IV.6

Introduction to Volume 4 of the Collected Papers of Oscar Zariski

(*with B. Teissier*)

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Introduction by J. Lipman and B. Teissier

The vision toward which the papers in this volume point is this: to seek a natural way of stratifying any algebraic or complex analytic variety X so that X is *equisingular along the strata,* i.e., the singularities which X has at the various points of each stratum are *equivalent* in some convincing sense.

An excellent introduction to the theories of equisingularity and saturation created in [81,82,85,89,90,93,97], is provided by Zariski himself in the expository papers [80,86,88,91,95]. In addition to reviewing salient features, we will indicate here some of the developments that have grown out of those theories. For more along these lines, see the report of Teissier.[31]

One of Zariski's basic ideas is that the equisingularity of a *hypersurface* $X \subset \mathbf{C}^N$ along a nonsingular subspace $Y \subset X$ around a point $0 \in Y$ should be defined (inductively, on the codimension of Y in X) by the equisingularity of the branch locus (i.e., reduced discriminant variety) $B_\pi \subset \mathbf{C}^{N-1}$ along $Y_\pi = \pi(Y)$ around $\pi(0)$, $\pi: X \to \mathbf{C}^{N-1}$ being a suitably general finite projection. The underlying feeling of a strong link between singularities and their "generic branch loci" can be traced back to Zariski's papers on fundamental groups [16–20,28,29]. (It is interesting to compare Zariski's ideas in these papers with recent proofs of the local and global versions of the Zariski-Lefschetz theorem given by Cheniot,[6] Hamm-Lê,[8] and Varchenko.[36]) The theory of saturation received some of its initial motivation from Whitney's work on topological triviality of analytic varieties along smooth subvarieties (cf. [85,§6]). Altogether, topology plays an important backstage role in Zariski's theory; and the conjunction of discriminant and topology provides the basis for many connections between his work on algebro-geometric equisingularity and recent work of others on monodromy, singularities of differentiable mappings, and analysis on singular varieties (cf. Teissier,[34] Varchenko,[40] and the end of section 3 below).

A really satisfactory theory of equisingularity exists only for the case when Y has codimension one in X. Here equisingularity of X along Y at 0 means that for some π, B_π is nonsingular. (It should be understood that we are thinking always of reduced *algebroid* varieties, or of *germs* of reduced complex analytic spaces). This case, studied in detail in [82], serves as a model for all further work in the area. There are many different criteria for equisingularity in codimension one, of algebro-geometric, differential-geometric, or topological nature. Each of these provides a theme for further development in higher codimension. But there some of the beautiful interconnections between the

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criteria vanish, and of others only a shadow is visible; the search for more substance remains a major challenge.

More specifically (cf. [82]), when B_π is nonsingular, Zariski showed that the singular locus $S = \mathrm{Sing}(X)$ is mapped isomorphically to B_π by π and that $\pi^{-1}(B_\pi) = S$ (this is the "non-splitting principle", cf. Teissier, p. 616[31]) and in particular S is nonsingular and of codimension one. Taking $Y = S$, we see that there exist retractions $X \to Y$ and that each of them displays X as a family of reduced plane curve germs parametrized by Y. The nonsingularity of B_π means that the irreducible components of any two curves in this family can be matched up in such a way that corresponding components have the same characteristic Puiseux exponents, and corresponding pairs of components have the same intersection multiplicity. (This germinal fact is implicit in Jung's work on local uniformization.[11]) This in turn means that the curves are *equivalent* from the viewpoint of resolution of singularities or, topologically, as embedded germs in $\mathbf{C}^2$. Conversely, any family of plane curve germs that has all its fibers reduced and equivalent in one of the above senses has singular locus (say Y) isomorphic to the parameter space (which is assumed nonsingular), and all projections $X \to Y \times \mathbf{C}$ "transversal" to X (i.e., in a direction not tangent to X at the origin) have nonsingular branch locus. Finally, given a hypersurface X whose singular locus Y is nonsingular and of codimension one, X is equisingular along Y if and only if the famous conditions (a) and (b) of Whitney[45] hold for the pair $(X - Y, Y)$. It can be shown in numerous ways that for any nonsingular Y of codimension one in X, the points of Y where X is equisingular form a dense Zariski-open subset of Y.

Complete as the theory is in codimension one, it does not exhaust all plausible notions of "non-variation of singularity type." For instance there are examples of Pham (cf. also Berthelot[4]) showing that the topology of a plane curve does not determine the topology of the versal unfolding of an equation of this curve and that very simple geometric features of the discriminant of this unfolding can change as the curve is deformed in an equisingular way.[21]

The codimension-one theory does not work as it stands for varieties over fields of characteristic >0, cf. Abhyankar.[1a,b] Perhaps in positive characteristic the concept of equivalence of plane curve singularities given in [81] (and further developed by Lejeune-Jalabert,[46] Moh,[47] and Fischer[48]) is not the definitive one.

We will now describe briefly some attempts to adapt the various equivalent ways of looking at codimension-one equisingularity to the case where the smooth subvariety Y has arbitrary codimension in the hypersurface X and also to the case where X is not a hypersurface. (Schickhoff even looks at some of these matters in the context of Banach spaces.[24])

1. Branch loci

The significance of the existence of equisingular projections π (those for which B_π is equisingular along Y_π) is not entirely clear. The theorem in [94] (cf. also Speder, p. 574[26]) is encouraging. Stronger positive evidence is provided first of all by the theorem of Varchenko, which states that if a family of hypersurface germs admits an equisingular projection (Y being the family of origins), then the family itself is topologically isomorphic to $X_0 \times Y$ for a suitable embedded germ X_0.[36,37,38] Secondly, Speder has shown that if X is "generically equisingular" along Y, then the pair (X-Sing X, Y) satisfies the Whitney conditions (i.e., X is "differentially equisingular" along Y [88], definition 2).[26] ("Generic" equisingularity is defined inductively by the condition that for "almost all" π, B_π is generically equisingular along Y_π).

For families X of isolated singularities of surfaces in $\mathbf{C}^3$, with smooth singular locus Y (of codimension two), Briançon[3] and Speder[27] show that the existence of *one* transversal equisingular projection already implies differential equisingularity. This also follows from a result of Lipman (unpublished) to the effect that the existence of such a projection implies the existence of a strong simultaneous resolution of the singularities of the family X, and Teissier's result:[33] "strong simultaneous resolution implies that the Whitney conditions hold" (the first result is proved only for families of surfaces, while the second holds quite generally). On the other hand, there are the following two examples of Briançon and Speder:

(1) Let $X \subseteq \mathbf{C}^4$ be given by

$$z^5 + ty^6z + y^7x + x^{15} = 0,$$

and let Y be the singular locus $x = y = z = 0$. For the projection $\pi(x,y,z,t) = (z,y,t)$, B_π is equisingular along Y_π. But X is *not* differentially equisingular along Y. Hence *no* transversal projection is equisingular; this answers negatively problem 1 on the second last page of [88].

(2) $X\colon z^3 + tx^4z + x^6 + y^6 = 0$

$Y\colon x = y = z = 0.$

Here X *is* differentially equisingular along Y; but again it can be shown that no transversal projection is equisingular.

The idea of using discriminants also appears from a completely different direction in the study of singularities, namely when it is realized that the number of vanishing cycles $\mu^{(N+1)}(X,0)$—or Milnor number[17]—of a hypersurface germ $(X,0) \subset (\mathbf{C}^{N+1},0)$ with isolated singularity is the order of vanishing of the discriminant of some map $(\mathbf{C}^{N+1},0) \to (\mathbf{C},0)$ having $(X,0)$ as fiber and that the discriminant of a general projection $f:(X,0) \to (\mathbf{C},0)$ vanishes to order $\mu^{(N+1)}(X,0) + \mu^{(N)}(f^{-1}(0),0)$ (cf. Teissier, section 5.5[34]). Actually $\mu^{(N)}(f^{-1}(0),0)$ does not depend on f, so we write $\mu^{(N)}(X,0)$ instead. Now it is easy to prove that $\mu^{(N+1)}(X,0)$ depends only on the topological type of the hypersurface $(X,0)$, but this is *not* so for $\mu^{(N)}(X,0)$. In fact, in the above example (1) of Briançon and Speder, considered as a family X_t of surfaces (with parameter t), the topological type of X_t does not depend on t (for t small), whereas $\mu^{(2)}(X_t,0)$ is different for $t \neq 0$ than for $t = 0$. In particular this shows that the topological type of a hypersurface germ does *not* determine the topological type of its general hyperplane section. To come back to $\mu^{(N+1)}$, Lê and Ramanujam proved that if it is constant in a family of hypersurfaces $(X_t,0)$ with isolated singularities and $N \neq 2$, then the fibers $(X_t,0)$ all have the same topological type.[12] Timourian showed that this implies that the family is locally topologically trivial.[35] However, as we have just seen, topological triviality does not imply that the family of discriminants of general projections $(X_t,0) \to (\mathbf{C},0)$ is trivial.

Apropos, there is a remarkable equivalence between *differential equisingularity* of a family of hypersurfaces with isolated singularities and *constancy of the sequence of Milnor numbers* of the members of the family together with their general linear sections of various dimensions. This area of investigation was opened up by Teissier[30] and further developed by Briançon and Speder. In fact, one of the ideas introduced in Teissier is the relationship between Zariski's discriminant conditions and the feeling that a "good" notion of equisingularity should have the following property:[31] if a hypersurface $X \subset \mathbf{C}^N$ is equisingular along Y, then for a sufficiently general nonsingular hypersurface $H \subset \mathbf{C}^N$ with $H \supset Y$, the intersection $X \cap H$ is equisingular along Y.

2. Saturation

There is nevertheless a fascinating theory when B_π is equisingular along Y_π in the most trivial sense, viz. B_π is *analytically a product* along Y_π. This is the theory of *equisaturation*. In order to capture algebraically the topological type in situations more general than that of plane curves, Zariski invented the notion of the *saturation* $\tilde{\mathfrak{o}}$ of a local ring $\mathfrak{o}$. For brevity, we deal here with "absolute saturation," in which case $\tilde{\mathfrak{o}}$ is defined by Zariski only when $\mathfrak{o}$ is either the local ring of a point on a hypersurface ([85, theorem 8.2],* [93, theorem 3.4]), or $\mathfrak{o}$ is one-dimensional ([93, appendix A]; cf. also Lipman[15] and Böger, Satz 5[5]). This $\tilde{\mathfrak{o}}$ is a local ring between $\mathfrak{o}$ and its normalization, and $\tilde{\mathfrak{o}}$ is radicial over $\mathfrak{o}$ [85,§4] (also Lipman[16]), so that in the analytic case the germs X and $\tilde{X}$ corresponding to $\mathfrak{o}$ and $\tilde{\mathfrak{o}}$ are locally homeomorphic, [85,§5] (also Seidenberg[25]). If $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are the local rings of two hypersurface germs X_1 and X_2, and if the saturations $\tilde{\mathfrak{o}}_1$ and $\tilde{\mathfrak{o}}_2$ are isomorphic, then X_1 and X_2 are topologically equivalent as embedded germs [85,§6]; and the converse is true if X_1 and X_2 are plane curves [90,§7]. Given $X \supset Y$ as before, and a retraction $\rho: X \to Y$, (X,ρ) is *equisaturated* along Y if the fibers $\rho^{-1}(y)$ $(y \in Y)$ have isomorphic saturations at their origins (this is a loose translation of [85, definition 7.3]). A basic fact is that X is equisaturated along Y if and only if for some sufficiently general π, B_π is analytically a product along Y_π [85,§7]. In particular, when Y has codimension one in X then equisaturation of X along Y is equivalent with equisingularity of X along Y.

Equisaturation of X (with local ring $\mathfrak{o}$) along Y also means that the germ $\tilde{X}$ corresponding to the local ring $\tilde{\mathfrak{o}}$ is analytically a product along Y. Since X and $\tilde{X}$ are homeomorphic, this implies that X is topologically a product along Y. Zariski proves more, namely that the *pair* $X \subset \mathbf{C}^N$ is topologically trivial along Y [85,§7].

Here a rather curious thing happened. Pham and Teissier tried to interpret Zariski's work, starting from the idea that topological triviality should be proved by integrating Lipschitz vector fields on X since they have the property of being integrable, of course, but also of extending locally to the ambient $\mathbf{C}^N$, by a pretty result of Banach. They were encouraged by the fact that Zariski's computations in [85] looked like the use of Lipschitz conditions. Therefore Pham and Teissier introduced a purely algebraic description, using the concept of integral dependence on ideals, of the sheaf of *locally Lipschitz meromorphic functions* $\tilde{\mathcal{O}}_X$ on a reduced space $(X,\mathcal{O}_X)$ and defined the absolute Lipschitz saturation of $\mathcal{O}_{X,x}$ as $\tilde{\mathcal{O}}_{X,x}$.[23] They indicated that in the case of *hypersurfaces*, Zariski saturation and Lipschitz saturation *coincide* (a counter-example in the non-hypersurface case was given by Zariski [93, introduction]). The relation between analytic triviality of B_π along Y_π and topological triviality of X along Y had then the following simple analytical explanation: any vector field on Y_π extends to a vector field on $\mathbf{C}^{N-1}$ tangent to B_π, and this can be lifted to a vector field on X with coefficients

* Theorem 8.3 of [85] does not hold as stated, but it does hold for hypersurfaces (cf. first paragraph in introduction to [85], and Böger, p. 247[5]).

which are *meromorphic* but satisfy (locally) a Lipschitz inequality and therefore are bounded; this lifted vector field extends to the ambient space, and if we take a basis of constant vector fields on Y to start with, the integration of corresponding vector fields in $\mathbf{C}^N$ provides a topological (even Lipschitz) trivialization of $X \subset \mathbf{C}^N$ along Y.

Lipschitz and Zariski saturation also coincide in the one-dimensional case (Lipman, p. 808, remark ii[16]). In [89] and [90] Zariski investigates thoroughly the structure and automorphisms of one-dimensional saturated local rings. The simplest version of the "structure theorem" [85, theorem 1.12] may be interpreted as saying that complete equicharacteristic one-dimensional saturated (i.e., equal to their saturation) local domains over, say, $\mathbf{C}$, are of the form $\mathbf{C}[[t^{a_1}, t^{a_2}, \ldots, t^{a_n}]]$ for suitable integers $a_1, \ldots, a_n$ (cf. *Math Reviews*, vol. 38, no. 5775). A class of one-dimensional rings which includes the saturated ones is studied by Lipman.[14] Notable among the algebraic features of one-dimensional saturated rings is the existence of "many automorphisms" ([93, appendix A]; also Böger, Satz 5[5]). This reflects the fact that saturation kills the *moduli* [92] of plane curve germs, in the previously mentioned sense that such germs have equivalent singularities at their origins if and only if they have isomorphic saturations. It is particularly enlightening, compared to the "Lipschitz" definition of saturation, to look at this result geometrically. Consider a nonplanar curve germ Γ. "Almost all" plane projections of Γ will have the same saturation as Γ. (This is the geometric meaning of the *existence of saturators* [85, proposition 1.6], a fact which also underlies the equality of one-dimensional Lipschitz and Zariski saturation.) Hence these plane projections have equivalent singularities at their origins. But conversely, *all* plane branches belonging to the same equivalence class can be obtained—up to isomorphism—as sufficiently general projections of a *single* Γ, namely the germ whose local ring is the saturation of the local ring of any one of the equivalent plane curves.

The algebraic theory of Lipschitz-saturation was taken up and improved on by Lipman[15,16] and Böger.[5,5a] Stutz provided new insight into the meaning of Lipschitz equisaturation when X is no longer a hypersurface, but Y is still of codimension one.[28,29] The joint theory of Zariski and Lipschitz saturation has been used by Nobile to prove an interesting theorem that implies in particular that any germ of reduced complex analytic surface is Lipschitz equivalent to an algebraic surface germ;[18] and by Teissier to give an algebraic proof of the fact that the constancy of Milnor's number in a family of plane curve germs implies that the family is equisingular (a result proved topologically by Lê-Ramanujam, see above).[32] To conclude on this topic, let us note that for a reduced complex analytic space X, the existence of a partition of X into nonsingular constructible strata X_α such that X is locally Lipschitz-trivial (and not just topologically trivial) along each stratum, seems to be completely open when X has dimension ≥ 3. However, Verdier has proved the existence of a stratification with "rugose" triviality, and rugosity is a Lipschitz-like condition, but relative to the stratification.[41]

3. Simultaneous resolution of singularities

Zariski proposes to geometers the following program for resolving singularities. Find for each complex analytic variety $X \subset \mathbf{C}^N$ a natural stratification with the following property: if $H \subset \mathbf{C}^N$ is a smooth hypersurface which avoids the "exceptional points" (i.e., the zero-dimensional strata), and cuts all the positive-dimensional strata transversally, then some suitable process of resolving the singularities of $X \cap H$ should propagate along the strata to resolve all the singularities of X outside the exceptional points. Thus, by induction on dimension, one would resolve almost all singularities of X. The remaining step would be to transform exceptional points into nonexceptional ones. For surfaces in $\mathbf{C}^3$, this idea is realized in [78] (cf. also [84, introduction]); the underlying fact is that the singularities of an equisingular family of plane curves can be simultaneously resolved by monoidal transformations [82, theorem 7.4]. A big problem in higher dimensions is that no *canonical* process for resolving singularities is known. Nevertheless, one would hope at least to be able to find natural stratifications which are "monoidally stable" in the sense that a certain class of permissible blow-ups $f: X' \to X$ would be *stratified* relative to the canonical stratifications on X' and X (i.e., f maps any stratum of X' smoothly onto some stratum of X).

There are several possible definitions of simultaneous resolution (cf. Teissier[33]), the strongest of which, as mentioned above, implies differential equisingularity. (Note: the family given in the above example (1) of Briançon-Speder admits a "weak" simultaneous resolution, though it is not differentially equisingular.) Several authors have approached equisingularity from the point of view of simultaneous resolution, and in some cases have succeeded in showing the nonsingularity of the local moduli space, a result which is not obvious even for plane curves (cf. Wahl,[42,43] Nobile,[19,20] and Teissier [92, appendix]).

The problem of finding criteria for simultaneous resolution in more general situations has its interest enhanced by the discovery by Arnol'd(§11)[2], Kushnirenko,[13] and Varchenko,[39,40] of very interesting "equiresolvable" families of functions having isolated critical points where the resolution can be explicitly constructed by a toroidal map $Z \to \mathbf{C}^{N+1}$ from the datum of the Newton polyhedron of the function, the families in question being made of "almost all" (in a precise sense) functions having a given Newton polyhedron. They also computed from the Newton polyhedron many invariants of the singularities, which are in fact invariants of the resolution, e.g., the zeta function of the monodromy, the "initial exponent" of the Fuchsian equation corresponding to the Gauss-Manin connection associated to the singularity, etc.

4. Differential equisingularity (X not necessarily a hypersurface)

For non-plane-curve germs, there is no satisfactory theory of equivalence of singularities. For example, any two irreducible curve germs in $\mathbf{C}^N$ ($N > 2$) are topologically equivalent. Nevertheless Stutz was able to generalize large portions of Zariski's equisingularity theory to the case where X is no longer a hypersurface, but $Y = \mathrm{Sing}(X)$ is still nonsingular and of codimension one;[28] and he brought out connections between equisingularity and the tangent cones C_4, C_5 of Whitney (§3).[44] We mentioned above the equivalence of equisingularity and differential equisingularity for families of plane curves. The following generalization is a distillation of results in Stutz[28] and Abhyankar.[1]

Theorem: Let X be a reduced d-dimensional complex analytic variety, let $Y = \mathrm{Sing}(X)$ be smooth and of dimension $d - 1$, and let $0 \in Y$. The following are equivalent:

(1) The *Whitney conditions* (a), (b) hold for the pair $(X - Y, Y)$ at every point in a neighborhood of 0 in Y.

(2) Every irreducible component X' of X at 0 contains Y (at least near 0), and the Zariski tangent cone T' ($\equiv$ Whitney's C_3) of X' at 0 is a d-plane; furthermore, if a sequence of points $x_i \in X' - Y$ approaches 0 and the tangent planes $T_{x_i}X'$ have a limit, then that limit is T'. (In other words $C_3 = C_4$ at 0; this is essentially a generalization of the *Jacobian criterion* [82,§5].)

(3) (*Simultaneous resolution*) If $\nu: \bar{X} \to X$ is the normalization of X, then (after replacing X by a suitable neighborhood of 0) we have:
(a) X is equimultiple along Y;
(b) $\bar{X}$ is nonsingular; and
(c) ν induces an etale covering $(\nu^{-1}(Y))_{\text{reduced}} \to Y$.

Moreover, when these equivalent conditions hold, then for every projection $\pi: X \to \mathbf{C}^d$ transversal to X, the branch locus B_π is nonsingular, and $\pi^{-1}(B_\pi) = Y$ (near 0). Conversely, if there exists a projection π with B_π nonsingular of dimension $d - 1$, and if every component of X contains and is equimultiple along $\pi^{-1}(B_\pi)$ near 0 (which is automatically so if X is a hypersurface), then the above conditions hold.

Teissier studies a refinement of differential equisingularity, called "c-cosecance,"[34] that grew out of ideas of Hironaka.[9] This equisingularity condition *is* stable for generic hypersurface sections (cf. end of section 1 above).

We mention also, in closing, a still open question of Zariski [88]: if two hypersurface germs have the same (embedded) topological type, do they have the same multiplicity? A step toward an affirmative answer is taken by Ephraim.[7] In this vein there is an intriguing result of Hironaka's that in a Whitney stratification of an arbitrary X, the closure of any stratum has the same multiplicity (possibly zero) at all points of any other fixed stratum.[10]

5. Paper [97] (a general theory of equisingularity): Branch loci revisited

In this fundamental paper, Zariski introduces the concept of the dimensionality type $\text{d.t.}_k(V,Q)$ of an algebroid hypersurface V at a point Q of V, with respect to a fixed coefficient field k of the local ring $\mathfrak{o}$ of V at its closed point P. The definition is by induction on the dimension, as follows: Set $\mathfrak{o} = k[[x_1,\ldots,x_{r+1}]] = k[[X_1,\ldots,X_{r+1}]]/(f)$, where $f(X_1,\ldots,X_{r+1}) = 0, f \in k[[X_1,\ldots,X_{r+1}]]$ is the equation of an algebroid hypersurface. Zariski introduces a new algebraic concept of a "generic projection" by adding infinitely many new independent variables $u_{i,A} = (u)$ where $A \in Z_0^{r+1}$, $1 \leq i \leq r$, and considering formal power series

$$x_i^* = \sum_{A \in Z_0^{r+1}, |A| \geq 1} u_{i,A} x^A \qquad (1 \leq i \leq r)$$

which in a precise sense define a generic projection into the affine space $A_{k^*}^r$ over the field k^* generated over k by the $u_{i,A}$. Zariski then defines the discriminant Δ_u^* of V with respect to this generic projection π_u; Δ_u^* is an algebroid hypersurface defined over k^*, and he defines the "image" Q^* of Q by π_u: He then defines inductively

$$\text{d.t.}_k(V,Q) = 1 + \text{d.t.}_{k^*}(\Delta_u^*, Q^*)$$

and $\text{d.t.}_k(V,Q) = 0$ if Q is a simple point of V. Intuitively $\text{d.t.}_k(V,P)$ is the codimension in V of the equisingularity stratum of P in V. The main theorem in [97] is that indeed the subsets $V(\sigma)$ of V consisting of the points of V where the dimensionality type of V is equal to a given integer σ form a stratification of V (by nonsingular subvarieties). Given now an algebraic hypersurface V over a field k, and defining the dimensionality type via the completions of the local rings, Zariski can then define an equisingular stratification for any algebraic hypersurface (an important fact, the semicontinuity of the dimensionality type for the Zariski topology along any algebraic subvariety W/k of V, is proved by Hironaka[49]). One of the beauties of Zariski's stratification is that, being defined by the constancy of a numerical invariant, it is uniquely defined (once k is fixed). Indeed the question of the independence of $\mathrm{d.t.}_k(V,P)$ on the field of representatives k is still open in general, although Zariski himself has important (unpublished) partial results. Another outstanding question is whether the dimensionality type can be computed by using only generic *linear* projections. In the complex-analytic framework this has been proved in the case where $\dim V = 3$ and V has a singular locus of dimension 1 by Briançon and Henry.[50]

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