closure of xA' in A' (remark (a), § 2), whence $A' : xA' = Ix^{-1} = (A')^I$, which is an Arf ring (Lemma 2.3 and remark (d), § 2). Therefore also $C' \subseteq A' : xA'$, i.e. $A + xC' \subseteq A'$.

Now *the radical of the ring $A + xC'$ is xC'*: indeed if Q is any maximal ideal of $A + xC'$ then $Q \supseteq M$, whence $Q \supseteq M(xC') = (xC')^2$, so $Q \supseteq xC'$; and on the other hand we have a surjective homomorphism $A/M \to (A + xC')/xC'$, so that $(A + xC')/xC'$ is a direct product of fields. Since $(xC')^2 = x(xC')$, the radical xC' is stable in $A + xC'$ (Lemma 1.11(i)), so the ring obtained from $A + xC'$ by blowing up its radical is $xC' : xC' = C' : C' = C'$. Since xC' is stable and C' is an Arf ring, it follows from (iii) of Theorem 2.2 (and cf. Corollaries 1.4(iv) and 1.10) that $A + xC'$ is an Arf ring. But $A \subseteq A + xC' \subseteq A'$, and so $A' = A + xC'$, $M' = xC'$, and

$$(A')^{M'} = C' = (A^M)'. \qquad \text{Q.E.D.}$$

For developing some consequences of 3.4 and 3.5 we need to recall the notion of the *branch sequence of A along a maximal ideal N of $\bar{A}$.* As before, we consider the sequence of semilocal rings

$$A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq \bar{A}$$

in which, for each $i \geqq 0$, A_{i+1} is the ring obtained from A_i by blowing up the radical of A_i. Let B_i $(= B_i(N))$ be the local ring $(A_i)_{N \cap A_i}$. The sequence $B_0, B_1, B_2, \cdots$ is called the **branch sequence** of A along N.

The members of any such branch sequence are, by definition, infinitely near to A, and every local ring B infinitely near to A occurs in *some* branch sequence. Let us denote the multiplicity of such a B by μ_B. If M_B is the maximal ideal of B and $f: A \to B$ is the canonical map, then B/M_B is an algebraic field extension of $A/f^{-1}(M_B)$, of finite degree δ_B. If $B_i = B_i(N)$ is as above $(i = 0, 1, 2, \cdots)$ then the sequence of pairs

$$(\mu_{B_i}, \delta_{B_i})_{0 \leqq i < \infty}$$

will be called the **multiplicity sequence** of A along N.

Proposition 3.6. *Let N be a maximal ideal in $\bar{A}$. If $B_0, B_1, B_2, \cdots$ is the branch sequence of A along N, then $(B_0)', (B_1)', (B_2)', \cdots$ is the branch sequence of A' along N, where $'$ denotes, as usual, Arf closure.*

Proof. The assertion is a straightforward consequence of the commutativity of Arf closure with quadratic transform and localization (Theorem 3.5 and Corollary 3.3), account being taken of Theorem 3.4(i). Q.E.D.

From Theorem 3.4, we now obtain:

COROLLARY 3.7. *If A, A', N are as in Proposition 3.6, then the multiplicity sequences of A and A' along N are the same.*

COROLLARY 3.8. (Du Val [3]). *If A is a complete local domain with algebraically closed residue field, so that the multiplicity sequence of A along the unique maximal ideal of $\bar{A}$ can be written $(\mu_0, 1), (\mu_1, 1), (\mu_2, 1), \cdots$, then A is an Arf ring if and only if the semigroup of values of elements of A (for the discrete valuation with valuation ring $\bar{A}$) is*

$$\{0, \mu_0, \mu_0 + \mu_1, \mu_0 + \mu_1 + \mu_2, \cdots\}.$$

Proof. Note that $\bar{A}$ is a finitely generated A-module, and that the branch sequence $A = A_0, A_1, A_2, \cdots$ of A is such that $A_n = \bar{A}$ for large n. Keep in mind that A and A' have the same multiplicity sequence (Corollary 3.7). It is easily checked that $A = A'$ if and only if A and A' have the same semigroup, and hence it is enough to show that $0, \mu_0, \mu_0 + \mu_1, \cdots$ is the semigroup of A'. This is clearly true if $A' = \bar{A}$; by induction (on the least n such that $A'_n = \bar{A}$) we may therefore assume that the statement holds for the ring $(A')^{M'} = M'x^{-1}$, where M' is the maximal ideal of A', and x is M'-transversal (cf. Lemma 1.11). But the value of x is

$$\lambda(\bar{A}/x\bar{A}) = \lambda(\bar{A}/M'\bar{A}) = \mu_0$$

(cf. proof of Theorem 3.4), and so the statement follows for A' Q.E.D.

Under our standing assumption that $\bar{A}$ is a finitely generated A-module, it is easily seen that the A-module $\bar{A}/A$ has finite length, say L. If A is the local ring of a rational point P on an algebraic curve Γ, then L is the increase in the arithmetic genus of Γ over that of the curve Γ' obtained from Γ by resolving the singularity at P. The next theorem gives some information about L.

THEOREM 3.9. *Let μ, δ be as in the remarks preceding Proposition 3.6. Then*

$$\lambda_A(\bar{A}/A) \geqq \sum_B \delta_B(\mu_B - 1)$$

where B runs through all the branch sequences of A along maximal ideals of $\bar{A}$; and equality holds if and only if A is an Arf ring.

Proof. Let the rings $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq \bar{A}$ be as in the definition of "branch sequence" (preceding Proposition 3.6). Fix an $i \geqq 0$, let

$M_1, M_2, \cdots, M_t$ be the maximal ideals of A_i, and let B_j be the local ring $(A_i)_{M_j}$ $(1 \leqq j \leqq t)$. We shall show that

$$\lambda_A(A_{i+1}/A_i) \geqq \sum_{j=1}^{t} \delta_{B_j}(\mu_{B_j} - 1)$$

with equality *if and only if* $\operatorname{emdim}(B_j) = \mu_{B_j}$ for all j. Then, in view of (iii) of Theorem 2.2, summation over all i will give the desired result.

Now since A_{i+1}/A_i is of finite length, we have an isomorphism of A_i-modules

$$A_{i+1}/A_i \cong \prod_{j=1}^{t} C_j/B_j$$

where $C_j = B_j \otimes_{A_i} A_{i+1}$. C_j is obtained from B_j by blowing up the maximal ideal M_jB_j (Corollary 1.2). Hence by Theorem 1.5 the B_j-module C_j/B_j has length $\geqq \mu_{B_j} - 1$, with equality if and only if M_jB_j is stable, i.e. $\operatorname{emdim}(B_j) = \mu_{B_j}$ (Corollary 1.10). The conclusion follows easily. Q. E. D.

COROLLARY 3.10. *Let A' be, as usual, the Arf closure of A. If B is a ring between A and $\bar{A}$, then $B \subseteq A'$ if and only if A and B have the same multiplicity sequence along each maximal ideal of $\bar{A}$. In particular, A' is the largest subring of $\bar{A}$ among those which contain A and have the same multiplicity sequences as A.*

Proof. If A and B have the same multiplicity sequences then so do A' and B' (Corollary 3.7). Since $A' \subseteq B'$, 3.9 gives

$$\lambda_{A'}(\bar{A}/B') \geqq \lambda_{B'}(\bar{A}/B') = \lambda_{A'}(\bar{A}/A')$$

whence $B' = A'$.

Conversely if $B \subseteq A'$, i.e. $B' = A'$, then A and B have the same multiplicity sequences (Corollary 3.7).

4. Strict closure of rings. We define the "strict closure" of one ring in another, study some basic properties of this operation (4.1, 4.2) and examine its behavior under flat extensions (4.3, 4.4, 4.10). The main results (4.5, 4.6, 4.11) bring out the relation between Arf rings and rings which are strictly closed in their integral closure.

Let A be *any* ring (*commutative, and having a multiplicative identity* $1 = 1_A$) and let B be a subring of A (with $1_A \in B$). For any ring C such that $B \subseteq C \subseteq A$ there is a canonical surjective ring homomorphism

$$\alpha (= \alpha_{A,B,C}) : A \otimes_B A \to A \otimes_C A$$

whose kernel is the ideal in $A \otimes_B A$ generated by the set of elements

$$\{c \otimes 1 - 1 \otimes c \mid c \in C\}.$$

In particular α is an isomorphism if and only if $C \subseteq B^*$, where B^* is the subring of A given by

$$B^* = \{x \in A \mid x \otimes 1 = 1 \otimes x \text{ in } A \otimes_B A\}.$$

It is immediate that, always, $B^* \subseteq C^*$ (use $\alpha_{A,B,C}$), and that $(B^*)^* = B^*$ since α_{A,B,B^*} is an isomorphism).

We say that B is **strictly closed** in A *if* $B = B^*$.[3] For any B, then, B^* is the smallest strictly closed (in A) ring between B and A; we call B^* the **strict closure** of B **in** A.

LEMMA 4.1. *Let $B \subseteq C \subseteq A$ be as above, and suppose that the canonical map*

$$\alpha \colon A \otimes_B A \to A \otimes_C A$$

is an isomorphism. Let P be a prime ideal in B. Then there is at most one prime ideal Q in C with the properties that $Q \cap B = P$ and $QA \cap C = Q$. If such a Q exists, then the fraction field of B/P is canonically isomorphic to that of C/Q.

Proof. The condition that α be an isomorphism "respects change of base" in the following sense: if B' is a B-algebra, and for any B-algebra E we let E' be the B'-algebra $E \otimes_B B'$, then, *if α is an isomorphism, so also is the canonical map*

$$\alpha' \colon A' \otimes_{B'} A' \to A' \otimes_{C'} A'.$$

(Let $C \xrightarrow{f} C' \xrightarrow{g} A'$ be the canonical maps. If $c \otimes 1 = 1 \otimes c$ in $A \otimes_B A$ for all c in C, then $c' \otimes 1 = 1 \otimes c'$ in $A' \otimes_{B'} A'$, where $c' = g(f(c))$; consequently $x' \otimes 1 = 1 \otimes x'$ in $A' \otimes_{B'} A'$ *for* all x' in the B'-subalgebra $g(C')$ of A' generated by all such c'.)

Furthermore, if B' is a *field*, and α' is an isomorphism, then *the canonical images of B' and C' in A' are equal.* (For, if $x \in A'$ and $x \notin$ image of B', then $x \otimes 1$ and $1 \otimes x$ are linearly independent over B' in $A' \otimes_{B'} A' = A' \otimes_{C'} A'$, so that $x \otimes 1 \neq 1 \otimes x$, whence $x \notin$ image of C'.)

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[3] In the terminology of Grothendieck, this means that the inclusion map $B \to A$ is a *strict monomorphism* in the category of commutative rings with identity. In case A is *integral* over B, it also means that $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ is an *effective epimorphism* of schemes.

Now take B' to be the field of fractions of B/P. Then $B' = B_S/PB_S$ where S is the multiplicative system $B - P$ in B, $C' = C_S/PC_S$, $A' = A_S/PA_S$, and the preceding assertion about the equality of the images of B' and C' in A' leads to the conclusion that if *α is an isomorphism, then the canonical homomorphism*

$$\beta: B_S/PB_S \to C_S/PA_S \cap C_S$$

is surjective. Thus if $PA_S \cap C_S \neq C_S$, then β is an isomorphism and $PA_S \cap C_S$ is a *maximal* ideal of C_S.

Finally, if Q is a prime ideal in C such that $Q \cap B = P$ and $QA \cap C = Q$, then

$$PA_S \cap C_S \subseteq QA_S \cap C_S = Q\, C_S \neq C_S$$

so that the *maximal* ideal $PA_S \cap C_S$ is equal to QC_S. It follows that Q is uniquely determined by P, which is the first assertion of the lemma, and the above map β gives the isomorphism in the last assertion. Q. E. D.

Remark. It is easily seen that Q exists if and only if $PA \cap B = P$.

THEOREM 4.2. *Let B be a noetherian local ring and let A be a subring of the total ring of fractions of B such that $A \supseteq B$ and A is a finitely generated B-module. Let B^* be the strict closure of B in A, and let C be a ring such that $B \subseteq C \subseteq B^*$. Then:*

(i) *For each prime ideal P of B there is a unique prime ideal Q of C such that $Q \cap B = P$. In particular, C is a (noetherian) local ring.*

(ii) *For each P, Q as in* (i), *the fraction fields of B/P and C/Q are canonically isomorphic.*

(iii) *The local rings B and C have the same multiplicity.*

Proof. Since A is integral over B, there exists a prime ideal P' in A such that $P' \cap B = P$; setting $Q = P' \cap C$, we have $Q \cap B = P$, $QA \cap C = Q$; hence the first two assertions follow from Lemma 4.1. (C is obviously noetherian, and C is local because for every maximal ideal M_C of C, $M_C \cap B = M_B$, the unique maximal ideal of B; so there is just one such M_C.)

As for (iii), the proof of Lemma 4.1 shows that $M_C = M_B A \cap C$, so that $M_C A = M_B A$. Furthermore, since C is a local ring with the same residue field as B, we have for any C-module M of finite length,

$$\lambda_C(M) = \lambda_B(M) \qquad (\lambda = \text{length}).$$

(It is enough to check that every C-module of length one is also a B-module of length one; but C/M_C is the only such C-module.) Now by [4; §(23.4)],

$$\mu_B(M_B) = \mu_B(M_BA) \qquad (\mu = \text{multiplicity})$$
$$\mu_C(M_C) = \mu_C(M_CA)$$

and the preceding remarks show that

$$\mu_B(M_BA) = \mu_C(M_BA) = \mu_C(M_CA).$$

Thus,

$$\mu_B(M_B) = \mu_C(M_C). \qquad \text{Q.E.D.}$$

"Strict closure" is well-behaved with respect to *flat* base change:

PROPOSITION 4.3. *Let A be a ring, let B be a subring of A, and let B' be a flat B-algebra. If B is strictly closed in A then B' is strictly closed in $A' = A \otimes_B B'$, and the converse is true if B' is faithfully flat over B.*

Proof. Let τ_1, τ_2 be the B-module homomorphisms from A into $A \otimes_B A$ defined by $\tau_1(x) = x \otimes 1$, $\tau_2(x) = 1 \otimes x$. To say that B is strictly closed in A is to say that the sequence of B-module homomorphisms

$$B \xrightarrow{\text{inclusion}} A \xrightarrow{\tau_2 - \tau_1} A \otimes_B A$$

is *exact*. Tensoring this sequence with B' we get (modulo canonical isomorphisms) the similar sequence for B' and A', and the conclusion follows.

COROLLARY 4.4. *Let A be a ring, let B be a subring of A with strict closure B^* in A, and let B' be a flat B-algebra. Then the strict closure $(B')^*$ of B' in $A \otimes_B B'$ is $B^* \otimes_B B'$.*

Proof. As in the beginning of the proof of 4.1 (with $C = B^*$) we have $C' = B^* \otimes_B B' \subseteq (B')^*$. But by 4.3, C' is strictly closed in $A \otimes_B B'$.

Q.E.D.

We turn next to the relation between strict closure and Arf closure.

PROPOSITION 4.5. (Zariski). *Let A, $\bar{A}$ be as in §1. If A is strictly closed in $\bar{A}$ then A is an Arf ring.*

Proof. If A/P were infinite for every maximal ideal P in A, the proof would be quite simple: given $x, y, z \in A$ with x regular and $y/x \in \bar{A}$, $z/x \in \bar{A}$ we have (in $\bar{A} \otimes_A \bar{A}$)

$$\frac{yz}{x} \otimes 1 = \frac{y}{x} \otimes x\frac{z}{x} = \frac{y}{x}x \otimes \frac{z}{x} = 1 \otimes \frac{yz}{x},$$

so if A is strictly closed in $\bar{A}$ then $yz/x \in A$; thus condition (#) of Definition 2.1 holds for A.

For the general case, we use the following fact: *if I is an integrally closed open ideal in A, and if A is strictly closed in A^I, then I is stable in A.* (In view of Theorem 2.2(ii), this will prove Proposition 4.5, because A is strictly closed in $\bar{A}$, hence *a fortiori* in A^I for all such I.)

To prove the above fact we may replace I by $IA(X)$, $A(X)$ being as in the proof of Theorem 1.9 (remark (e) in §2 shows that $IA(X)$ is integrally closed, and Proposition 4.3 shows that $A(X)$ is strictly closed in $A^I \otimes_A A(X) = A(X)^{IA(X)}$ (Corollary 1.2); finally I is stable if and only if $IA(X)$ is stable (Corollary 1.4(iii)). So we may assume that there exists an I-transversal element x. If $y \in I$, $z \in I$, then $y/x \in A^I$, $z/x \in A^I$, (Lemma 1.8(i)), and we conclude, just as in the beginning of the proof of 4.5, that $yz/x \in A$. But since $yz/x^2 = (y/x)(z/x) \in \bar{A}$, we see at once that yz/x is integral over $xA \subseteq I$, whence $yz/x \in I$. Thus $I^2 \subseteq xI$, i.e. I is stable (Lemma 1.11(i)).

Q.E.D.

In the next theorem, we prove the converse of Proposition 4.5, under the assumption that A contains a field. It would be nice to get rid of this assumption (or, failing this, to show that the assumption is necessary).

THEOREM 4.6. *Let A and $\bar{A}$ be as in* §1, *and assume that A contains a field F with $1_A \in F$. If A is an Arf ring, then A is strictly closed in $\bar{A}$.*

Proof. We consider the sequence of rings

$$A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq \bar{A}$$

where A_{i+1} is the ring obtained by blowing up the radical of A_i ($i \geqq 0$). We will show that (i): $\bigcup_{n \geqq 0} A_n$ $(= \varinjlim_n A_n) = \bar{A}$, and that (ii): A is strictly closed in A_n for each $n \geqq 0$. It follows (since $\otimes$ commutes with $\varinjlim$) that

$$\bar{A} \otimes_A \bar{A} = \varinjlim_n (A_n \otimes_A A_n)$$

and hence that A is strictly closed in $\bar{A}$.

To prove (i), let $x, y \in A$, with x regular and $y/x \in \bar{A}$. We must show that $y/x \in A_n$ for some $n \geqq 0$. We do this by induction on the length $\lambda_A(A/xA)$. If this length is zero, then x is a unit in A and $y/x \in A_0$. If the length is not zero, then x lies in some maximal ideal P of A, and since y is integral over $xA \subseteq P$, also $y \in P$. Since the radical M is of the form PQ for a suitable ideal Q, and MA_1 $(= (PA_1)(QA_1))$ is principal, therefore PA_1

is principal, say $PA_1 = zA_1$ with z regular in A_1 (cf. proof of Proposition 1.1). Then z is not a unit in A_1 (since A_1 is integral over A), and $xz^{-1}, yz^{-1} \in A_1$. Using remark (b) following Corollary 1.10 we see that

$$\lambda_A(A/xA) = \lambda_A(A_1/xA_1) \geqq \lambda_{A_1}(A_1/xA_1) > \lambda_{A_1}(A_1/xz^{-1}A_1).$$

By the inductive hypothesis, we have for some $n > 0$

$$y/x = yz^{-1}/xz^{-1} \in (A_1)_{n-1} = A_n,$$

and so (i) is proved.

We prove (ii) by induction on n, there being nothing to prove if $n = 0$. Let us assume then that A is strictly closed in A_n, and deduce that A is strictly closed in A_{n+1}.

To begin with, we may assume that A is local, because first of all localization at a maximal ideal of A "commutes" with the sequence $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ (cf. proof of Lemma 2.4); secondly A_P is an Arf ring if A is (Corollary 2.5); and finally (as is easily checked), if A_P is strictly closed in $(A_P)_n = A_n \otimes_A A_P$ for all maximal ideals P of A, then A is strictly closed in A_n.

Nsxt, let A^* be the strict closure of A in A_{n+1}; by Lemma 4.1 and its proof, A^* is a local ring, with maximal ideal $M^* = MA_{n+1} \cap A^*$, and the canonical map $A/M \to A^*/M^*$ is an isomorphism. Let $x \in A^*$, i.e. $x \otimes 1 = 1 \otimes x$ in $A_{n+1} \otimes_A A_{n+1}$. We wish to show that $x \in A$. The preceding remarks show that $x - y \in M^*$ for some y in A, so we may assume that $x \in M^*$. Since A is an Arf ring, there exists an M-transversal element z in A, and

$$zA_{n+1} = MA_{n+1} = M^*A_{n+1}$$

(Lemma 1.8(i)). Hence, in particular, $x/z \in A_{n+1}$.

If we could show that $x/z \in A_1$, then we would be done, because $A_1 = Mz^{-1}$ (Lemma 1.11(iii)), whence $x \in M$. Since A_1 is an Arf ring (Lemma 2.3), the inductive hypothesis allows us to assume that A_1 is strictly closed in A_{n+1}, so it would be enough to show that, in $A_{n+1} \otimes_{A_1} A_{n+1}$

$$x/z \otimes 1 - 1 \otimes x/z = 0.$$

Now, in $A_{n+1} \otimes_A A_{n+1}$ we have

$$x \otimes 1 - 1 \otimes x = z[(x/z) \otimes 1 - 1 \otimes (x/z)] = 0.$$

Our conclusion will follow then, from:

LEMMA 4.7. *Let A, $\bar{A}$ be as in §1, with A local, and assume that A contains a field. Assume also that the maximal ideal M of A is stable, and* *let* $A_1 = A^M = Mz^{-1}$, *z being M-transversal. Let C be a ring with* $A_1 \subseteq C \subseteq \bar{A}$. *Then the kernel of the canonical map* $\alpha\colon C \otimes_A C \to C \otimes_{A_1} C$ *is the annihilator of z in* $C \otimes_A C$.

Proof. The kernel of α, being generated by elements of the form $y \otimes 1 - 1 \otimes y$, $y \in A_1$, clearly annihilates z (since $yz \in A$). To prove the converse, we may asumme that A is complete: for, in the first place, the maximal ideal $M\hat{A}$ of $\hat{A}$ is stable (Corollary 1.4(iii)), and clearly z is $M\hat{A}$-transversal; secondly, since $\hat{A}$ is flat over A, we have $\hat{A}_1 = (\hat{A})^{M\hat{A}} = \hat{A} \otimes_A A_1$; and finally we have a commutative diagram (with $\hat{C} = C \otimes_A \hat{A}$)

$$\begin{tikzcd}
C \otimes_A C \arrow[r, "\alpha"] \arrow[d] & C \otimes_{A_1} C \arrow[d, "\beta"] \\
\hat{C} \otimes_{\hat{A}} \hat{C} \arrow[r, "\hat{\alpha}"] & \hat{C} \otimes_{\hat{A}_1} \hat{C}
\end{tikzcd}$$

with β injective (since $\hat{C} \otimes_{\hat{A}_1} \hat{C} = (C \otimes_{A_1} C) \otimes_A \hat{A}$), which shows that it suffices to prove Lemma 4.7 for α.

Assuming now that A is complete, let k be a field of representatives of A, so that A contains the power series ring $k[[z]]$. Let $w = \sum_{i=1}^{p} a_i \otimes_A b_i$ $(a_i, b_i \in C)$ be such that $zw = 0$ in $C \otimes_A C$. Then, in $C \otimes_{k[[z]]} C$, we have

$$z \sum_{i=1}^{p} a_i \otimes b_i = \sum_{j=1}^{q} c_j \otimes d_j (x_j \otimes 1 - 1 \otimes x_j) \qquad (c_j, d_j \in C;\, x_j \in A).$$

After subtracting from the x_j suitable elements of k, we may assume that $x_j \in M$, whence $x_j/z \in C$, so that (still in $C \otimes_{k[[z]]} C$)

$$z\Big(\sum a_i \otimes b_i - \sum c_j \otimes d_j \Big(\frac{x_j}{z} \otimes 1 - 1 \otimes \frac{x_j}{z}\Big)\Big) = 0.$$

If we know that z is a *regular* element in $C \otimes_{k[[z]]} C$, then it follows at once that $\sum a_i \otimes_{A_1} b_i = 0$, and we are done. But z is regular in C, i.e. C is a *flat* $k[[z]]$-module, so $C \otimes_{k[[z]]} C$ is also a flat $k[[z]]$-module, and hence z is regular in $C \otimes_{k[[z]]} C$. Q.E.D.

COROLLARY 4.8. *Let A, $\bar{A}$ be as in Theorem 4.6, and assume that $\bar{A}$*

is a finitely generated A-module. Then the Arf closure of A coincides with the strict closure of A in $\bar{A}$.

COROLLARY 4.9. *Let A and* $\bar{A}$ *be as in Theorem* 4.6, *and let B be a flat A-algebra such that B is a one-dimensional semi-local Macaulay ring such that* $B \otimes_A \bar{A}$ *is integrally closed in its total ring of fractions. Then:*

(i) *If A is an Arf ring, so is B.*

(ii) *If* $\bar{A}$ *is a finitely generated A-module, and A′ is the Arf closure of A, then* $B \otimes_A A'$ *is the Arf closure of B.*

(4.8) *and* 4.9 *follow easily from* 4.3, 4.4, 4.5 *and* 4.6).

Remarks. Corollary 4.9(i) is stronger than Proposition 2.8, provided that A contains a field. For, as in the proof of 4.6, we have $\bar{A} = \bigcup_{n\geqq 0} A_n$, and similarly the integral closure $\bar{B}$ of B is $\bigcup_{n\geqq 0} B_n$; but under the conditions of 2.8 (cf. proof of 2.8) $B_n = B \otimes_A A_n$, whence $\bar{B} = B \otimes_A \bar{A}$.

Does 4.9(i) *still hold when A does not contain a field?* It might be easier to answer this question than the corresponding one for Theorem 4.6.

* * *

The next lemma indicates that one-dimensional local rings should play a special role in questions about strict closure. Recall that a noetherian ring A is said to satify condition (S_1) if the ideal (0) has no embedded associated prime ideals, and A satisfies condition (S_2) if A satisfies (S_1) and furthermore no principal ideal generated by a regular element in A has embedded associated primes.

LEMMA 4.10. *Let A be a noetherian ring satisfying condition* (S_2), *and let* $\bar{A}$ *be the integral closure of A in its total ring of fractions. Then:*

(i) *A is strictly closed in* $\bar{A}$ *if and only if* A_P *is strictly closed in* $\bar{A}_P = \bar{A} \otimes_A A_P$ *for every prime ideal P in A such that* A_P *has Krull dimension one.*

(ii) *A satisfies the condition* (#) *in Definition* 2.1 *if and only if* A_P *satisfies* (#) *for every P as in* (i).

Proof. (i). If A is strictly closed in $\bar{A}$ then A_P is strictly closed in $\bar{A}_P$ by 4.3. Suppose, conversely, that A_P is strictly closed in $\bar{A}_P$ whenever $\dim. A_P = 1$. Let $x \in \bar{A}$ be such that $x \otimes 1 = 1 \otimes x$ in $\bar{A} \otimes_A \bar{A}$; we want to show that $x \in A$. Set $x = u/v$ ($u, v \in A$; v regular) and let I be the ideal

$\{w \in A \mid wx \in A\}$. Then $uI \subseteq vA$, so if $x \notin A$, i.e. $u \notin vA$, then I is contained in some associated prime ideal P of vA; since A satisfies (S_2) and v is not a unit in A, we have $\dim. A_P = 1$. But $(x/1) \otimes 1 = 1 \otimes (x/1)$ in $\bar{A}_P \otimes_{A_P} \bar{A}_P$ whence, A_P being strictly closed in $\bar{A}_P$, $x/1 \in A_P$, i.e. $I \not\subseteq P$. This contradiction shows that $x \in A$.

(ii) We prove first that *if A satisfies (#) and if S is a multiplicatively closed subset of A, then also the ring of fractions A_S satisfies (#).* Indeed, let $x = a/s$, $y = b/s$, $z = c/s$ be elements of A_S $(a, b, c \in A;\ s \in S)$ such that y and z are integral over the ideal xA_S. We must show that *if x is regular in A_S, then $yz \in xA_S$.* It is easily seen that for some s' in S, $s'b$ and $s'c$ are integral over the ideal aA, and so, since A satisfies (#), our conclusion would follow immediately *if a were regular in A.*

It suffices to show that $a + d$ is regular for some element d in the kernel K of the canonical map $A \to A_S$ since $a/s = (a + d)/s$ for any such d. Let $p_1, p_2, \cdots, p_n$, $p_1', p_2', \cdots, p_{n'}'$, $p_1'', p_2'', \cdots, p_{n''}''$, be all the minimal prime ideals in A, the notation being such that

$$K \subseteq p_1, p_2, \cdots, p_n; \qquad K \not\subseteq p_1', p_2', \cdots, p_{n'}', p_1'', \cdots, p_{n''}'';$$
$$a \in p_1', \cdots, p_{n'}'; \qquad a \notin p_1'', \cdots, p_{n''}''.$$

Note that $a \notin p_i$ $(1 \leqq i \leqq n)$, since otherwise a would be a zerodivisor in $A/K \subseteq A_S$, contrary to the assumption that $x = a/s$ is regular in A_S. Now let d be such that

$$d \in K \cap p_1'' \cap p_2'' \cap \cdots \cap p_{n''}''$$
$$d \notin p_1' \cup p_2' \cup \cdots \cup p_{n'}'.$$

Then $a + d$ does not lie in any minimal prime ideal of A and so, since A satisfies (S_1), $a + d$ is regular in A, as required.

Thus, if A satisfies (#), then A_P satisfies (#) for all prime ideals P in A.

Next, assume that A_P satifies (#) whenever $\dim. A_P = 1$. Let $x, y, z \in A$ be such that x is regular and both y and z are integral over the ideal xA; we want to show that $yz \in xA$. Let I be the ideal $\{w \in A \mid wyz \in xA\}$. Then $yzI \subseteq xA$, so if $yz \notin xA$, then I is contained in some associated prime ideal P of xA; since A satisfies (S_2), $\dim. A_P = 1$, so A_P satisfies (#) and hence $yzA_P \subseteq xA_P$, i.e. $I \not\subseteq P$. This contradiction completes the proof. Q.E.D.

Remarks. (a) It is easy to check in 4.10(i), that $\bar{A}_P$ is the integral closure of A_P in its total ring of fractions. (Use the fact that A satisfies (S_1)).

(b) If P is a non-maximal prime ideal of A, then A/P is infinite, since every finite integral domain is a field. Hence the local ring A_P has an infinite residue field, and if also $\dim. A_P = 1$, then A_P satisfies (#) if and only if A_P is an Arf ring.

Thus, and in view of 4.5 and 4.6, 4.10 gives:

THEOREM 4.11. *Let A, $\bar{A}$ be as in 4.10, and assume that A contains a field F (with $1_A \in F$). Then A is strictly closed in $\bar{A}$ if and only if A_P is an Arf ring for every P as in 4.10; and in particular if A has no maximal ideal P such that A/P is finite and $\dim. A_P = 1$, then A is strictly closed in $\bar{A}$ if and only if A satisfies the condition (#) in Definition 2.1.*

5. Connections with saturated rings.

We first recall some terminology from Zariski's theory of saturation of rings (cf. [6]).

Let A be a one-dimensional Macaulay local ring, with maximal ideal M, and let $\bar{A}$ be the integral closure of A in its total ring of fractions F. We suppose that there is given a subfield K of F, containing the element 1 of F, such that F is finite-dimensional as a K-vector space and such that A is integral over $R = A \cap K$. R has a unique maximal ideal, namely $M_R = M \cap K$; we shall suppose also that R/M_R is an *infinite* field.

Let Ω be an algebraic closure of K. There exist finitely many K-algebra homomorphisms $\psi: F \to \Omega$. For each such ψ, $\psi(F)$ is a subfield of Ω containing K, and $\psi(A)$ is a one-dimensional local subring of $\psi(F)$ containing, and integral over, R. Let $F^\#$ be the compositum of all the fields $\psi(F)$ (ψ running through $\mathrm{Hom}_{K\text{-alg}}(R, \Omega)$, and let $v_1, v_2, \cdots, v_n$ be all the (non-trivial) valuations of $F^\#$ which are non-negative on R. The v_i are discrete, rank one, valuations, *finite in number*, because the integral closure $R^\#$ of R in $F^\#$ is the same as the integral closure of any $\psi(A)$, and hence (by the theorem of Krull-Akizuki, cf. [2, Ch. 7; pp. 29-31]) $R^\#$ is a semi-local Dedekind domain.

For elements η, ζ of F, we say that η **dominates** ζ (with respect to K) if, for any v_i as above, and any two K-homomorphisms ψ_j, ψ_k of F into Ω (hence into $F^\#$) we have:

$$v_i[\psi_j(\eta) - \psi_k(\eta)] \geqq v_i[\psi_j(\zeta) - \psi_k(\zeta)].$$

Finally, A is **saturated** with respect to K if A contains every element of $\bar{A}$ which dominates an element of A.

PROPOSITION 5.1 (Zariski). *With notation as above, if A is saturated* *with respect to K then A is strictly closed in $\bar{A}$ (i.e. if $x\in\bar{A}$ and $x\otimes 1 = 1\otimes x$ in $\bar{A}\otimes_A\bar{A}$, then $x\in A$).*

Proof. Let $x\in\bar{A}$ be such that $x\otimes 1=1\otimes x$ in $\bar{A}\otimes_A\bar{A}$; it suffices to show that x dominates some element of A. The kernel of the canonical map $\bar{A}\otimes_R\bar{A}\to\bar{A}\otimes_A\bar{A}$ consists of all sums of elements of the form

$$ac\otimes b-a\otimes cb=(a\otimes b)(c\otimes 1-1\otimes c) \qquad (a,b\in\bar{A};c\in A).$$

Thus, in $\bar{A}\otimes_R\bar{A}$, we have

$$x\otimes 1-1\otimes x=\sum_{t=1}^{h}(a_t\otimes b_t)(c_t\otimes 1-1\otimes c_t) \qquad (a_t,b_t\in\bar{A};c_t\in A).$$

From any two K-homomorphisms $\psi_j,\psi_k\colon F\to\Omega$, we obtain a homomorphism $\psi_j\otimes\psi_k\colon\bar{A}\otimes_R\bar{A}\to\Omega$, which applied to the preceding equation gives

$$\psi_j(x)-\psi_k(x)=\sum_{t=1}^{h}\psi_j(a_t)\psi_k(b_t)[\psi_j(c_t)-\psi_k(c_t)]$$

It follows at once that x dominates $c\in A$, where c is given by:

LEMMA 5.2. *With A, R, M_R as above, let $c_1,c_2,\cdots,c_h$ be any elements of A. Then there exists an element $c\in A$ such that c is dominated by each one of $c_1,c_2,\cdots,c_h$.*

Proof. (cf. [6; p. 971, Prop. 1.6]). Let V be the R-submodule of A generated by $c_1,c_2,\cdots,c_h$. For each v_i, ψ_j, ψ_k as in the definition of domination, set

$$V_{ijk}=\{z\in V\mid v_i(\psi_j(z)-\psi_k(z))>\min_{1\leq t\leq h}[v_i(\psi_j(c_t)-\psi_k(c_t))]\}.$$

Then V_{ijk} is an R-submodule of V, and $V_{ijk}\neq V$ since at least one c_t is not in V_{ijk}.

Now V/M_RV is a vector space over the field R/M_R (which is an *infinite* field, by assumption); and if $f\colon V\to V/M_RV$ is the canonical map then, by Nakayama's lemma, $f(V_{ijk})\neq V/M_RV$, so that

$$V/M_RV\neq\bigcup_{i,j,k}f(V_{ijk})$$

whence

$$V\neq\bigcup_{i,j,k}V_{ijk}.$$

Any c in V lying outside $\cup V_{ijk}$ will be as required. Q.E.D.

COROLLARY 5.3. *If A is saturated with respect to K, then A is an Arf ring.*

Proof. This results from Propositions 4.5 and 5.1. A more direct proof, avoiding altogether the notion of "strictly closed", follows easily from the identity

$$\psi_j\left(\frac{yz}{x}\right)-\psi_k\left(\frac{yz}{x}\right)=\psi_j\left(\frac{y}{x}\right)[\psi_j(z)-\psi_k(z)]+\psi_k\left(\frac{z}{x}\right)[\psi_j(y)-\psi_k(y)]$$
$$-\psi_j\left(\frac{y}{x}\right)\psi_k\left(\frac{z}{x}\right)[\psi_j(x)-\psi_k(x)].$$

(Cf. Definition 2.1; and use Lemma 5.2 with $c_1=x$, $c_2=y$, $c_3=z$).

Q. E. D.

* * *

Suppose now, for simplicity, that A is a complete one-dimensional local domain. It follows from 5.3 that the saturation $\tilde{A}_K$ of A (= smallest subring of $\bar{A}$ containing A and saturated with respect to K) always contains the Arf closure A'. A' may or may not be equal to $\tilde{A}_K$, as is shown (in a strong way) by the following example:

Let $k[[t]]$ be the power series ring over an algebraically closed field k of characteristic zero, and let $A=k[[x,y]]\subseteq k[[t]]$, with

$$x=t^p$$
$$y=t^q+a_1t^{q+1}+a_2t^{q+2}+\cdots \qquad (a_i\in k),$$

where $p<q$ are two relatively prime positive integers. (A is the local ring of an algebroid plane curve with one characteristic pair.) The saturation $\tilde{A}$ of A with respect to $k((t^p))$ is the *smallest* ring among those of the form $\tilde{A}_{k((z))}$ with $0\neq z\in M$ (cf. [6; Corollary 1.10]). One checks that

$$\tilde{A}=k[[t^p,t^q,t^{q+1},t^{q+2},\cdots]].$$

Now $A'=\tilde{A}$ if and only if A' and $\tilde{A}$ have the same semigroup of values. By 3.7 and 3.8, the semigroup of A' is

$$0,\mu_0,\mu_0+\mu_1,\mu_0+\mu_1+\mu_2,\cdots$$

where $\mu_0,\mu_1,\mu_2,\cdots$ is the multiplicity sequence of A. This sequence is found as follows: let

$$q=ap+b \qquad 0<b<p$$
$$p=cb+d \qquad 0\leqq d<b$$
$$\vdots$$

(Euclidean algorithm). Then

$$\mu_0 = \mu_1 = \cdots \mu_{a-1} = p$$
$$\mu_a = \mu_{a+1} = \cdots = \mu_{a+c-1} = b$$
$$\mu_{a+c} = \cdots \qquad = d \qquad \text{etc. etc.}$$

From this we conclude easily that: $A' = \tilde{A}$ *if and only if* $q \equiv \pm 1 \pmod p$.

*　　*　　*

We conclude with an example of a family of equivalent, non-isomorphic, plane curve singularities whose Arf closures are equally non-isomorphic. Recall that two plane algebroid curves have equivalent singularities if and only if they have isomorphic saturations [6; § 2]. The example shows, then, that the operation of "Arf closure" does not serve, as saturation does, to "kill the moduli" of equivalent plane singularities.

Let k be a field of characteristic zero, and let $k[[t]]$ be the power series ring in one variable over k. Let n be a positive integer and let $\alpha = (a_1, a_2, \cdots, a_{n-1})$ be a sequence of elements of k. Set

$$A = A_\alpha = k[[x, y]]$$

where

$$x = t^{2n+1}$$
$$y = t^{3n+2} + a_1 t^{3n+3} + a_2 t^{3n+4} + \cdots + a_{n-1} t^{4n+1}.$$

Then A represents a plane algebroid curve with one characteristic pair $(2n+1, 3n+2)$. If A_1 is the quadratic transform of A, A_2 of A_1, A_3 of A_2, etc. . ., then the multiplicities of $A, A_1, A_2, A_3, \cdots$ are easily seen to be $2n+1, n+1, n, 1, \cdots$ ($A_r = k[[t]]$ for $r \geqq 3$). From this (cf. 3.7 and 3.8)—or by direct computation—we find that the Arf closure of A is

$$A' = A_\alpha' = k[[x, y]] + t^{4n+2} k[[t]].$$

Now let $\beta = (b_1, b_2, \cdots, b_{n-1})$ be another sequence of elements in k. We write $\alpha \equiv \beta$ if there exists an element $c \neq 0$ in k such that

$$(a_1, a_2, \cdots, a_{n-1}) = (cb_1, c^2 b_2, \cdots, c^{n-1} b_{n-1}).$$

It is clear that if $\alpha \equiv \beta$ then A_β is mapped isomorphically onto A_α, as A_β' is onto A_α', by the k-automorphism of $k[[t]]$ which sends t to ct. Conversely:

If $\alpha \not\equiv \beta$, *then* A_α' *and* A_β' *are not* k*-isomorphic* (*and* a fortiori A_α *and* A_β *are not* k*-isomorphic*).

Indeed, if A_α' and A_β' are k-isomorphic, then there exists a power series τ of order 1 in $k[[t]]$ such that

$$\text{(i)}: \quad \tau^{2n+1} \in A_\alpha'$$

and

$$\text{(ii)}: \quad \tau^{3n+2} + b_1\tau^{3n+3} + \cdots + b_{n-1}\tau^{4n+1} \in A_\alpha'.$$

Setting

$$\tau = ct(1 + d_1t + d_2t^2 + \cdots + d_{n+1}t^{n+1} + \cdots)$$

we see at once, from (i), that $d_1 = d_2 = \cdots = d_n = 0$, and hence that

$$\tau^{3n+2} + b_1\tau^{3n+3} + \cdots + b_{n-1}\tau^{4n+1} \equiv c^{3n+2}(t^{3n+2} + cb_1t^{3n+3} + \cdots + c^{n-1}b_{n-1}t^{4n+1}) \quad (\text{modulo } t^{4n+2}).$$

From (ii) it then follows that

$$(a_1, a_2, \cdots, a_{n-1}) = (cb_1, c^2b_2, \cdots, c^{n-1}b_{n-1}). \qquad \text{Q. E. D.}$$

One last remark: if k is algebraically closed and if $(a_1, a_2, \cdots, a_{n-1}, b_1, b_2, \cdots, b_{n-1})$ are independently transcendental over the prime subfield of k, then A_α' and A_β' are isomorphic, since there exists an automorphism θ of k with $\theta(a_i) = b_i$ $(i = 1, 2, \cdots, n-1)$, and θ can be extended to an automorphism of $k[[t]]$ leaving t fixed. On the other hand, if $n \geqq 3$ then $\alpha \not\equiv \beta$, so that A_α' and A_β' are not k-isomorphic. It follows that *there is no automorphism of A_α' extending θ^{-1}*. Thus Arf rings can be less "rich" in automorphisms than saturated rings (cf. [6, Theorem 1.16]).

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