UNIQUE FACTORIZATION IN COMPLETE LOCAL RINGS

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§0. Introduction: U.F.D.'s and algebraic geometry.

In this lecture I will report on a number of themes which can be traced back to a large extent to the work of Samuel in the early 1960's on the topic of unique factorization domains. Through his work, and especially through some fertile conjectures, Samuel stimulated a great deal of research in an area which was more or less dormant. As will become evident , a remarkable feature of the subsequent research was the extent to which methods of algebraic geometry were employed to give deep insight into what were apparently purely algebraic questions. (Major credit for this profitable synthesis belongs to Grothendieck.)

Recall that a unique factorization domain (U.F.D.), or factorial ring, is a commutative integral domain in which every non-zero element can be factored into irreducible ones in an essentially unique way. Every U.F.D. is normal (integrally closed in its field of fractions). We will deal only with noetherian U.F.D.'s. A noetherian normal domain is a U.F.D. if and only if every height one prime ideal is principal.

We can reformulate this criterion for noetherian U.F.D.'s in the following useful way. To each noetherian normal domain R we associate the group of divisors Div(R), which is defined to be the free abelian group generated by the height one prime ideals in R. Among the divisors we have the principal divisors, which are those of the form

$$(f) = \sum_{p} v_p(f) \qquad \text{(p - a height one prime ideal of R)}$$

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where f is a non-zero element of the fraction field of R, and for each height one prime p, v_p is the associated discrete valuation (with valuation ring R_p). We have

$$(f/g) = (f) - (g),$$

so the principal divisors form a subgroup of $\mathrm{Div}(R)$. The quotient

$$C\ell(R) = \mathrm{Div}(R)/(\text{principal divisors})$$

is called the divisor class group of R. To say that every height one prime is principal is to say that all divisors are principal; thus:

$$R \text{ is a U.F.D.} \iff C\ell(R) = (0)$$

(For details, cf. [2, §3]).

* * *

The concept of U.F.D. arose in connection with number theory (Euler, Gauss, Kummer,...), and has played an important role in that subject up to the present day. We have to pass by this line of development, in favor of the geometric aspects of the study of U.F.D.'s. Here is one:

Let V be a projectively normal closed subvariety of some projective space over a field k (so that the corresponding homogeneous coordinate ring $k[V]$ is a normal domain). Let R be the local ring at the vertex of the projecting cone over V (R is obtained from $k[V]$ by localizing at the maximal ideal generated by all homogeneous elements of positive degree). Samuel showed [30, §2] that

$$C\ell(R) \cong C\ell(V)/(\text{hyperplane section}).$$

[$C\ell(V)$ is the group of linear equivalence classes of codimension one cycles on V; and we are factoring out the subgroup "generated" by a hyperplane section, i.e. the least subgroup containing the equivalence classes of those cycles which are (scheme-theoretically) the complete intersection of V with a hypersurface of the ambient projective space.] Consequently:

R is a U.F.D.

$\iff$ (#) every irreducible codimension one subvariety of V is cut out (scheme-theoretically) by a hypersurface of the ambient projective space.

In case V is non-singular (or even locally factorial) the condition (#) means that every invertible $\mathcal{O}_V$-module is isomorphic to $\mathcal{O}_V(n)$ for some integer n. Some examples of non-singular projectively normal V's having this property are:

-(i) Grassmann varieties.

(These were treated by Severi and, over fields of positive characteristic, by Igusa. Cf. also [20, p. 124].)

-(ii) Non-singular complete intersections of dimension ≥ 3.

(This is the theorem of Lefschetz, mentioned in §3 of Hartshorne's talk on equivalence of cycles (these Proceedings); cf. also §5 below.)

-(iii) "Most" non-singular two-dimensional complete intersections.

(Lefschetz' generalization of Noether's theorem; cf. Hartshorne's talk and also [11, exposé XIX].)

Assuming that V is projectively normal, non-singular, and satisfies (#), one can show that for the completion $\hat{R}$ to be a U.F.D. a sufficient condition (which is also necessary if char.k = 0, or if dim.V = 1) is that

(##) $$H^1(V, \mathcal{O}_V(n)) = 0 \qquad \text{for all } n > 0$$

(cf. [7, §4])[2]. (##) is satisfied e.g. if V is a complete intersection and dim.V ≥ 2.

So we see that questions about local U.F.D.'s can have non-trivial global geometric significance.

§1. Unique factorization in formal power series rings.

It is elementary (going back to Gauss) that if R is a U.F.D. then so is any polynomial ring $R[X_1, X_2, \ldots, X_n]$.

In this section, we survey some highlights in the history of the corresponding question for formal power series rings:

For which noetherian U.F.D.'s R is it true that:
(*) any power series ring $R[[X_1, X_2, \ldots, X_n]]$ is a U.F.D.?

The story begins in 1905 with Lasker's proof that (*) is true if R is an infinite field. In 1938, Krull [21] showed the same thing with R any

[2]When k = complex numbers and dim.V ≥ 2, we have $H^1(V, \mathcal{O}_V(-n)) = 0$ for $n > 0$ (Kodaira's vanishing theorem), and (#) implies that $H^1(V, \mathcal{O}_V) = 0$ (since V has discrete Picard scheme). So (##) says that $H^1(V, \mathcal{O}_V(n)) = 0$ for all $n \in \mathbb{Z}$, which means precisely that depth(R)(= depth($\hat{R}$)) ≥ 3.

discrete valuation ring having infinite residue field, but he could not settle the finite residue field case[3]. This was done by Cohen in 1946 [6, p. 94, Theorem 18]. Another proof of Cohen's result, making more explicit use of the Weierstrass preparation theorem - and thus quite close to the proofs of Lasker and Krull - is given by Bourbaki in [2, §3.9, Proposition 8]. (What Krull missed was a simple lemma on automorphisms of power series rings [2, §3.7, Lemma 3]). Krull also expressed doubt that (*) would hold for R an arbitrary principal ideal domain [21, p. 770]; but in this he was mistaken [2, §3, exercise 9]. In fact in 1961 Samuel [29, pp. 3-4] and Buchsbaum [5, p. 753] showed more generally that (*) is true for any locally regular U.F.D. R. (The main ingredient of their proofs is the Auslander-Buchsbaum theorem that every regular local ring is a U.F.D.)

At the same time, Samuel gave the following result [29, p. 5]:

(I) If R is a U.F.D. whose localizations at maximal ideals are all Cohen-Macaulay, and if the power series ring in one variable $R_p[[X]]$ is a U.F.D. for all height two prime ideals p in R, then R[[X]] is a U.F.D.

(Samuel's proof of (I) is complicated, and is said by Danilov [9, p. 368] to be incomplete. However, Danilov [loc. cit] gives a more general result, which we will describe in §3.)

(I) brings the study of power series rings R[[X]] over two-dimensional local U.F.D.'s R to the foreground. Samuel found examples of such R for which R[[X]] is not a U.F.D [29], [30]. These examples were, however, suspect, in that their completions were not U.F.D.'s: to a geometer, properties of local rings which are not preserved under completion somehow lack authenticity. Anyway, Samuel conjectured [30, p. 171]:

(II) If R is a complete local U.F.D., then so is R[[X]] (?)

The first progress on (II) was due to Scheja, who showed [32, p. 128, Satz 2] that (II) holds whenever R has depth ≥ 3. A nice consequence of this is that if R[[X]] is a U.F.D., (R as in (II) then so is $R[[X_1,X_2,\ldots,X_n]]$ for any n. (For, if depth $R[[X]] < 3$, then R is either a field or a discrete valuation ring.)

Of course Scheja's result says nothing about (II) in case R is two-dimensional. One difficulty in attacking this case was that when (II) was formulated, there was only one example known of a non-regular two-dimensional henselian local U.F.D., namely the analytic ring $\mathbb{C}\{\{x,y,z\}\}/(x^2 + y^3 + z^5)$,

[3]Krull's idea, to deduce the finite case from the infinite one [21, p. 778], works well once one has available - which Krull didn't - some basic facts on faithfully flat ring extensions. (Cf. [12, p. 35, Cor. 6.11].)

which Mumford had investigated with transcendental methods [25, §IV].[4] Such was the state of knowledge (or rather ignorance) ten years ago.

Scheja tried to find a two-dimensional counterexample to (II). He proved that for any three-dimensional regular local ring S with maximal ideal generated by u, v, w, the two-dimensional ring $R = S/(u^2 + v^3 + w^5)$ is a U.F.D., but so is $R[[X]]$. He then discovered a number of previously unknown complete local two-dimensional U.F.D.'s; but each one of them satisfied Samuel's conjecture (cf. [32]). So after a while, he gave up. Apparently he did so too soon, for, as it turned out, he had actually found essentially all the two-dimensional R for which (II) holds - there are very few[5], whereas there are many others for which (II) fails.

The first counterexample to (II) was found by Salmon [28] - it is the ring

$$R = k(U)[[X,Y,Z]]/(X^2 + Y^3 + UZ^6)$$

where k is any field, and U is an indeterminate. Here $R[[X]]$ is not a U.F.D., whereas R is. But, if in the description of R we replace the field $k(U)$ by its algebraic closure, the resulting ring is no longer a U.F.D.. This again should arouse the scepticism of any geometer; it means that R is not a genuine U.F.D., in that $C\ell(R)$ has many non-zero elements which happen to be thinly concealed, i.e. defined over an algebraic extension of the residue field! Later on, a whole series of counterexamples was given by Danilov [8, §1] and Grothendieck [unpublished]; these all had the same deficiency: they lost their U.F.D. property when the residue field was extended to its algebraic closure.

And indeed, (II) is true if the residue field of R is algebraically closed.

In fact, a better result holds. To formulate it, we need the notion of a discrete divisor class group (DCG). If R is any normal noetherian domain, there is a canonical map $i:C\ell(R) \to C\ell(R[[X]])$ defined as follows: for any height one prime ideal p in R, $pR[[X]]$ is a height one prime ideal in $R[[X]]$; there is a unique map $Div(R) \to Div(R[[X]])$ sending each p to the corresponding divisor $pR[[X]]$;

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[4]Subsequently Brieskorn showed that there are no other analytic examples! [4]. (In [22, §25], Brieskorn's result is extended to arbitrary two-dimensional henselian local rings with algebraically closed residue field.) A propos, the ring in question has a distinguished history, going back to Klein's lectures on the icosahedron.

[5]Precisely those which have rational singularities (provided the singularity is resolvable), cf. Theorem 3 in §3. A complete list of rational U.F.D.'s is given in [22, §25].

one checks that this map takes principal divisors of R to principal divisors of R[[X]], and hence induces a map $i: C\ell(R) \to C\ell(R[[X]])$ (cf. [12, §6]). i is easily seen to be injective [12, p. 35, Cor. 6.13] (in fact i has a left inverse [ibid, p. 130, remarks]). R is said to have DCG if i is bijective.

Note that if R is a U.F.D. (i.e. $C\ell(R) = (0)$), and R has DCG, then R[[X]] is a U.F.D. (and conversely).

So (II) may be looked at from a more general point of view (due to Danilov): study rings with DCG.[6]

Here is the result:

THEOREM 1. Let R be a complete normal noetherian local ring with algebraically closed residue field, and suppose that $C\ell(R)$ is a finitely generated abelian group. Then R has DCG. (In particular, if R is a U.F.D., then so is R[[X]].)

The Theorem was proved by Danilov [10], with some additional restrictions on R, in the equicharacteristic case (i.e. when R contains a field);[7] his proof uses resolution of singularities and the theory of the Picard scheme of a scheme which is proper over a field. A much more elementary proof, using the theorem of Ramanujam-Samuel (cf. §5 below), was given by Storch [34], under the assumptions that R contains a field and that the residue field of R is uncountable and of cardinality strictly greater than that of $C\ell(R)$. Following some hints of Grothendieck, I worked out a theory of Picard schemes for schemes proper over any complete local ring; with this machinery, it was possible to make Danilov's and Storch's arguments apply to any R whose singularities can be resolved [23, Theorem 1']. Finally, Boutot showed how to get rid of the ungainly condition of resolvable singularities [23, Theorem 1].

§2. The local Picard scheme.

To get a proper feeling for the preceding results (and in particular for the terminology "DCG") one can adopt a philosophy due to Grothendieck. Let R be a normal complete local noetherian ring. Assume for simplicity that Spec(R) is regular outside the closed point, and that R contains a field of representatives k. Following some work of Mumford [25] on two-dimensional analytic local rings Grothendieck proposed a method for giving $C\ell(R)$ a natural structure of locally algebraic group - the "local Picard scheme" - over the residue field k of R. (For details, cf. [16, pp. 189-191]. Roughly speaking, there should be a locally algebraic k-group P, and

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[6] A survey of Danilov's excellent work on rings with DCG is given in §19 of [12].

[7] Under these conditions he also proves the converse: DCG ⇒ $C\ell(R)$ finitely generated.

for k-algebras K a natural (K-functorial) map

$$\theta_K : C\ell(R \hat{\otimes}_k K) \to P(K) \quad (= \text{K-valued points of } P)$$

which, as a first approximation, can be thought of as being bijective. (This is not true for all K, but never mind; at this moment we are just describing a philosophy, and don't want to get involved with technicalities.)

If P is discrete (i.e. zero-dimensional) then $P(k) = P(k[[X]])$, so $C\ell(R) = C\ell(R[[X]])$ and R has DCG. Conversely, if P has positive dimension, then there is a tremendous number of points in the kernel of $P(k[[X]]) \to P(k)$ (think of them as little analytic arcs on P emerging from the zero-point). So $C\ell(R[[X]])$ is bigger than $C\ell(R)$, i.e. R does not have DCG.

Now R is a U.F.D. if $P(k)$ $(= C\ell(R)) = (0)$, i.e if P has just one k-rational point. If k is algebraically closed, this means that P is zero-dimensional, so R has DCG. (This is the "explanation" of Theorem 1, §1).

If k is not algebraically closed, then $P(k)$ can be (0) even if P has positive dimension. In Salmon's example, for instance, P is the projective plane cubic curve defined over $k(U)$ by

$$X^2Z + Y^3 + UZ^3 = 0,$$

whose only $k(U)$-rational point is $(1,0,0)$. The above-mentioned counter-examples of Danilov and Grothendieck were constructed by completing the local rings at the vertices of projecting cones over certain curves, namely principal homogeneous spaces over elliptic curves having just one rational point over their field of definition; then P turns out to be the elliptic curve itself. As mentioned before, in each of these examples, the U.F.D. property is destroyed by extending the residue field to its algebraic closure; this appears now as a reflection of the fact that an elliptic curve over an algebraically closed field has many rational points.

Thus we can say that (II) (§1) is basically a geometric statement; the counterexamples of Salmon, Danilov and Grothendieck have arithmetic, but not geometric, significance.

It should be emphasized that the local Picard scheme, *per se*, plays no role, except for motivation, in the proof of Theorem 1. However the main lines of the proof are similar to - or suggested by - those in Grothendieck's proposed construction.

Actually, it is only recently that the theory of local Picard schemes has really been developed, by Boutot [3], who uses a different approach than the one outlined by Grothendieck. Boutot considers a local ring R with maximal ideal $\underline{m}$, such that R contains a field of representatives k (i.e. k maps canonically onto $R/\underline{m}$). Let $U = \text{Spec}(R) - \{\underline{m}\}$. (Note that for normal R, $C\ell(R) = \text{Pic}(U)$ if R has an isolated singularity, or more generally if U is locally factorial.) The idea is to make Pic(U) into a locally algebraic k-group in a natural way. To this end, for each noetherian k-algebra A, let $\hat{R}_A$ be the $\underline{\underline{m}}$-adic completion of $R \otimes_k A$, let $\hat{U}_A$ be the inverse image of U in $\mathrm{Spec}(\hat{R}_A)$, and consider the functor

$$P(A) = \mathrm{Pic}(\hat{U}_A).$$

Using M. Artin's representability criteria, Boutot shows ([3], and oral communication):

> Assume that depth $R \geq 2$ and that the k-vector space $H^1(U, \mathcal{O}_U)$ is finite-dimensional (these assumptions hold, for example, if R is complete, normal, and of dimension ≥ 3). Then the étale sheaf associated to P is a locally algebraic k-group, whose Zariski tangent space at the origin is $H^1(U, \mathcal{O}_U)$.

It seems reasonable to anticipate further interesting developments in the study of local Picard schemes. After all, they should be no less important for local rings than global Picard schemes are for varieties. One can hope, for example, that the local Pic. will enter in a significant way into the theory of classification and deformation of singularities.

§3. Depth and discrete divisor class groups.

This section is centered around Danilov's generalization of Samuel's theorem (I) (§1). Danilov's result [9, p. 374, Theorem 1] asserts that the DCG property lives in depth 2:

THEOREM 2. If the normal noetherian ring R is such that the localizations R_p have DCG for all prime ideals p such that $\mathrm{depth}(R_p) = 2$, then R itself has DCG.

(Danilov's proof of Theorem 2 uses some of the deep cohomological results of [16], and he needs some additional mild hypothesis on R; but there is a quite elementary proof which does not require this additional condition [24].)

Recall that a normal noetherian ring R satisfies the Serre condition (S_3) if for any prime ideal p with $\dim.(R_p) \geq 3$ we have also $\mathrm{depth}(R_p) \geq 3$.

COROLLARY 1. If the noetherian normal ring R satisfies (S_3), and if all its two-dimensional localizations R_p have DCG - for example if they are regular - then R has DCG.

As before, this focuses attention on the two-dimensional case, for which Danilov shows:

THEOREM 3 [9, §4]. For a two-dimensional normal local ring R with resolvable singularity and perfect residue field the following are equivalent:

(i) R has DCG.

(ii) R has a rational singularity.

(iii) The strict henselization of R has a finite divisor class group[8].

Danilov also shows [9, §5] that:

The converse of Corollary 1 holds if R is excellent and contains a field of characteristic zero.

(Here again he uses some of the heavy machinery from [16], plus the following fact: if R is an excellent $\mathbb{Q}$-algebra and $f: X \to \mathrm{Spec}(R)$ is a resolution of singularities, then R has DCG if and only if $H^1(X, \mathcal{O}_X) = (0)$.[9] (Hence the DCG property localizes for excellent $\mathbb{Q}$-algebras; whether this is true for more general R doesn't seem to be known.))

In view of Scheja's result (immediately following (II)(§1)), the preceding gives:

If R is a complete equicharacteristic zero U.F.D. of depth ≥ 3, then R satisfies (S_3). Furthermore, the two-dimensional localizations of R have at worst rational singularities (so they are explicitly known [22, §25]).

Here, if the residue field is algebraically closed (char. 0), the "depth ≥ 3" hypothesis is superfluous. For, the U.F.D. property implies that Boutot's local Picard scheme (§2) is zero-dimensional (it has just one rational point), so its tangent space $H^1(U, \mathcal{O}_U) = 0$, i.e. R has depth ≥ 3. Alternatively, as indicated just after Theorem 1 (§1), Danilov showed that a normal local ring R with algebraically closed char. 0 residue field, and with finitely generated divisor class group, has DCG provided it is complete (or, more generally, provided that a reduction to the complete case via Artin's approximation theorems is possible, for example if R is the

[8] Cf. also [22, §17]. The resolvability of the singularity of R is equivalent to analytic normality [ibid. §16.2]. Without this assumption, and without any assumption on the residue field, it is still true that R has DCG if and only if R has a pseudo-rational singularity [ibid, §9].

[9] "Explanation": $H^1(X, \mathcal{O}_X)$ is the tangent space at the origin of the local Picard scheme. (Danilov does not use this.)

local ring of an algebraic variety over $\mathbb{C}$, or if R is analytic). Hence, by the converse of Corollary 1, any such R satisfies (S_3).

A different approach to this last result is given by Hartshorne and Ogus, in [18, §2]. For excellent normal local rings R with residue field $\mathbb{C}$, whose completion is algebraizable, they show that the (S_3) property follows from the vanishing of $H^1(X, \mathcal{O}_X)$, where $f: X \to \mathrm{Spec}(R)$ is a resolution of singularities. Their methods are analytic, being based on a dualized version of the Grauert-Riemenschneider vanishing theorem.

In conclusion, we note that for the converse of Corollary 1, characteristic zero is essential. Serre has given examples (over fields of positive characteristic) of non-singular projective surfaces V whose Picard scheme is discrete (zero-dimensional) but not reduced. Danilov points out [9, pp. 376-377] that the projecting cone over a suitable projective embedding of such a V has a vertex whose completed local ring is normal, with DCG, and three-dimensional, but of depth two (so it doesn't satisfy (S_3)).

§4. U.F.D.'s which are not Cohen-Macaulay.

In Samuel's theorem (I)(§1) one assumption on the U.F.D. R is that the localizations of R are Cohen-Macaulay (C-M). At the end of the paper [29] where (I) is given, Samuel states that all the examples of U.F.D.'s known to him are locally C-M, and asks whether this is true in general.

In this connection, Murthy showed [26] (cf. also [12, §12] or [19, (7.18)]) that any C-M U.F.D. R which is a homomorphic image of a regular local ring is in fact a Gorenstein ring. The reason for this is that the "canonical" (or "dualizing") module of R is a reflexive fractionary ideal [19, (6.7), (7.29)], hence invertible, since R is a U.F.D.; and the invertibility of the canonical module characterizes Gorenstein rings [ibid, (5.9)].

Using a variant of local duality, Hartshorne and Ogus [18, §1] have improved Murthy's result by weakening the hypothesis that R be C-M to the two conditions:

(i) R satisfies (S_3) and

(ii) $\mathrm{depth}(R_p) \geq \frac{1}{2}\dim.(R_p) + 1$ for all primes p with $\dim.(R_p) \geq 5$.

For example, if R is a complete local U.F.D. with algebraically closed residue field of characteristic zero, then R satisfies (S_3) (cf. §3); hence if R has dimension ≤ 4, then both (i) and (ii) are satisfied, so R is a Gorenstein ring. (Cf. [18, p. 428]; this result is originally due to M. Raynaud (unpublished).)

The first example of a U.F.D. which is not locally C-M was given by Bertin [1]; it is the ring of invariants of a cyclic group G of order 4 acting linearly on a polynomial ring B in 4 indeterminates over a field k of characteristic 2. The U.F.D. property is established by a "Galois descent" technique of Samuel [12, §16], which gives an injective map of the divisor class group of the fixed ring B^G into $H^1(G,B^*)$ (B^* = units of B); but $B^* = k^*$, on which G acts trivially, so that

$$H^1(G,B^*) = H^1(G,k^*) = \mathrm{Hom}_{\mathbb{Z}}(G,k^*) = 0$$

(since k^* has no 2-torsion). (The same argument applies to a cyclic group of order p^r acting on any polynomial ring over a field of char.p.)

To show that her example was not locally C-M, Bertin constructed explicitly a system of homogeneous parameters which is not a regular sequence (cf. [12, §16.8]). Then Hochster and Roberts noticed [20, p. 127] that Bertin's example was very closely related to some surfaces previously studied by Serre (the same surfaces used in Danilov's example at the end of §3), and that Serre's computations led to another proof of the failure of the C-M property; since Serre's surfaces exist over fields of any characteristic ≥ 5, one gets examples similar to Bertin's over any such field.

In analogy with the situation discussed in §1, one could still reasonably ask whether any complete local U.F.D. is C-M.

This question was open until quite recently, when the answer was found to be "no". Freitag and Kiehl constructed a class of analytic local rings (over the complex numbers) which are U.F.D.'s of dimension 60 and depth 3, hence certainly not C-M. [15, p. 144, Thm. 5.8] (These examples arise in connection with the study of the cusps of Hilbert modular groups associated with totally real Galois extensions of $\mathbb{Q}$, whose Galois group is the alternating group A_5 (of order 60) on five elements; the methods used involve complex analysis, cohomology of groups, number theory,...). An argument of Danilov [10, p. 235, remarks 5 and 3], making use of Artin's analytic approximation theorem, implies that the completions of the Freitag-Kiehl examples are U.F.D.'s (which are not C-M).

Over fields of characteristic > 0, some complete local non C-M U.F.D.'s have been found even more recently by Fossum and Griffith. In fact they show that the completion of Bertin's example is a U.F.D. [13]. (Bertin's example is a graded subring of a polynomial ring, and completion is with respect to the powers of the irrelevant ideal.) More generally, in [14] they treat the following situation:

Let k be a field of characteristic $p > 0$, let $n > 0$, and let G be a cyclic group of order p^n operating on the polynomial ring $B = k[X_1,X_2,\ldots,X_{p^n}]$ by cyclically permuting the indeterminates. This action of G extends to the power series ring $\hat{B} = k[[X_1,X_2,\ldots,X_{p^n}]]$. Then:

(i) The fixed ring B^G is a graded U.F.D. which is not C-M.

(Proof: essentially the same as for the above-mentioned Hochster-Roberts examples.)

(ii) The fixed ring $\hat{B}^G$ is the completion of B^G at its irrelevant ideal (so that $\hat{B}^G$ is not C-M.)

(Proof: elementary.)

(iii) $C\ell(\hat{B}^G) \subseteq H^1(G,\hat{B}^*)$ ($\hat{B}^*$ = units of B)

(Proof: by Samuel's Galois descent.)

(iv) (Main Result). If $p^n = 4$, or $n = 1$ and $p \geq 5$, then $H^1(G,\hat{B}^*) = 0$ (whence, by (ii) and (iii), $\hat{B}^G$ is a complete local U.F.D. which is not C-M).

(Proof: elaborate computation.)

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I hope the matter will not rest here, for there does not seem to be any real insight yet into the connection between the U.F.D. and C-M properties for complete local rings (not even for those obtained from cones over non-singular projective varieties (§0)). By real insight, I mean the sort of understanding which the local Picard scheme gives us for the subjects treated in §1.

§5. Parafactoriality

The notion of parafactoriality was introduced by Grothendieck in [16], and used effectively there to study factorial rings.

Recall that a noetherian local ring R with maximal ideal $\underline{m}$ is *parafactorial* if depth $R \geq 2$ and $\mathrm{Pic}(\mathrm{Spec}(R) - \{\underline{m}\}) = 0$.

The connection of this notion with factoriality is the following simple fact [16, p. 130, Cor. 3.10]: *when* $\dim.R \geq 2$, R *is factorial if and only if*: R *is parafactorial and* R_p *is factorial for every prime ideal* $p \neq \underline{m}$.

An early application was to the following *conjecture of Samuel* [30, p. 172], which generalizes a classical global theorem of Lefschetz (cf. §0, example (ii)): *a local complete intersection which is factorial in codimension ≤3 is factorial*. In [16, p. 132] Grothendieck showed:

A local complete intersection of dimension ≥ 4 is parafactorial.

(From this, and from the above characterization of factorial rings, Samuel's conjecture follows immediately, by induction on the dimension.)

A result along similar lines has been proved by Ogus [27, p. 350, Cor. 3.14]. Let R, $\underline{m}$ be as above, with $R/\underline{m}$ of characteristic zero. Let $d = \dim.(R)$, and assume that R is a homomorphic image of a regular local ring A of dimension $d + r$. Let $U = \mathrm{Spec}(R) - \{\underline{m}\}$, $V = \mathrm{Spec}(A) -$ {closed point}. Assume further that:

(i) U is locally a complete intersection in V.

(ii) $\mathrm{depth}(R) \geq 3$.

(iii) $r \leq d - 3$ ("small embedding codimension").

Then, under these conditions, R is parafactorial.

Hartshorne and Ogus derive a number of corollaries from this result [18, §3]. Here is one: if, furthermore, R is locally factorial in codimension ≤ 3 (so that by assumption (i) and Grothendieck's above theorem, U is locally factorial) then R itself is factorial.

* * *

In response to a conjecture of Grothendieck, Samuel proved the following theorem [31]:

Let R, $\underline{m}$, be as above, and assume that the completion $\hat{R}$ is normal. Let A be the power series ring $R[[X]]$, and let p be the prime ideal $\underline{m}A$. Then the canonical map of divisor class groups $C\ell(A) \rightarrow C\ell(A_p)$ is bijective.

The same result for power series rings in any finite number of variables was proved independently by Ramanujam [33, appendix]. (He reduces the question to the one-variable case, then uses -as does Samuel- the Weierstrass preparation theorem.)

An easy consequence is that A is parafactorial. (In fact one can derive many other parafactorial rings, cf. [17, §(21.14)].)

This Ramanujam-Samuel theorem plays an important role in the proof of Theorem 1 (§1). It is also essential in Boutot's theory of local Pic. (§2), for showing that the zero-section of the "local Picard functor" P is represented by a closed immersion (this being one of Artin's representability conditions). In fact Boutot proves and uses the following strong form of the Ramanujam-Samuel theorem:

Let R and A be noetherian local rings, $\underline{m}$ the maximal ideal of R, $\phi: R \rightarrow A$ a local homomorphism making A formally smooth over R (for the usual topologies), and such that the residue field of A is finite over that of R. Let $q \in \mathrm{Spec}(A)$ be such that $q \not\subset \underline{m}A$ and depth $A_q \geq 3$. Then A_q is parafactorial.

Having used this as input for the existence of local Pic., Boutot then obtains further refinements as output, for example:

Let $\phi: R \rightarrow A$ be a local homomorphism of equicharacteristic local rings making A formally smooth over R. Then A is parafactorial under either of the following two sets of conditions:

(i) $\dim.(A) > \dim.(R)$ and $\mathrm{depth}(A) \geq 3$.

(ii) $\dim.(A) = \dim.(R)$, R is parafactorial and strictly henselian.

As a corollary of this last generalized form of Ramanujam-Samuel's theorem, Boutot gets the following result, which is manifestly related to the subject matter of §1:

Let $f:X \to S$ *be a regular morphism of equi-characteristic locally noetherian schemes. Suppose that the strict henselization of the local ring of each point of* S *is factorial. Then the same holds for* X.

Details will appear in Boutot's thesis.

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