PICARD SCHEMES OF FORMAL SCHEMES; APPLICATION TO RINGS WITH DISCRETE DIVISOR CLASS GROUP

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Introduction.

We are going to apply scheme-theoretic methods - originating in the classification theory for codimension one subvarieties of a given variety - to questions which have grown out of the problem of unique factorization in power series rings.

Say, with Danilov [D2], that a normal noetherian ring A has discrete divisor class group (abbreviated DCG) if the canonical map of divisor class groups $\bar{i}:C(A) \to C(A[[T]])$ is bijective[(2)]. In §1, a proof (due partially to J.-F. Boutot) of the following theorem is outlined:

THEOREM 1. Let A be a complete normal noetherian local ring with algebraically closed residue field. If the divisor class group C(A) is finitely generated (as an abelian group), then A has DCG.

(1) Supported by National Science Foundation grant GP-29216 at Purdue University.

(2) For the standard definition of $\bar{i}$, cf. [AC, ch. 7, §1.10]. (Note that the formal power series ring A[[T]] is noetherian [AC, ch. 3, §2.10, Cor. 6], integrally closed [AC, ch. 5, §1.4], and flat over A [AC, ch. 3, §3.4, Cor. 3].)

The terminology DCG is explained by the fact that in certain cases (cf. [B];[SGA 2, pp. 189-191]) with A complete and local, C(A) can be made into a locally algebraic group over the residue field of A, and this locally algebraic group is discrete (i.e. zero-dimensional) if and only if $\bar{i}$ is bijective.

A survey of results about rings with DCG is given in [F, ch. V].

Recall that A is factorial if and only if $C(A) = (0)$ [AC, ch. 7, §3]. Also, A local ⇒ $A[[T]]$ local, with the same residue field as A; and A complete ⇒ $A[[T]]$ complete [AC, ch. 3, §2.6]. Hence (by induction):

COROLLARY 1. If A (as in Theorem 1) is factorial, then so is any formal power series ring $A[[T_1, T_2, \ldots, T_n]]$.

When the singularities of A are resolvable, more can be said:

THEOREM 1'. Let A be as in Theorem 1, with C(A) finitely generated, and suppose that there exists a proper birational map $X \to \mathrm{Spec}(A)$ with X a regular scheme (i.e. all the local rings of points on X are regular). Let B be a noetherian local ring and let $f: A \to B$ be a local homomorphism making B into a formally smooth A-algebra (for the usual maximal ideal topologies on A and B).(3) Then B is normal, and the canonical map $C(A) \to C(B)$ is bijective.

Some brief historical remarks are in order here. Corollary 1 was conjectured by Samuel [S2, p. 171];(4) however Samuel did not

(3) "Formal smoothness" means that the completion $\hat{B}$ is A-isomorphic to a formal power series ring $\bar{A}[[T_1, T_2, \ldots, T_n]]$, where $\bar{A}$ is a complete local noetherian flat A-algebra with maximal ideal generated by that of A (cf. [EGA 0_{IV}, §§19.3, 19.6, 19.7]). In particular, B is flat over A.

(4) For some earlier work on unique factorization in power series rings cf. [S1] and [K].

assume that the residue field of A was algebraically closed, and without this assumption, the conjecture was found by Salmon to be false [SMN]. Later, a whole series of counterexamples was constructed by Danilov [D1] and Grothendieck [unpublished].(5) Danilov's work led him to the following modification of Samuel's conjecture [D1, p. 131]:

If A is a local ring which is "geometrically factorial" (i.e. the strict henselization of A is factorial) *then also A[[T]] is geometrically factorial.*

In this general form, the conjecture remains open, though some progress has been made by Boutot [unpublished].

The study of Samuel's conjecture evolved into the study of rings with DCG. A complete normal noetherian local ring A has been shown to have DCG in the following cases(6):

(*i*) (Scheja [SH]). A is factorial and depth $A \geq 3$.

(*ii*) (Storch [ST2]) A contains a field, and the residue field of A is algebraically closed and uncountable, with cardinality greater than that of C(A).

[Actually, for such A, Storch essentially proves Theorem 1' *without* needing any desingularization $X \to \mathrm{Spec}(A)$. Storch's proof uses a theorem of Ramanujam-Samuel (cf. proof of Theorem 1' in §1) and an elementary counting argument.]

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(5) In these counterexamples the locally algebraic group of footnote (2) above has dimension > 0, but has *just one point* - namely zero - *rational over the residue field of A.*

(6) For some investigations in the context of analytic geometry, cf. [ST1] and [P].

(*iii*) (*Danilov* [D3]) If

either (a) A contains a field of characteristic zero

or (b) A contains a field, the residue field of A is separably closed, and there exists a projective map $g: X \to \mathrm{Spec}(A)$ with X a regular scheme, such that g induces an isomorphism

$$X - g^{-1}(\{\underline{m}\}) \xrightarrow{\approx} \mathrm{Spec}(A) - \{\underline{m}\}$$

($\underline{m}$ = maximal ideal of A)

then C(A) finitely generated $\Rightarrow$ A has DCG.

[Danilov uses a number of results from algebraic geometry, among them the theory of the Picard scheme of schemes proper over a field, and the resolution of singularities (by Hironaka in case (a), and by assumption in case (b)).]

Significant simplifications have been brought about by *Boutot*. His lemma (§1) enabled him to eliminate all assumptions about resolution of singularities in the above-quoted result of Danilov, and also to modify the proof of Theorem 1' to obtain the proof of Theorem 1 which appears in §1 below.

Our proof of Theorem 1' is basically a combination of ideas of Danilov and Storch, except that in order to treat the case when A does not contain a field, we need a theory of *Picard schemes for schemes proper over a complete local ring of mixed characteristic.* This theory - which is the main underlying novelty in the paper - is given in §§2-3.

§1. Proofs of Theorems 1 and 1'.

The two theorems have much in common, and we will prove them together. Let A, B be as in Theorem 1'; for Theorem 1 we will simply take $B = A[[T]]$. Since A is local and B is faithfully flat over A, the canonical map $C(A) \to C(B)$ is injective [F, Prop. 6.10]; so we need only show that $C(A) \to C(B)$ is surjective.

Both B and its completion $\hat{B}$ are normal: when $B = A[[T]]$ this is clear; and under the assumption of Theorem 1', since B and $\hat{B}$ are formally smooth over A, it follows from the existence of the "desingularization" $X \to \mathrm{Spec}(A)$ [L1, Lemma 16.1]. As above, since $\hat{B}$ is faithfully flat over B, $C(B) \to C(\hat{B})$ is injective, and consequently we may assume that $B = \hat{B}$ $(= \bar{A}[[T_1, T_2, \ldots, T_n]]$, cf. footnote (3) in the Introduction). [Note here that if $R \subseteq S \subseteq T$ are normal noetherian rings with S flat over R and T flat over S (and hence over R), then the composition of the canonical maps

$$C(R) \to C(S) \to C(T)$$

is the canonical map $C(S) \to C(T)$.]

Let M be the maximal ideal of A. Then $M\bar{A}$ is the maximal ideal of $\bar{A}$, and by the theorem of Ramanujam-Samuel [F, Prop. 19.14],

$$C(B) \to C(B_{MB})$$

is bijective. Furthermore [EGA 01, p. 170, Cor. (6.8.3)], there exists a complete local noetherian flat B_{MB}-algebra B^* such that B^*/MB^* is an algebraically closed field. B^* is formally smooth over A (footnote (3) above) so under the hypotheses of Theorem 1', B^* is normal; furthermore B^* is faithfully flat over B_{MB}, so that, as before

$$C(B_{MB}) \to C(B^*)$$

is injective. Thus for Theorem 1' it suffices to show that $C(A) \to C(B^*)$ is surjective.

To continue the proof of Theorem 1', let U_A be the domain of definition of the rational map inverse to $X \to \mathrm{Spec}(A)$. Then U_A is isomorphic to an open subscheme of X, so we have a <u>surjective</u> map $\mathrm{Pic}(X) \to \mathrm{Pic}(U_A)$ [EGA IV, (21.6.11)]; furthermore the codimension of $\mathrm{Spec}(A) - U_A$ in $\mathrm{Spec}(A)$ is ≥ 2, so there is a <u>natural isomorphism</u> $\mathrm{Pic}(U_A) \xrightarrow{\sim} C(A)$ [<u>ibid</u>, (21.6.12)]. Similar considerations hold with B^* in place of A, and $X^* = X \otimes_A B^*$ in place of X. (The projection $X^* \to \mathrm{Spec}(B)$ is proper and birational, and X^* is a regular scheme [L1, Lemma 16.1].) There results a commutative diagram

$$\begin{array}{ccccc} \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(U_A) & \xrightarrow{\sim} & C(A) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Pic}(X^*) & \longrightarrow & \mathrm{Pic}(U_{B^*}) & \xrightarrow{\sim} & C(B^*) \end{array}$$

Since $\mathrm{Pic}(X^*) \to \mathrm{Pic}(U_{B^*})$ is surjective, it will be more than enough to show that <u>$\mathrm{Pic}(X) \to \mathrm{Pic}(X^*)$ is bijective</u>.

The corresponding step in the proof of Theorem 1 is more involved, and goes as follows. Let $B = A[[T]]$, let B^* be as above, and let I be a divisorial ideal in B. We will show below that there exists an open subset U_A of $\mathrm{Spec}(A)$ whose complement has codimension ≥ 2, and such that, with

$$U_B = (U_A)\otimes_A B \quad (\subseteq \mathrm{Spec}(B)), \qquad U^* = (U_A)\otimes_A B^* \ (\subseteq \mathrm{Spec}(B^*))$$

we have that

(i) IB_q is a principal ideal in B_q for all prime ideals $q \in U_B$, and

(ii) the canonical map $\nu:\mathrm{Pic}(U_B) \to \mathrm{Pic}(U^*)$ is injective.

Now there is a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(U_A) & \xrightarrow{\ \lambda\ } & \mathrm{Pic}(U_B) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ C(A) & \longrightarrow & C(B) \end{array}$$

cf. [EGA IV, (21.6.10)]. Since B is flat over A, it is immediate (from the corresponding property for U_A) that the complement of U_B in $\mathrm{Spec}(B)$ has codimension ≥ 2; hence (i) signifies that the element of $C(B)$ determined by I is of the form $\mu_B(\xi)$ for some $\xi \in \mathrm{Pic}(U_B)$. So if we could show that ξ lies in the image of λ, then we would have the desired surjectivity of $C(A) \to C(B)$.

At this point we need:

LEMMA (J.-F. Boutot)[1]. <u>There exists a projective birational map $\phi: X \to \mathrm{Spec}(A)$ such that ϕ induces an isomorphism $\phi^{-1}(U_A) \xrightarrow{\sim} U_A$, and such that ξ lies in the image of the canonical map $\mathrm{Pic}(X \otimes_A B) \to \mathrm{Pic}(U_B)$.</u>

(Here X may be taken to be <u>normal</u>, but not necessarily regular.) Setting $X^* = X \otimes_A B^*$, we have a natural commutative diagram

$$\begin{array}{ccccc} \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X \otimes_A B) & \longrightarrow & \mathrm{Pic}(X^*) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Pic}(U_A) & \xrightarrow{\lambda} & \mathrm{Pic}(U_B) & \xrightarrow{\nu} & \mathrm{Pic}(U^*) \end{array}$$

with ν injective (cf. (ii) above). A simple diagram chase shows then that for ξ to lie in the image of λ, it more than suffices that <u>$\mathrm{Pic}(X) \to \mathrm{Pic}(X^*)$ be bijective</u>.

Let us finish off this part of the argument by constructing U_A satisfying (i) and (ii). [It will then remain - for proving both Theorems 1 and 1' - to examine the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^*)$.]

Let

$$U_A = \{p \in \mathrm{Spec}(A) \mid A_p \text{ is a regular local ring}\}.$$

By a theorem of Nagata [EGA IV (6.12.7)], U_A is open in $\mathrm{Spec}(A)$; and certainly, A being normal, the codimension of $\mathrm{Spec}(A) - U_A$ in $\mathrm{Spec}(A)$ is ≥ 2. Since the fibres of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are regular [EGA IV, (7.5.1)], therefore B_q is regular for all $q \in U_B$ [EGA 0_{IV}, (17.3.3)], and (i) follows.

(1) The proof, which will appear in Boutot's thèse, was presented at a seminar at Harvard University in January, 1972.

As for (ii), setting $U' = U_A \otimes_A B_{MB}$ (M = maximal ideal of A) we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(U_B) & \longrightarrow & \mathrm{Pic}(U') \\ \downarrow & & \downarrow \\ C(B) & \longrightarrow & C(B_{MB}) \end{array}$$

in which the vertical arrows are <u>isomorphisms</u> [EGA IV, (21.6.12)], and also $C(B) \to C(B_{MB})$ is an isomorphism (cf. above); so we have to show that <u>$\mathrm{Pic}(U') \to \mathrm{Pic}(U^*)$ is injective.</u> Since $\mathrm{Pic}(U')$ is isomorphic to $C(B_{MB})$, this injectivity amounts to the following statement:

(#) <u>Let I be a divisorial ideal of B_{MB}, and let $\mathcal{I}^*$ be the coherent ideal sheaf on Spec(B*) determined by the ideal IB*. If $\mathcal{I}^*|U^* \cong \mathcal{O}_{U^*}$, then I is a principal ideal.</u>

Since B_{MB} is local, and B* is faithfully flat over B_{MB}, we have

$$I \text{ principal} \Leftrightarrow I \text{ invertible} \Leftrightarrow IB^* \text{ invertible}.$$

Now I is a reflexive B_{MB}-module [CA, p. 519, Ex. (2)], and therefore IB* is a reflexive B*-module [<u>ibid</u>, p. 520, Prop. 8]. Since B* is flat over B_{MB}, it follows (from the corresponding property of U') that for every prime ideal P in B* such that $P \notin U^*$, the local ring B^*_P has depth ≥ 2. This being so, if $i: U^* \to \mathrm{Spec}(B^*)$ is the inclusion map, then the natural map

$$\mathcal{O}_{\mathrm{Spec}(B^*)} \to i_*(\mathcal{O}_{U^*})$$

is an <u>isomorphism</u> [EGA IV, (5.10.5)]. Since IB^* is reflexive, application of $\mathrm{Hom}_{B^*}(\cdot, B^*)$ to a "finite presentation"

$$(B^*)^n \to (B^*)^m \to \mathrm{Hom}_{B^*}(IB^*, B^*) \to 0,$$

gives an exact sequence

$$0 \to IB^* \to (B^*)^m \to (B^*)^n,$$

whence a commutative diagram, with exact rows,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}^* & \longrightarrow & \mathcal{O}^m & \longrightarrow & \mathcal{O}^n \\ & & \downarrow & & \downarrow \wr & & \downarrow \wr \\ 0 & \to & i_*(\mathcal{I}^*|U^*) & \to & i_*(\mathcal{O}^m_{U^*}) & \to & i_*(\mathcal{O}^n_{U^*}) \end{array} \qquad [\mathcal{O} = \mathcal{O}_{\mathrm{Spec}(B^*)}]$$

from which we conclude that the canonical map

$$\mathcal{I}^* \to i_*(\mathcal{I}^*|U^*)\,[\cong i_*(\mathcal{O}_{U^*})]$$

is an isomorphism. Thus $\mathcal{I}^*$ is isomorphic to $\mathcal{O}_{\mathrm{Spec}(B^*)}$, and (ii) is proved.

The rest of the discussion applies to both Theorems (1 and 1'). We must now examine the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^*)$.

The kernel of the surjective map $\mathrm{Pic}(X) \to \mathrm{Pic}(U_A)$ consists of the linear equivalence classes of those divisors on X which are supported on $X - U_A$; hence (X being assumed to be normal) this kernel is isomorphic to a subgroup of the free abelian group generated by those irreducible components of $X - U_A$ having codimension one in X; since $Pic(U_A) \subseteq C(A)$, and $C(A)$ is finitely generated, therefore _Pic(X) is finitely generated._

Let k (resp. k^*) be the residue field of A (resp. B^*). There is an obvious map $k \to k^*$. In §2 we will show that

(1.1) _There exists a k-group-scheme P and a commutative diagram_

$$\begin{array}{ccc} P(k) & \longrightarrow & P(k^*) \\ \downarrow \wr & & \downarrow \wr \\ Pic(X) & \longrightarrow & Pic(X^*) \end{array} .$$

Here $P(k) \to P(k^*)$ is the map from k-valued points of P to k^*-valued points corresponding to the map $k \to k^*$; and the vertical maps are _isomorphisms_.

Furthermore, in §3 it will be shown that

(1.2) _There exists a closed irreducible k-subgroup P^o of P, whose underlying subspace is the connected component of the zero point of P, and such that:_

(i) _P^o is the inverse limit of its algebraic_ (= finite type over k) _quotients; moreover if $\bar{P}$ is such a quotient, then $P(k) \to \bar{P}(k)$ is surjective._

(ii) $P/P^o = \varprojlim_{n>0} Q_n$, <u>where Q_n is a discrete</u> (= reduced and zero-dimensional) <u>locally algebraic k-group; moreover</u> $P(K) \to (P/P^o)(K)$ <u>is surjective for any algebraically closed field</u> $K \supseteq k$.

To show that $\mathrm{Pic}(X) \to \mathrm{Pic}(X^*)$ is bijective, it will then suffice to show that <u>P^o is infinitesimal</u> [in other words, every algebraic quotient of P^o is zero-dimensional, so that $P^o(k) = P^o(k^*) = 0$, whence $\mathrm{Pic}(X) \to \mathrm{Pic}(X^*)$ can be identified with the map

$$\varprojlim_{n} (Q_n(k) \to Q_n(k^*))$$

which is obviously bijective].

But since $P^o(k) \subseteq P(k)$ is finitely generated, so is $\bar{P}(k)$ for any algebraic quotient $\bar{P}$ of P^o. By the structure theorem for connected reduced commutative algebraic groups over an algebraically closed field, we know that $\bar{P}_{red}$ has a composition series whose factors are multiplicative groups, additive groups, and abelian varieties. It follows easily that if $\bar{P}(k) = \bar{P}_{red}(k)$ is finitely generated, then $\bar{P}(k) = 0$, i.e. $\bar{P}$ is zero-dimensional.

§2. <u>The Picard Scheme of a Formal Scheme.</u>

In this section we establish the existence of a <u>natural group-scheme structure on $\mathrm{Pic}(\mathfrak{X})$</u> for certain formal schemes $\mathfrak{X}$. (If

$p\mathcal{O}_{\mathfrak{X}} = (0)$ (cf. (2.2)) there will be nothing new here. For the case $p\mathcal{O}_{\mathfrak{X}} \neq (0)$, most of the work is carried out in [L2], whose results will be quoted and used.) From this we will obtain (1.1). However, for completeness, we prove more general results than are required in the proof of Theorems 1 and 1'.

DEFINITION (2.1). A formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is weakly noetherian if $\mathfrak{X}$ has a fundamental system of ideals of definition $\mathcal{J}_0 \supseteq \mathcal{J}_1 \supseteq \mathcal{J}_2 \supseteq \cdots$ such that for each $n \geq 0$ the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_n)$ is noetherian.

It amounts to the same thing to say: in the category of formal schemes,

$$\mathfrak{X} = \varinjlim_{n \geq 0} X_n$$

where $X_0 \to X_1 \to X_2 \to \ldots$ is a sequence of immersions of noetherian schemes X_n, the underlying topological maps being homeomorphisms (cf. [EGA 0I, §10.6, pp. 411-413]).

Any noetherian formal scheme is weakly noetherian [ibid, middle of p. 414].

If $\mathfrak{X}$ is weakly noetherian and $\mathcal{J}$ is any ideal of definition, then $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is a noetherian scheme; indeed, $\mathcal{J} \supseteq \mathcal{J}_n$ for some n (since $\mathfrak{X}$ is quasi-compact) so that $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is a closed subscheme of the noetherian scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_n)$. In particular, taking $\mathcal{J}$ to be the largest ideal of definition of

$\mathfrak{X}$, we see that we may - and, for convenience, we always will - assume that <u>the scheme</u> $\mathfrak{X}_{red} = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_0)$ <u>is reduced</u>. (Cf. [EGA 01, p. 172 (7.1.6)].)

Next, let k be a perfect field of characteristic $p \geq 0$. For $p > 0$ let W(k) be the ring of (infinite) Witt vectors with coefficients in k; and for $p = 0$ let W(k) be the field k itself. W(k) is complete for the topology defined by the ideal pW(k); the corresponding formal scheme Spf(W(k)) will be denoted by $\mathfrak{W}_k$.

(2.2) <u>In what follows we consider a triple</u> $(\mathfrak{X}, k, f)$ <u>with</u>:

(i) $\mathfrak{X}$ a weakly noetherian formal scheme.

(ii) k a perfect field of characteristic $p \geq 0$.

(iii) $f: \mathfrak{X} \to \mathfrak{W}_k$ a morphism of formal schemes such that for every ideal of definition $\mathcal{J}$ of $\mathfrak{X}$, the induced map of schemes

$$f_{\mathcal{J}}: (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}) \to \mathrm{Spec}(W(k))$$

is <u>proper</u>[(1)].

<u>Remarks.</u> Morphisms $f: \mathfrak{X} \to \mathfrak{W}_k$ are in one-one correspondence with <u>continuous homomorphisms</u> $i: W(k) \to H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ [EGA 01, p. 407, (10.4.6)][(2)]. The above map $f_{\mathcal{J}}$ corresponds to the composed

(1) For (iii) to hold it suffices that $f_{\mathcal{J}}$ be proper for <u>one</u> $\mathcal{J}$ (cf. (2.6) below).

(2) The existence of such an i implies that p is topologically nilpotent in $H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ (since the image of a topologically nilpotent element under a continuous homomorphism is again topologically nil-potent). On the other hand, if p is topologically nilpotent in $H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, then clearly <u>every</u> ring homomorphism $W(k) \to H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is continuous.

homomorphism

$$W(k) \xrightarrow{\ i\ } H^{o}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \xrightarrow{\text{canonical}} H^{o}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}).$$

It is practically immediate that <u>$f_{\mathcal{J}}(\mathfrak{X})$ is supported in the closed point of Spec(W(k)).</u>

<u>Example.</u> Let R be a complete noetherian local ring with maximal ideal M and residue field k (perfect, of characteristic $p \geq 0$); let $g: X \to \mathrm{Spec}(R)$ be a proper map; and let $\mathfrak{X}$ be the formal completion of X along the closed fibre $g^{-1}(\{M\})$. The structure theory of complete local rings gives the existence of a (continuous) homomorphism $W(k) \to R$; composing with the map

$$R \to H^{o}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \ [= H^{o}(X, \mathcal{O}_{X})]$$

determined by g, we obtain $i: W(k) \to H^{o}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, whence a triple $(\mathfrak{X}, k, f)$ as above.

(2.3) For any k-algebra A let $W_n(A)$ (resp. W(A)) be the ring of Witt vectors of length n (resp. of infinite length) with coefficients in A. ($W_n(A) = W(A) = A$ if $p = 0$.) We consider $W_n(A)$ to be a discrete topological ring, and give W(A) the topology for which $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ is a fundamental system of neighborhoods of 0, K_n being the kernel of the canonical map $W(A) \to W_n(A)$ $(n \geq 1)$; then, in the category of topological rings,

$$W(A) = \varprojlim_{n \geq 1} W_n(A).$$

It is not hard to see that $K_1^2 = pK_1$, whence

$$K_1^{n+1} = p^n K_1 \subseteq K_n \ ;$$

so $W(A)$ is an "admissible" ring, and we may let $\mathfrak{W}_A$ be the affine formal scheme

$$\mathfrak{W}_A = \mathrm{Spf}(W(A)).$$

In particular, for $A = k$, we get the same $\mathfrak{W}_k$ as in (2.1). If B is an A-algebra, then $W(B)$ is in an obvious way a topological $W(A)$-algebra, so that $\mathfrak{W}_A$ varies functorially with A.

With $f: \mathfrak{X} \to \mathfrak{W}_k$ as in (2.2), we set

$$\mathfrak{X}_A = \mathfrak{X} \times_{\mathfrak{W}_k} \mathfrak{W}_A = \mathfrak{X} \hat{\otimes}_{W(k)} W(A)$$

(product in the category of formal schemes). We have then the <u>covariant functor of k-algebras</u>

$$A \to \mathrm{Pic}(\mathfrak{X}_A).$$

What we show below is that <u>the fpqc sheaf P associated to this functor is a k-group scheme,</u> and that furthermore <u>the canonical map $\mathrm{Pic}(\mathfrak{X}_A) \to P(A)$ is bijective if A is an algebraically closed field.</u>

<u>Example</u> (continued from (2.2)). Suppose that $\mathfrak{X}$ is obtained from a proper map $g: X \to \mathrm{Spec}(R)$ as in the example of (2.2). For

any k-algebra A, setting $R_A = R\hat{\otimes}_{W(k)}W(A)$ (completed tensor product, R being topologized as usual by its maximal ideal M), we have

$$\mathfrak{X}_A = \mathfrak{X}\hat{\otimes}_{W(k)}W(A) = \mathfrak{X}\hat{\otimes}_R R_A .$$

Now if A is a perfect field, then R_A has the following properties, which characterize R_A as an R-algebra (up to isomorphism): R_A is a complete local noetherian flat R-algebra such that $R_A/MR_A \cong A$ (cf. [EGA 01, p. 190, (7.7.10)] and [EGA 0_{IV}, (19.7.2)]). Furthermore, $\mathfrak{X}_A$ is then the completion of the scheme $X_A = X\otimes_R R_A$ along the closed fibre of the projection $g_A: X_A \to \mathrm{Spec}(R_A)$. Hence Grothendieck's algebrization theorem [EGA III, (5.1.6)] gives that "completion" is an equivalence from the category of coherent $\mathcal{O}_{X_A}$-modules to the category of coherent $\mathcal{O}_{\mathfrak{X}_A}$-modules. Since an $\mathcal{O}_X$-module is invertible if and only if so is its completion(3), we deduce a <u>natural isomorphism</u>

$$\mathrm{Pic}(X_A) \cong \mathrm{Pic}(\mathfrak{X}_A).$$

Hence, restricting our attention to those A which are <u>algebraically closed fields</u>, we will have an A-functorial isomorphism

$$\mathrm{Pic}(X_A) \cong P(A).$$

(3) This follows easily from the fact that the completion $\hat{B}_I$ of a noetherian ring B w.r.t. an ideal I is faithfully flat over the ring of fractions B_{1+I}, so that if J is a B-ideal with $J\hat{B}_I$ a projective $\hat{B}_I$-module, then JB_{1+I} is a projective B_{1+I}-module.

This gives us the diagram (1.1) which is needed in the last step of the proof of Theorems 1 and 1'.

(2.4) We fix a fundamental system $\mathcal{J}_0 \supseteq \mathcal{J}_1 \supseteq \mathcal{J}_2 \supseteq \cdots$ of defining ideals of $\mathfrak{X}$, and for $n \geq 0$ let X_n be the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_n)$. For any k-algebra A, let $X_{n,A}$ be the scheme

$$X_{n,A} = X_n \otimes_{W(k)} W_n(A).$$

The ringed spaces $X_{0,A}, X_{1,A}, \ldots, X_{n,A}, \ldots$ and $\mathfrak{X}_A$ all have the same underlying topological space, say X, and on this space X we have $\mathcal{O}_{\mathfrak{X}_A} = \varprojlim_n \mathcal{O}_{X_{n,A}}$. Hence there is a natural map

$$(*) \qquad \mathrm{Pic}(\mathfrak{X}_A) \to \varprojlim_n \mathrm{Pic}(X_{n,A}).$$

LEMMA. <u>Let A be a k-algebra, and if $p > 0$ assume that $A^p = A$</u> (i.e. the Frobenius endomorphism $x \to x^p$ of A is surjective). <u>Then the above map (*) is bijective.</u>

<u>Remark.</u> When $p > 0$ and $A^p = A$, or when $p = 0$, then $X_{n,A} = X \otimes_{W(k)} W(A)$.

<u>Proof of Lemma.</u> Say that an open subset U of X is <u>affine</u> if $(U, \mathcal{O}_{\mathfrak{X}_A}|U)$ is an affine formal scheme. The affine open sets form a base for the topology of X.

For each n, let $\mathcal{F}_n$ be the sheaf of multiplicative units in the sheaf of rings $\mathcal{O}_{X_{n,A}}$ (on the topological space X) and let

$$\mathcal{F} = \varprojlim_n \mathcal{F}_n = \text{sheaf of units in } \mathcal{O}_{\mathfrak{X}_A}.$$

For $m \geq n$, the kernel of $\mathcal{O}_{X_{m,A}} \to \mathcal{O}_{X_{n,A}}$ is nilpotent; so a simple argument ([L2, Lemma (7.2)], with the Zariski topology in place of the étale topology) shows that for affine U the canonical maps

$$H^i(U, \mathcal{F}_m) \to H^i(U, \mathcal{F}_n)$$

are <u>bijective</u> if $i > 0$, and <u>surjective</u> if $i = 0$. Applying [EGA 0_{III}, (13.3.1)], we deduce that for all $i > 0$, the maps

$$H^i(X, \mathcal{F}) \to \varprojlim_n H^i(X, \mathcal{F}_n)$$

are <u>surjective</u>. Furthermore, in order that

$$\begin{array}{ccc} H^1(X, \mathcal{F}) & \to & \varprojlim_n H^1(X, \mathcal{F}_n) \\ \| & & \| \\ \mathrm{Pic}(\mathfrak{X}_A) & & \varprojlim_n \mathrm{Pic}(X_{n,A}) \end{array}$$

be <u>bijective</u>, it is sufficient that the inverse system $H^0(X, \mathcal{F}_n)_{n\geq 0}$ satisfies the Mittag-Leffler condition (ML); and for this it is enough that the inverse system $H^0(X, \mathcal{O}_{X_{n,A}})$ should satisfy (ML); that is, for each fixed n, if I_{mn} $(m \geq n)$ is the image of $H^0(X, \mathcal{O}_{X_{m,A}}) \to H^0(X, \mathcal{O}_{X_{n,A}})$, then the sequence

(**) $$I_{n,n} \supseteq I_{n+1,n} \supseteq I_{n+2,n} \supseteq \cdots$$

should stabilize (i.e. $I_{N,n} = I_{N+1,n} = I_{N+2,n} = \cdots$ for some N).

For $p > 0$ it is shown in [L2, Corollary (0.2) and Theorem (2.4)] that the fpqc sheaf $\underline{\underline{H}}_n$ associated to the functor

$$A \to H^0(X, \mathcal{O}_{X_{n,A}})$$

(of k-algebras A) is an affine algebraic k-group; furthermore [ibid, Corollary (4.4)] the canonical map

$$H^0(X, \mathcal{O}_{X_{n,A}}) \to \underline{\underline{H}}_n(A)$$

is bijective whenever $A^p = A$; and finally, for $m \geq n$, if $\underline{\underline{I}}_{mn}$ is the image (in the category of algebraic k-groups) of the natural map $\underline{\underline{H}}_m \to \underline{\underline{H}}_n$, and if $A^p = A$, then the canonical map

$$\underline{\underline{H}}_m(A) \to \underline{\underline{I}}_{mn}(A)$$

is surjective, so that $I_{mn} = \underline{\underline{I}}_{mn}(A)$ [cf. ibid, last part of proof of (6.3)]. Similar facts when $p = 0$ are well-known (and more elementary).

Now the sequence

$$\underline{\underline{I}}_{n,n} \supseteq \underline{\underline{I}}_{n+1,n} \supseteq \underline{\underline{I}}_{n+2,n} \supseteq \cdots$$

of closed subgroups of $\underline{\underline{H}}_n$ must stabilize, whence so must the sequence (**). Q.E.D.

(2.5) Before stating the basic existence theorem we need some more notation. For any scheme Y, Br(Y) will be the cohomological Brauer group of Y:

$$\mathrm{Br}(Y) = H^2_{\text{étale}}(Y, \text{multiplicative group}).$$

For any ring R we set:

$$\mathrm{Br}(R) = \mathrm{Br}(\mathrm{Spec}(R))$$
$$\mathrm{Pic}(R) = \mathrm{Pic}(\mathrm{Spec}(R))$$
$$R_{red} = R/\text{nilradical of } R.$$

For any defining ideal $\mathcal{J}$ of $\mathfrak{X}$ and any k-algebra A:

$$\mathfrak{X}_{\mathcal{J}} = \text{the scheme } (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$$

$$\mathfrak{X}_{\mathcal{J},A} = \mathfrak{X}_{\mathcal{J}} \otimes_{W(k)} W(A).$$

Finally, we set

$$k_0 = H^0(\mathfrak{X}_{red}, \mathcal{O}_{\mathfrak{X}_{red}}).$$

Since $\mathfrak{X}_{red}$ is proper over k (cf (2.2)), therefore k_0 is a finite product of finite field extensions of k.

Now for any $\mathcal{J}$, we have (cf (2.2)) a proper map

$$f_{\mathcal{J}}: \mathfrak{X}_{\mathcal{J}} \to \mathrm{Spec}(W(k))$$

whose image is supported in the closed point of Spec(W(k)).

Hence, when $p > 0$, [L2, Theorem (7.5)] gives us a k-group-scheme $P_{\mathcal{J}}$ and, for all k-algebras A with $A^p = A$, an exact A-functorial sequence

$$0 \to \mathrm{Pic}(k_0 \otimes_k A_{red}) \to \mathrm{Pic}(\mathfrak{X}_{\mathcal{J},A}) \to P_{\mathcal{J}}(A)$$
$$\to \mathrm{Br}(k_0 \otimes_k A_{red}) \to \mathrm{Br}(\mathfrak{X}_{\mathcal{J},A}) \quad .$$

A similar result is well-known for $p = 0$, or more generally when $p\mathcal{O}_{\mathfrak{X}_{\mathcal{J}}} = (0)$, with no condition on A, since then $\mathfrak{X}_{\mathcal{J}}$ is proper over the field k (cf [GR, Cor. 5.3]).

Also, if $\mathcal{J} \subseteq \mathcal{J}'$, then the canonical map

$$P_{\mathcal{J}} \to P_{\mathcal{J}'}$$

is <u>affine</u> ([SGA 6, Expose XII, Prop. (3.5)] when $p = 0$, and [L2, Prop. (2.5)] when $p > 0$). Thus $P = \varprojlim_{\mathcal{J}} P_{\mathcal{J}}$ <u>exists as a k-group-scheme</u> (cf. [EGA IV, §8.2]).

Now, in view of Lemma (2.4), a simple passage to inverse limits gives the desired result:

THEOREM. <u>There exists a k-group scheme P, and for k-algebras A such that $A^p = A$</u> (the condition $A^p = A$ is vacuous when $p = 0$) <u>an exact sequence, varying functorially with A,</u>

$$0 \to \mathrm{Pic}(k_0 \otimes_k A_{red}) \to \mathrm{Pic}(\mathfrak{X}_A) \to P(A) \to$$
$$\to \bigcap_{\mathcal{J}} \ker[\mathrm{Br}(k_0 \otimes_k A_{red}) \to \mathrm{Br}(\mathfrak{X}_{\mathcal{J},A})] .$$

COROLLARY. _If A is an algebraically closed field, then the above map $\mathrm{Pic}(\mathfrak{X}_A) \to P(A)$ is bijective._

For, then $\mathrm{Pic}(k_0 \otimes_k A_{red}) = \mathrm{Br}(k_0 \otimes_k A_{red}) = (0)$.[4]

Remarks. 1. The k-group-scheme P is _uniquely determined_ by the requirements of the Theorem. Indeed, since for every k-algebra A there exists a faithfully flat A-algebra $\bar{A}$ with $\bar{A}^p = \bar{A}$ [L2, Lemma (0.1)], and since every element in $\mathrm{Pic}(k_0 \otimes_k A_{red})$ or in $\mathrm{Br}(k_0 \otimes_k A_{red})$ is locally trivial for the étale topology on A, it follows easily that _P is the fpqc sheaf associated to the functor_ $A \to \mathrm{Pic}(\mathfrak{X}_A)$ _of k-algebras A._

2. P^0, the connected component of zero in P, is described in (3.2) below. The remarks following (1.2) suggest that the following conjecture - or some variant - should hold:

Conjecture: P^0 is infinitesimal if and only if the natural (split injective) map

$$\mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(\mathfrak{X} \hat{\otimes}_W W[[T]]) \qquad (W = W(k))$$

is bijective.

(4) The Corollary, which is what we need for Theorems 1 and 1', could be proved more directly, using [L2, §1, comments on part II]; then we could do without our Lemma (2.4), and without introducing "Br". In a similar vein it can be deduced from the Theorem - or shown more directly - that if K is a normal algebraic field extension of k such that every connected component of $\mathfrak{X}_{red}$ has a K-rational point, and if A is any perfect field containing K, then $\mathrm{Pic}(\mathfrak{X}_A) \to P(A)$ is bijective.

(2.6) <u>(Appendix to §2)</u>. The following proposition is meant to give a more complete picture of how our basic data $(\mathfrak{X}, k, f)$ can be defined. It will not be used elsewhere in this paper.

To begin with, observe that if $(\mathfrak{X}, k, f)$ is as in (2.2), then f induces a <u>proper map</u>

$$f_{\mathcal{J}_0} : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_0) = \mathfrak{X}_{red} \to \mathrm{Spec}(k)$$

(cf. (2.2)). Hence $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$ is a finite k-module (equivalently: a finite $W(k)$-module) and - <u>a fortiori</u> - a finite $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ - module. Conversely:

PROPOSITION. <u>Let $\mathfrak{X}$ be a weakly noetherian formal scheme, and assume that the $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$-module $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$ is finitely generated. Let k be a perfect field of characteristic $p \geq 0$, and let</u>

$$f_0 : \mathfrak{X}_{red} \to \mathrm{Spec}(k)$$

<u>be a proper map of schemes. Then f_o extends (uniquely, if $p > 0$) to a map of formal schemes $f: \mathfrak{X} \to \mathfrak{W}_k$. Furthermore, all the maps $f_{\mathcal{J}}$ (cf. (2.2)) are proper.</u>

<u>Proof.</u> (Sketch) f_o corresponds to a homomorphism $i_o : k \to H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$; the problem is to lift i_o to a continuous homomorphism

$$i : W(k) \to H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

Let $\mathcal{J}_0 \supseteq \mathcal{J}_1 \supseteq \mathcal{J}_2 \supseteq \cdots$ be a fundamental system of defining ideals of $\mathfrak{X}$ (cf. (2.1)), and let $H_o = H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})/H^o(\mathfrak{X}, \mathcal{J}_o)$. We will show below that:

(*) <u>the canonical map</u> $H_o \xrightarrow{\pi} H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$ <u>is bijective.</u>

Then the existence of the lifting i follows (since W(k) is formally smooth over its subring $\mathbb{Z}_{p\mathbb{Z}}$) from [EGA 0_{IV}, (19.3.10)] (with $\mathscr{I} = H^o(\mathfrak{X}, \mathcal{J}_o)$). For the uniqueness when $p > 0$, cf. [<u>loc. cit.</u> (20.7.5) or (21.5.3)(ii)]. {Or else note that $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$, being reduced and finite over k, is perfect, and argue as in [SR, p. 48, Prop. 10], using the following easily proved fact in place of [<u>ibid</u>., p. 44, Lemme 1]:

If $a, b \in H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ satisfy $a \equiv b \pmod{H^o(\mathfrak{X}, \mathcal{J}_n)}$, then for some N depending only on n we have

$$a^{p^N} \equiv b^{p^N} \pmod{H^o(\mathfrak{X}, \mathcal{J}_{n+1})}.\}$$

Now (*) simply says that $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$ is surjective, and to prove this we may assume that $\mathfrak{X}$ is <u>connected</u>; then $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$, being finite over k, is a perfect field, as is its subring H_o (since $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$ is finite over H_o, by assumption), say $H_o = K$. As above, the identity map $K \to K$ lifts to a homomorphism $W(K) \to H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and thereby, for every ideal of definition $\mathcal{J}$, the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is a W(K)-scheme. For $\mathcal{J} = \mathcal{J}_o$ the structural map $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_o) \to \mathrm{Spec}(W(K))$ factors as

$$(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_o) = \mathfrak{X}_{red} \to \mathrm{Spec}(H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})) \xrightarrow{\text{finite}} \mathrm{Spec}(K) \hookrightarrow \mathrm{Spec}(W(K)).$$

Note that $\mathfrak{X}_{red}$, being proper over k, is proper over $H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_{red}})$, and hence also over K. Arguing as below, we see that $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_n)$ is proper over W(K), whence the kernel of $\pi_n: H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_n) \to H^o(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_o)$ is a W(K)-module of finite length. So by [EGA 0_{III}, (13.2.2)], $\pi = \varprojlim \pi_n$ will be surjective if π_n is surjective for all n. Let us show more generally for any scheme map $\phi: X \to \mathrm{Spec}(W(K))$ that if ϕ induces a proper map

$$Y = X_{red} \to \mathrm{Spec}(K) \subseteq \mathrm{Spec}(W(K))$$

then $H^o(X, \mathcal{O}_X) \to H^o(Y, \mathcal{O}_Y)$ is surjective.

Let $\bar{K}$ be an algebraic closure of K. Then $W(\bar{K})$ is a faithfully flat W(K)-algebra. In view of [EGA III, (1.4.15)]. (Künneth formula for flat base change) and the fact that

$$Y \otimes_{W(K)} W(\bar{K}) = Y \otimes_K \bar{K}$$

is reduced (K being perfect), we may replace X by $X \otimes_{W(K)} W(\bar{K})$, i.e. we may assume that K is algebraically closed. But then $H^o(Y, \mathcal{O}_Y)$ is a product of copies of K, one for each connected component of Y, so the assertion is obvious.

It remains to be shown that the maps $f_{\mathcal{J}}$ are all proper. $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is noetherian, and $\mathfrak{X}_{red} = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})_{red}$. By [EGA II (5.4.6) and EGA 01, p. 279, (5.3.1)(vi)] it suffices to show that $f_{\mathcal{J}}$ is locally of finite type; so what we need is that if A is a noetherian $W(k)$-algebra with a nilpotent ideal N such that A/N is finitely generated over $W(k)$, then also A is finitely generated over $W(k)$. But if $a_1, a_2, \ldots, a_r$ in A are such that their images in A/N are $W(k)$-algebra generators of A/N, and if $b_1, b_2, \ldots, b_s$ are A-module generators of N, then it is easily seen that

$$A = W(k)[a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s].$$

Q.E.D.

§3. <u>Structure of inverse limits of locally algebraic k-groups.</u>

In this section, we establish (1.2)- and a little more- for any group-scheme P of the form $\varprojlim P_n$, where (P_n, f_{mn}) (n, m, non-negative integers, $n \geq m$) is an inverse system of locally algebraic k-groups (k a field), the maps $f_{mn}: P_n \to P_m$ ($n \geq m$) being <u>affine</u> (cf. [EGA IV, §8.2]). (Note that the group-scheme P of §(2.5) is of this form.) This is more or less an exercise, and the results are presumably known, but I could not find them recorded anywhere.

(3.1) By [SGA 3, p. 315], $f_{mn}: P_n \to P_m$ ($n \geq m$) factors uniquely as

$$P_n \xrightarrow{u} P_{mn} \xhookrightarrow{v} P_n$$

where v is a closed immersion and u is affine, faithfully flat, and finitely presented. (P_{mn} is the image, or coimage, of f_{mn}.) For $n_1 \geq n_2$, P_{mn_1} is a closed subgroup of P_{mn_2}, and we can set

$$\bar{P}_m = \bigcap_{n \geq m} P_{mn} = \varprojlim_{n \geq m} P_{mn}.$$

$\bar{P}_m$ is a closed subgroup of P_m, its defining ideal in $\mathcal{O}_{P_m}$ being the union of the defining ideals of the P_{mn}. Clearly f_{mn} induces a map $\bar{f}_{mn}: \bar{P}_n \to \bar{P}_m$, so we have an inverse system $(\bar{P}_n, \bar{f}_{mn})$.

PROPOSITION. (i) <u>P (together with the natural maps $\bar{f}_n: P \to \bar{P}_n$) is equal to</u> $\varprojlim \bar{P}_n$.

(ii) <u>The maps $\bar{f}_{mn}: \bar{P}_n \to \bar{P}_m$ and $\bar{f}_m: P \to \bar{P}_m$ are affine, faithfully flat, and universally open.</u>

(iii) <u>If K is any algebraically closed field containing k, then</u>

$$\bar{f}_m(K): P(K) \to \bar{P}_m(K)$$

<u>is surjective.</u>

(iv) <u>$\ker(\bar{f}_{mn})$ is a closed subgroup of $\ker(f_{mn})$.</u>

<u>Proof.</u> (i) and (iv) are left to the reader. It is clear that all the maps $\bar{f}_{mn}$ and $\bar{f}_m$ are affine. We show below that $\bar{f}_m$ is faithfully flat for all m. Since $\bar{f}_m = \bar{f}_{mn} \circ \bar{f}_n$ for $n \geq m$, it will follow that $\bar{f}_{mn}$ is faithfully flat [EGA IV, (2.2.13)]. This implies that $\bar{f}_{mn}$ is universally open [EGA IV,

(2.4.6)] and hence so is $\bar{f}_m$ [EGA IV, (8.3.8)], proving (ii). As for (iii), since $\bar{f}_{st}$ is locally of finite type and surjective, it follows that $\bar{f}_{st}(K)$ is surjective for all $t \geq s$; in particular, $\bar{f}_{n,n+1}(K)$ is surjective for all $n \geq m$, so any element of $\bar{P}_m(K)$ can be lifted back to $P(K) = \varprojlim_{n\geq m} \bar{P}_n(K)$, i.e. $\bar{f}_m(K)$ is surjective.

So let us show that $\bar{f}_m$ is faithfully flat. Let $y \in \bar{P}_m$, and let U be an affine open neighborhood of y in P_m. Since U is noetherian, we see that for some n_o

$$\bar{P}_m \cap U = P_{mn} \cap U \qquad \text{for all } n \geq n_o.$$

But f_{mn} induces a faithfully flat map

$$P_n \times_{P_m} U \to P_{mn} \times_{P_m} U = \bar{P}_m \cap U \qquad (n \geq n_o).$$

Since $P_n \times_{P_m} U$ and $\bar{P}_m \cap U$ are affine, and since for any ring R an inductive limit of faithfully flat R-algebras is still a faithfully flat R-algebra, we conclude that

$$P \times_{\bar{P}_m} (\bar{P}_m \cap U) = P \times_{P_m} U = \varprojlim_{n\geq n_o} (P_n \times_{P_m} U)$$

is faithfully flat over $\bar{P}_m \cap U$. Thus $\bar{f}_m$ is faithfully flat.

(3.2) Because of Proposition (3.1), we can <u>assume from now on that</u> $P_m = \bar{P}_m$ (so that all the maps f_{mn} ($= \bar{f}_{mn}$) are faithfully flat etc. etc.). Furthermore, certain additional conditions which may be imposed on the original f_{mn} (for example the condition that $\ker(f_{mn})$ be unipotent) will not be destroyed by this replacement of P_m by $\bar{P}_m$ (because of (iv) in Prop. (3.1)).

We examine now the connected component of the zero-point of P. Let P_n^o be the open and closed subgroup of P_n supported by the connected component of zero in P_n (cf. [DG, ch. II, §5, no. 1]). Then $f_{mn}:P_n \to P_m$ $(n \geq m)$ induces a map $f_{mn}^o:P_n^o \to P_m^o$, so we have an inverse system (P_n^o, f_{mn}^o). Set $P^o = \varprojlim P_n^o$.

PROPOSITION. (i) The maps f_{mn}^o are affine, faithfully flat and finitely presented; and $\ker(f_{mn}^o)$ is a closed subgroup of $\ker(f_{mn})$.

(ii) P^o is a closed irreducible subgroup of P, and the underlying subspace of P^o is the connected component of zero in P. Furthermore, if $x \in P^o$, then the canonical map of local rings $\mathcal{O}_{P,x} \to \mathcal{O}_{P^o,x}$ is bijective.

Proof. (i) is immediate except perhaps for the surjectivity of f_{mn}^o, which follows from the fact that the (topological) image of f_{mn}^o is open [EGA IV, (2.4.6)] and closed [DG, p. 249, (5.1)].

As for (ii), it is clear that P^o is a closed subgroup of P; and if Q is any connected subspace of P containing zero, then $f_n(Q) \subseteq P_n^o$ for all n ($f_n:P \to P_n$ being the natural map) whence $Q \subseteq P^o$ (since $P^o = \varprojlim P_n^o$ in the category of topological spaces [EGA IV, 8.2.9]). So for the first assertion of (ii), it remains to be shown that P^o is irreducible (hence connected). For this it suffices to show that P^o is covered by open irreducible subsets, any two of which have a non-empty intersection. P^o_o, being irreducible, has such a covering by irreducible affine subsets, and we can cover P^o by their inverse images. Since all the maps f^o_{on} are affine and each P^o_n is irreducible, we need only check that a direct limit of rings with irreducible spectrum has irreducible spectrum. But this is easily seen, since "A has irreducible spectrum" means that "for a, b $\in$ A, ab is nilpotent $\Leftrightarrow$ either a or b is nilpotent".

Finally, for $x \in P^o$, we have

$$\mathcal{O}_{P,x} = \varinjlim \mathcal{O}_{P_n, f_n(x)} = \varinjlim \mathcal{O}_{P^o_n, f_n(x)} = \mathcal{O}_{P^o,x} \quad .$$

Q.E.D.

Remark. Though P^o is not algebraic over k in general, it may nevertheless have certain finite-dimensional structural features. For example, when k is perfect, if A_n is the abelian variety which is a quotient of $(P^o_n)_{red}$ by its maximal linear subgroup L_n (structure theorem of Chevalley) then f_{mn} $(n \geq m)$ induces an epimorphism $A_n \to A_m$, with infinitesimal kernel. If furthermore the kernel of f_{mn} is unipotent (as would be the case, e.g. in (2.5) [L2; Cor. (2.11)]), then, writing

$$L_n = M_n \times U_n \qquad (M_n \text{ multiplicative, } U_n \text{ unipotent})$$

we find that f_{mn} induces an isomorphism $M_n \to M_m$.

(3.3) For each n, let $\pi_o(P_n)$ be the étale k-group P_n/P_n^o (cf. [DG, p. 237, Prop. (1.8)]). The natural map $q_n:P_n \to \pi_o(P_n)$ is faithfully flat and finitely presented (loc. cit). f_{mn} induces a map $\pi_o(f_{mn}):\pi_o(P_n) \to \pi_o(P_m)$, so we have an inverse system $(\pi_o(P_n), \pi_o(f_{mn}))$. We set $\pi_o(P) = \varprojlim \pi_o(P_n)$.

PROPOSITION. (i) The maps $\pi_o(f_{mn})$ are finite, étale, surjective; and $\ker(\pi_o(f_{mn}))$ is a quotient of $\ker(f_{mn})$.

(ii) The canonical map $q:P \to \pi_o(P)$ is faithfully flat and quasi-compact, with kernel P^o (so that the sequence

$$0 \to P^o \to P \to \pi_o(P) \to 0$$

is exact in the category of fpqc sheaves). The (topological) fibres of $P \to \pi_o(P)$ are irreducible, and they are the connected components of P. For any $x \in P$, the canonical map of local rings $\mathcal{O}_{P,x} \to \mathcal{O}_{q^{-1}q(x),x}$ is bijective. If K is an algebraically closed field containing k, then $P(K) \to \pi_o(P)(K)$ is surjective.

Proof. (i) Consider the commutative diagram (with $n \geq m$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_n^o & \longrightarrow & P_n & \xrightarrow{q_n} & \pi_o(P_n) & \longrightarrow & 0 \\ & & \downarrow f_{mn}^o & & \downarrow f_{mn} & & \downarrow \pi_o(f_{mn}) & & \\ 0 & \longrightarrow & P_m^o & \longrightarrow & P_m & \xrightarrow{q_m} & \pi_o(P_m) & \longrightarrow & 0 \end{array}$$

The maps in the rows are the natural ones, and the rows are exact in the category of fppf sheaves (when we identify k-groups with functors of k-algebras...). Since f^o_{mn} is an epimorphism of fppf sheaves (Prop. (3.2)), so therefore is the natural map $\ker(f_{mn}) \to \ker(\pi_o(f_{mn}))$, and we have the second assertion of (i).

f_{mn}, q_m, and q_n are all faithfully flat - hence surjective and quasi-compact, and then so is $\pi_o(f_{mn})$. Since $\pi_o(P_n)$ and $\pi_o(P_m)$ are étale over k, therefore the map $\pi_o(f_{mn})$ is étale. Thus the kernel of $\pi_o(f_{mn})$-being quasi-compact and étale over k - is finite over k, and it follows that the map $\pi_o(f_{mn})$ is finite.

(ii) For the last assertion, note that we have an inverse system of exact sequences

$$0 \to P^o_n(K) \to P_n(K) \to \pi_o(P_n)(K) \to 0$$

and that $P^o_{n+1}(K) \to P^o_n(K)$ is <u>surjective</u> for all n (Prop. (3.2)); so on passing to the inverse limit we obtain an exact sequence

$$0 \to P^o(K) \to P(K) \to \pi_o(P)(K) \to 0 \quad .$$

The exactness of $0 \to P^o \to P \overset{q}{\to} \pi_o(P)$ is straightforward. To show that q is flat let $x \in P$, $y = q(x)$, and let x_n, y_n be their images in P_n, $\pi_o(P_n)$ respectively. Then $\mathcal{O}_{P_n,x_n}$ is flat over $\mathcal{O}_{\pi_o(P_n),y_n}$, and passing to inductive limits, we see that $\mathcal{O}_{P,x}$ is flat over $\mathcal{O}_{\pi_o(P),y}$. Next let $z \in \pi_o(P)$, let

z_n be the image of z in $\pi_o(P_n)$, and let $Q = q^{-1}(z)$, $Q_n = q_n^{-1}(z)$. Note that Q_n is irreducible, and is a connected component of P_n. The Q_n form an inverse system of schemes, in which the transition maps are affine, and

$$Q = \varprojlim Q_n .$$

We show next that $Q_n \to Q_m$ <u>is surjective;</u> then it follows that Q is non-empty (so that q is surjective - hence faithfully flat) and the proof of Prop. (3.2) (ii) can be imitated to give all the assertions about the fibres of q.

Let $\bar{k}$ be the algebraic closure of k. By a simple translation argument, we deduce from the surjectivity of $P_n^o \to P_m^o$ that every component of $Q_n \otimes_k \bar{k}$ maps surjectively onto a component of $Q_m \otimes_k \bar{k}$; since every component of $Q_m \otimes_k \bar{k}$ projects surjectively onto Q_m, we find that $Q_n \to Q_m$ is indeed surjective.

It remains to be seen that q is quasi-compact. The fibres of the maps $\pi_o(f_n):\pi_o(P) \to \pi_o(P_n)$ $(n \geq 0)$ form a basis of open sets on $\pi_o(P)$ (since $\pi_o(P_n)$ is discrete as a topological space); furthermore these fibres are quasi-compact (since $\pi_o(f_n)$ is an affine map), and their inverse images in P are quasi-compact (the affine map $P \to P_n$ and the finitely presented map $P_n \to \pi_o(P_n)$ are both quasi-compact, so the composed map $P \to \pi_o(P_n)$ is quasi-compact); it follows that q is quasi-compact. Q.E.D.

Remarks.

1. Say that a k-group Q is <u>pro-étale</u> if it is of the form $\lim\limits_{\leftarrow} Q_n$, where (Q_n, g_{mn}) is an inverse system of the type we have been considering, with all the Q_n <u>étale</u> over k. For example $\pi_o(P)$ is pro-étale. It is immediate that if Q is pro-étale and $f:G \to Q$ is a map of k-groups, with G <u>connected</u>, then f is the zero-map. From this we see that, with P as above, <u>every map of P into a pro-étale k-group factors uniquely through</u> $P \to \pi_o(P)$.

2. Let (P_n, f_{mn}) be as above, and assume that the kernel of f_{mn} is <u>unipotent</u> for all m,n. Set $Q_n = \pi_o(P_n)$, $g_{mn} = \pi_o(f_{mn})$; by (i) of Proposition (3.3), the kernel of g_{mn} is étale and also unipotent (i.e. annihilated by p^t for some t, with p = char. of k). Assume also that the abelian group $Q_n(\bar{k})$ ($\bar{k}$ = algebraic closure of k) is <u>finitely generated</u> (for each n). (These assumptions hold in the situation described in (2.5), cf. [L2; Prop. (2.7), Cor. (2.11)].)

Let Q_n^t be the kernel of multiplication by p^t in Q_n. Then $Q_n^o \subseteq Q_n^1 \subseteq Q_n^2 \subseteq \ldots$, and since $Q_n(\bar{k})$ is finitely generated, we have, for large t, $Q_n^t = Q_n^{t+1} = \ldots$; so we can set

$$Q_n^{(p)} = \bigcup_t Q_n^t = Q_n^t \quad \text{for large } t.$$

Clearly $Q_n^{(p)}$ is finite étale over k, and unipotent; and the quotient $R_n = Q_n/Q_n^{(p)}$ is étale over k. Consider the commutative diagram $(n \geq m)$:

$$\begin{array}{ccccccccc} 0 & \to & Q_n^{(p)} & \to & Q_n & \to & R_n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Q_m^{(p)} & \to & Q_m & \to & R_m & \to & 0 \end{array} .$$

Straightforward arguments give that:

(i) Multiplication by p in R_n is a monomorphism.

(ii) $R_n \to R_m$ is an isomorphism.

(iii) $Q_n^{(p)} \to Q_m^{(p)}$ is an epimorphism.

Then, passing to the inverse limit, we obtain:

<u>There exists an exact sequence</u>

$$0 \to Q^{(p)} \to \pi_o(P) \to R \to 0$$

$Q^{(p)}$ = inverse limit of unipotent finite étale k-groups.

R = étale k-group such that the abelian group $R(\bar{k})$ ($\bar{k}$ = algebraic closure of k) is finitely generated and without p-torsion.

Here R is already determined by P_1.

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