COMPOSITIO MATHEMATICA, Vol. 38, Fasc. 1, 1979, pag. 37–43
Sijthoff & Noordhoff International Publishers – Alphen aan den Rijn
Printed in the Netherlands

DOUBLE POINT RESOLUTIONS OF DEFORMATIONS OF RATIONAL SINGULARITIES

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Let Y_0 be a normal algebraic surface over an algebraically closed field k, and let $f_0: Z_0 \to Y_0$ be the minimal resolution of singularities. There is a unique way to factor f_0 as $Z_0 \xrightarrow{g_0} X_0 \longrightarrow Y_0$ with a proper birational g_0 and a normal surface X_0, such that a reduced irreducible curve C on Z_0 which blows down to a point on Y_0 already blows down on X_0 iff C is non-singular and rational with self-intersection $C^2 = -2$ (cf. [**1**, p. 493, (2.7)]; or, for greater generality, [**7**, p. 275, (27.1)]). The singularities of X_0 are all **Rational Double Points.** X_0, which is uniquely determined by Y_0, will be called the **RDP-resolution** of Y_0.

Suppose now that Y_0 is affine and has just one singularity y, y being *rational.* There is a conjecture of Wahl [**9**], and Burns-Rapoport [**2**, 7.4] to the effect that the Artin component in the (formal or henselian) versal deformation space of y is obtained from the deformation space of Z_0 by factoring out a certain Coxeter group. According to Wahl [**10**], this would imply that the Artin component is smooth, and is in fact formally identical with the deformation space of X_0 (at least after things are suitably localised). Wahl also shows how the conjecture reduces to the statement that under "blowing down", the set of first order infinitesimal deformations of X_0 maps injectively into that of Y_0. Our purpose here is to outline a proof that *this injectivity* (and so the conjecture) *does indeed hold.*

Wahl has previously established some special cases of the conjecture by showing that a certain cohomology group vanishes [10]. I am grateful to him for all the information and motivation he has provided.

[1] AMS(MOS) subject classifications (1970). Primary 14J15, 32C45.
Research supported by the National Science Foundation.

Let $g: Y \to S$ be a flat map of finite type of noetherian schemes, such that for each $s \in S$ the fibre Y_s is a normal surface having only rational singularities. We assume – for simplicity only – that every closed point of Y maps to a closed point of S whose residue field is algebraically closed. By an RDP-resolution of $Y \to S$ is meant a proper map $f: X \to Y$ such that $g \circ f: X \to S$ is flat, and for each s the map $f_s: X_s \to Y_s$ is the RDP-resolution of Y_s (see above).

THEOREM: *Given $Y \to S$ as above, there exists at most one (up to Y-isomorphism)* RDP-*resolution.*

(For $S = \mathrm{Spec}(k[t]/t^2)$ we get the above-indicated injectivity statement.)

In fact we show the following:

Let $U \subseteq Y$ be the open set where g is smooth and let $i: U \to Y$ be the inclusion map. Set

$$\omega = i_*\Omega^2_{U/S} \quad (\Omega^2 = \text{relative 2-differentials})$$

and for all $n \geq 0$, let ω^n be the image of the natural map $\omega^{\otimes n} \to i_*((\Omega^2_{U/S})^{\otimes n})$. Then, if *an* RDP-*resolution $f: X \to Y$ exists,* we must have

$$X = \mathrm{Proj}(\bigoplus_{n \geq 0} \omega^n)$$

(with f the canonical map). In other words X is the scheme-theoretic closure of U in the projective bundle $\mathbf{P}(\omega)$ [4, *II*, (4.1.1)].

EXAMPLE (Suggested by Riemenschneider). Let Y_0 be the cone over a non-singular rational curve of degree 4 in $\mathbf{P}^4$ (say over the complex numbers $\mathbf{C}$). The versal deformation of the vertex has a one-dimensional *non-Artin* component, found by Pinkham: the corresponding deformation is $Y \to S = \mathrm{Spec}(\mathbf{C}[t])$, where $Y \subseteq \mathbf{C}^5 \times S$ is the zero-set of the 2×2 minors of

$$\begin{pmatrix} x_1 & x_2 & x_3 + t \\ x_2 & x_3 & x_4 \\ x_3 + t & x_4 & x_5 \end{pmatrix}$$

Here, of course, no RDP-resolution can exist. One computes that ω^n is isomorphic to J^n, where J is the fractionary ideal generated by

$(1/x_2,\ 1/(x_3+t),\ 1/x_4)$; and $\mathrm{Proj}(\bigoplus \omega^n)$ ($\cong$ blow-up of J) is non-singular, but *not* an RDP-resolution of Y. (The fibre over $t=0$ has two components!)

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After outlining the proof of some preliminary facts from duality theory (Lemma 1), we prove the Theorem in Lemmas 2 and 3 below.

Let $f: X \to Y$ be an RDP-resolution of $g: Y \to S$.

LEMMA 1: (cf. [5, p. 298] or 5′, Prop. 9) (i) *Since $g: Y \to S$ is flat, with Cohen–Macaulay fibres, the relative dualizing complex $g^!\mathcal{O}_S$ has just one non-zero cohomology sheaf $\omega_{Y/S}$, and $\omega_{Y/S}$ is flat over S. Since $g \circ f$ is flat, with Gorenstein fibres, the similarly defined $\mathcal{O}_X$-module $\omega_{X/S}$ is invertible.*

(ii) *For any map $S' \to S$, with S' noetherian, if $Y' = Y \times_S S'$ and $\pi: Y' \to Y$ is the projection, then $\omega_{Y'/S'} = \pi^*(\omega_{Y/S})$.*

PROOF: We may assume $S = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, with B a homomorphic image of a polynomial ring $R = A[\xi_1, \ldots, \xi_n]$ (cf. [5, p. 383, p. 388]). Then the first part of (i) states that

(i*): $\mathrm{Ext}^i_R(B, R) = 0$ for $i \neq n-2$, and $\mathrm{Ext}^{n-2}_R(B, R)$ is A-flat;

and (ii) says that

(ii*): for any noetherian A-algebra A', if $R' = R \otimes_A A'$ and $B' = B \otimes_A A'$, then the natural map

$$\mathrm{Ext}^{n-2}_R(B, R) \otimes_A A' \to \mathrm{Ext}^{n-2}_{R'}(B', R')$$

is an isomorphism.

(i*) and (ii*), together with Nakayama's lemma, reduce the proof of the last assertion in (i) to the well-known case where S is the spectrum of a field [5, p. 296, Prop. 9.3].

We first show that B *has homological dimension $n-2$ over R*. For this we may replace R by its localization at an arbitrary maximal ideal $\mathfrak{M}$, and A by its localization at $\mathfrak{M} \cap A$ [8, p. 188, Thm. 11]. Let $\mathfrak{m}$ be the maximal ideal of A, let $\bar{R} = R \otimes_A (A/\mathfrak{m})$ and $\bar{B} = B \otimes_A (A/\mathfrak{m})$, so that $\bar{B}$ is a normal two-dimensional homomorphic image of the regular n-dimensional local ring $\bar{R}$, and so $\bar{B}$ has homological dimension $n-2$ over $\bar{R}$. (What matters here and subsequently is that $\bar{B}$ is Cohen–Macaulay.) Let K be the residue field of R. Since R and B are A-flat, we have for any R-projective (hence A-flat) resolution $P.$ of B that the homology $H_j(P.\otimes_A(A/\mathfrak{m}))=\mathrm{Tor}_j^A(B, A/\mathfrak{m})=0$ for $j>0$, i.e. $P.\otimes_A(A/\mathfrak{m})$ is an $\bar{R}$-projective resolution of $\bar{B}$, whence, for all i,

$$(\#)\qquad \mathrm{Tor}_i^R(B, K)=\mathrm{Tor}_i^{\bar{R}}(\bar{B}, K);$$

and our assertion follows from [8, p. 193, Thm. 14].

Thus B has a finitely generated R-projective (hence A-flat) resolution

$$P.: 0\to P_{n-2}\to P_{n-3}\to\cdots\to P_0\to 0.$$

Let $Q.$ be the complex with

$$Q_i=\mathrm{Hom}_R(P_{n-2-i}, R)\quad (i\in Z).$$

For any A-algebra A' and any i we have

$$(\#\#)\qquad H_{n-2-i}(Q.\otimes_A A')=\mathrm{Ext}_R^i(B, R')=\mathrm{Ext}_{R'}^i(B', R').$$

(For the second equality cf. the proof of (#) above.)

Now (i*) results from the following Lemma. (We assume again, as we may, that A and R are local, and let $\mathfrak{m}$ be the maximal ideal of A.)

LEMMA 1a: *Let $Q.$ be an A-flat complex of finitely-generated R-modules, with $Q_i=0$ for $i<0$, and such that the homology $H_j(Q.\otimes_A(A/\mathfrak{m}))=0$ for all $j>0$. Then $H_0(Q.)$ is A-flat, and $H_j(Q.)=0$ for all $j>0$.*

The *proof* is left to the reader.

Finally, applying [4, III′, (7.3.1)(c) and (7.3.7)] to the homological functor $T.$ of A-modules M given by

$$T_p(M)=H_p(Q.\otimes_A M)=\mathrm{Ext}_R^{n-2-p}(B, R\otimes_A M)\quad (p\in Z)$$

($Q.$ as above) we see that for *every* A-module M and every i, there is a natural isomorphism

$$\mathrm{Ext}_R^i(B, R)\otimes_A M \xrightarrow{\sim} \mathrm{Ext}_R^i(B, R\otimes_A M).$$

In view of (##), taking $M=A'$ we get (ii*). Q.E.D.

LEMMA 2: *Let $\mathscr{L}$ be the invertible $\mathcal{O}_X$-module $\omega_{X/S}$. Then:*

(i) *$\mathscr{L}$ is very ample for f.*

(ii) *For every $n > 0$, the canonical map $(f_*\mathscr{L})^{\otimes n} \to f_*(\mathscr{L}^{\otimes n})$ is surjective.*

PROOF: We first show that $R^1 f_*(\mathcal{O}_X) = 0$: for any closed point y of Y, let $\mathfrak{m}_y$ be the maximal ideal of $\mathcal{O}_{Y,y}$; then for all $r \geq 0$, $\mathfrak{m}_y^r \mathcal{O}_X / \mathfrak{m}_y^{r+1} \mathcal{O}_X$ is an $\mathcal{O}_{f^{-1}(y)}$-module generated by its global sections; since Y_s ($s = g(y)$) has rational singularities, therefore $H^1(\mathcal{O}_{f^{-1}(y)}) = 0$, and since $f^{-1}(y)$ has dimension ≤ 1, we get $H^1(\mathfrak{m}_y^r \mathcal{O}_X / \mathfrak{m}_y^{r+1}) = 0$; conclude with [4, III, (4.2.1)].

Now by [7, p. 220, (12.1) and p. 211, proof of (7.4)] it will be enough to show for any reduced irreducible curve $C \subseteq f^{-1}(y)$ that $(\mathscr{L}.C)$ (the degree of $\mathscr{L}$ pulled back to C) is >0. Lemma 1 (ii) reduces us to the case $S = \mathrm{Spec}(k)$, k an algebraically closed field. Let $p: Z \to X$ be a minimal resolution of singularities, and let $\bar{C}$ be the component of $p^{-1}(C)$ which maps onto C. Then $(\mathscr{L}.C) = (p^{-1}\mathscr{L}.\bar{C})$. But $p^{-1}\mathscr{L}$ is a dualizing sheaf on Z [1, p. 493, (2.7)]; since (by definitior of RDP-resolution) $\bar{C}$ is not a nonsingular rational curve with $C^2 \geq -2$, therefore $(p^{-1}\mathscr{L}.\bar{C}) > 0$. Q.E.D.

Lemma 2(i) implies that $X = \mathrm{Proj}(\bigoplus_{n \geq 0} f_*(\mathscr{L}^{\otimes n}))$, cf. [4, II, (4.6.2), (4.6.3), (5.4.4)] or [4, III, (2.3.4.1)]; so we need only show that $f_*(\mathscr{L}^{\otimes n}) = \omega^n$ ($n > 0$). This is given by (ii) of Lemma 2 and by:

LEMMA 3: *Let $\mathscr{L} = \omega_{X/S}$. Then for all $n > 0$ there is a natural injective map*

$$\theta_n : f_*(\mathscr{L}^{\otimes n}) \hookrightarrow i_*((\Omega^2_{U/S})^{\otimes n});$$

and θ_1 is even bijective (i.e. $f_*\mathscr{L} = \omega$).

PROOF: Let $U \subseteq Y$ be as before, and let $j: f^{-1}(U) \to X$ be the inclusion map. f induces a proper map $f^{-1}(U) \to U$ which is fibrewise (over S) an isomorphism; hence $f^{-1}(U) \to U$ is an isomorphism. Now

$$\Omega^2_{U/S} = \omega_{U/S} = j^*\mathscr{L},$$

and θ_n is obtained by applying f_* to the natural map $\mathscr{L}^{\otimes n} \to j_* j^* \mathscr{L}^{\otimes n}$. For the injectivity, it suffices that the $(X - f^{-1}(U))$ depth of $\mathscr{L}$ be ≥ 1 [3, 1.9 and 3.8]. [4, IV, (11.3.8)] and (ii) of Lemma 1 above enable us to verify this fibrewise (over S); in other words, we need only check the simple case where $S = \mathrm{Spec}(k)$, k a field.

Similarly we see that the $(Y-U)$-depth of $\omega_{Y/S}$ is ≥ 2, and conclude that

$$\omega_{Y/S} = i_* i^* \omega_{Y/S} = \omega.$$

For the surjectivity of θ_1, it suffices (by Nakayama's Lemma) that $f_*\mathscr{L} \to \omega_s = \omega_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_s}$ be surjective for all $s \in S$. Since $f^*f_*\mathscr{L} \to \mathscr{L}$ is surjective (Lemma 2), so is $f^*f_*\mathscr{K} \to \mathscr{K}$, where $\mathscr{K}$ is the kernel of $\mathscr{L} \to \mathscr{L}_s = \mathscr{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_s}$; since $R^1 f_* \mathcal{O}_X = 0$ and f has fibres of dimension ≤ 1, therefore $R^1 f_* \mathscr{K} = 0$, and $f_*(\mathscr{L}) \to f_*(\mathscr{L}_s)$ is surjective. But $\mathscr{L}_s$ (resp. ω_s) is a dualizing sheaf on X_s (resp. Y_s), and since Y_s has rational singularities, therefore $f_*(\mathscr{L}_s) = \omega_s$ ([6, p. 606, 3.5] or, for greater generality, [7′, §2]). Q.E.D.

REMARK: For $S = \mathrm{Spec}(k[t]/t^2)$, Wahl has a proof of Lemma 3 which avoids duality theory [**10**, §2].

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(Oblatum 6–VI–1977 & 1–IX–1977)

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