AN ELEMENTARY THEORY OF GROTHENDIECK'S RESIDUE SYMBOL

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INTRODUCTION. Grothendieck has defined an intriguing homomorphism, the "residue symbol", and listed some of its basic properties (cf. [5, pp. 195-199], and also [1]). This symbol has found application in several areas ([2], [3], [7], [10]). Unfortunately, Grothendieck's treatment is embedded in a formidable global duality theory, which makes detailed proofs inaccessible to many who may find the symbol itself quite useful.

We outline here an approach to residues which requires only basic commutative and homological algebra. The feasibility of such an approach was suggested by Cartier fifteen or twenty years ago. It is both more elementary and more general than the one in [5]. It should be noted however that the formalism of residues does take on more meaning in the context of duality, from which it arose.

1. BASIC DEFINITIONS. All rings are commutative. We consider a ring A, and a homomorphism of A-algebras $\pi: R \to B$ where, as an A-module, B is finitely generated and projective. Let $S = R \otimes_A B$, and let $\gamma: S \to B$ be the unique A-algebra homomorphism such that $\gamma(r \otimes b) = \pi(r)b$. For any integer $q \geq 0$ we have natural maps

$$\mathrm{Ext}_R^q(B,B) = \mathrm{Ext}_S^q(B,\mathrm{Hom}_A(B,B)) \xrightarrow{\tau} \mathrm{Ext}_S^q(B,\mathrm{Hom}_A(B,A)) \to \mathrm{Hom}_A(\mathrm{Tor}_q^S(B,B),A).$$

(τ exists because of a natural B-isomorphism

$$\mathrm{Hom}_A(B,B) \otimes_S B \xrightarrow{\sim} \mathrm{Hom}_A(B,A)$$

induced by the A-linear <u>trace map</u> $\mathrm{tr}_{B/A}: \mathrm{Hom}_A(B,B) \to A$.)

Let $\Omega = \Omega_{R/A}$ be the R-module of Kähler A-differentials, and let J be the kernel of γ. There exist natural isomorphisms of exterior powers

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$$\Omega^q \otimes_R B = (\Lambda^q \Omega) \otimes_R B \xrightarrow{\sim} \Lambda^q(J/J^2) \qquad (q \geq 0)$$

and a natural homomorphism of graded B-algebras

$$\bigoplus_{q \geq 0} \Lambda^q(J/J^2) \longrightarrow \bigoplus_{q \geq 0} \mathrm{Tor}_q^S(B,B).$$

Combining all these maps, we obtain natural A-linear homomorphisms

$$\mathrm{Res}^q : \Omega^q \otimes_R \mathrm{Ext}_R^q(B,B) \to \mathrm{Tor}_q^S(B,B) \otimes_B \mathrm{Ext}_R^q(B,B) \to A \qquad (q \geq 0).$$

2. THE RESIDUE SYMBOL. Let $(f_1,\ldots,f_q)$ be a <u>regular sequence</u> in R. Suppose that B (as above) is R/I, where $I = (f_1,\ldots,f_q)R$. Then there are natural isomorphisms

$$\mathrm{Ext}_R^q(B,B) \xrightarrow{\sim} \mathrm{Hom}_B(\Lambda^q(I/I^2),B) \xrightarrow{\sim} B$$

so that Res^q gives an A-homomorphism

$$\mathrm{Res}_{R/A} : \mathrm{Hom}_B(\Lambda^q(I/I^2),\Omega^q/I\Omega^q) \to A.$$

For any $\omega \in \Omega^q$, let

$$\begin{bmatrix} \omega \\ f_1,\ldots,f_q \end{bmatrix} \in \mathrm{Hom}_B(\Lambda^q(I/I^2),\Omega^q/I\Omega^q)$$

be the map which takes $\bar{f}_1 \wedge \ldots \wedge \bar{f}_q$ $(\bar{f}_i = f_i \bmod I^2)$ to $\omega \bmod I\Omega^q$. We have then defined the symbol

$$\mathrm{Res}_{R/A} \begin{bmatrix} \omega \\ f_1,\ldots,f_q \end{bmatrix} \in A.$$

The formulation and proof of properties of this symbol corresponding to those listed in [5] is now a (quite involved) exercise in algebra.

3. EXAMPLE. Assume further that R is a polynomial or power series ring in q variables over A, so that $\Omega^q/I\Omega^q \cong B$. Then the B-linear map

$$\mathrm{Hom}_B(\Lambda^q(I/I^2),\Omega^q/I\Omega^q) \to \mathrm{Hom}_A(B,A)$$

corresponding to $\mathrm{Res}_{R/A}$ is an <u>isomorphism</u>, whose inverse is the map θ which is the main object of study in [8]. Theorem (4.2) of <u>loc. cit</u>. can be restated as follows:

$$\mathrm{Res}_{R/A}\begin{bmatrix} df_1 \wedge df_2 \wedge \ldots \wedge df_q \\ f_1, f_2, \ldots, f_q \end{bmatrix} = \mathrm{tr}_{B/A}(1).$$

This is part of (R6) on p. 198 of [5]. Scheja and Storch derive some interesting corollaries, for example concerning equality of Kähler and Dedekind differents.

4. LOCAL RESIDUES. In algebraic or analytic geometry we consider the case when A is a field, R is a q-dimensional local ring containing A and whose residue field is finite over A, and $B = R/I$, where I is an ideal whose radical is the maximal ideal M of R. There is then a commutative diagram

$$\begin{array}{ccc} \Omega^q \otimes_R \mathrm{Ext}^q_R(B,R) & \longrightarrow & \Omega^q \otimes_R H^q_M(R) = H^q_M(\Omega^q) \\ \downarrow & & \downarrow \rho \\ \Omega^q \otimes_R \mathrm{Ext}^q_R(B,B) & \xrightarrow{\ \mathrm{Res}^q\ } & A \end{array}$$

(cohomology supported in M)

where ρ, the <u>local residue</u> (or <u>trace</u>) <u>map</u>, does not depend on I. This ρ is a basic component of duality theory. For example, if V is a d-dimensional irreducible variety over a perfect field A, there is a <u>dualizing sheaf</u> on V [6, Chapter III, §7] whose stalk at a closed point $v \in V$, with local ring R, is the following R-module of meromorphic q-forms:

$$\{\omega/r \mid \omega \in \Omega^q,\ 0 \neq r \in R, \text{ and } \rho(\omega \otimes \lambda) = 0 \text{ for all } \lambda \in H^q_M(R) \text{ with } r\lambda = 0\}.$$

This dualizing sheaf is constructed in [4, Théorème 4.1] by means of Grothendieck's machinery. With our definition of residues, a more digestible treatment, in the spirit of [7], is anticipated.

The curve case $q = 1$ is presented in [9, Chapter IV, §3]. The connection with the foregoing may be clarified by an example:

5. EXAMPLE. Let A be a field, and $R = A[[X]]$ the formal power series ring in one variable. Let $K = R[1/X]$ be the fraction field of R.

Here

$$H^1_M(\Omega) = \Omega \otimes_R K$$

and for any $h = a_0 + a_1X + a_2X^2 + \ldots$ in R, and $s > 0$, we find that

$$\rho(hdX/X^s) = \mathrm{Res}_{R/A}\begin{bmatrix} hdX \\ X^s \end{bmatrix} = a_{s-1} ,$$

the coefficient of X^{-1} in h/X^s. This is of course the classical definition of the residue of hdX/X^s; as developed here, it is clearly independent of the choice of the "parameter" X (cf. [9, p. 25]).

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