REMARKS ON ADJOINTS AND ARITHMETIC GENERA OF ALGEBRAIC VARIETIES

By PIOTR BLASS and JOSEPH LIPMAN[1]

Let $\mathbf{P}^{n+1}$ be projective $(n+1)$-space over a fixed algebraically closed field k, let $F \subset \mathbf{P}^{n+1}$ be a (reduced) hypersurface of degree d, and let $f: \tilde{F} \to F$ be a desingularization. A classical characterization of the arithmetic genus of $\tilde{F}$:

$$p_a(\tilde{F}) = (-1)^n[\chi(\mathcal{O}_{\tilde{F}}) - 1] = h^n(\mathcal{O}_{\tilde{F}}) - h^{n-1}(\mathcal{O}_{\tilde{F}}) + \cdots + (-1)^{n-1}h^1(\mathcal{O}_{\tilde{F}})$$

is that

(#) *$p_a(\tilde{F}) - 1$ is the virtual (or "postulated") dimension of the linear system of hypersurfaces in $\mathbf{P}^{n+1}$ having degree $d - n - 2$ and adjoint to F.*

(For the case $n = 2$, cf. [**10**, p. 73]; for $n = 3$ see [**11**, p. 590, footnote 13].)

Our main observation is that this characterization is an immediate corollary of duality theory and the Grauert-Riemenschneider vanishing theorem, the latter being valid when the characteristic of k is zero [**2**], and, if $n \leq 2$, also for characteristic > 0 (combine [**8**, Proposition 2.6] with [**7**, Theorem 2.3]).

To see this, we restate the characterization in a form which depends only on F (*not its embedding in* $\mathbf{P}^{n+1}$), and is in fact meaningful for *any* complete algebraic variety (reduced and pure n-dimensional).

Let $\omega_{\tilde{F}}$ be a dualizing sheaf on $\tilde{F}$, and let ω_F be a dualizing sheaf on F (ω_F is determined up to isomorphism by its property of representing the functor $\mathrm{Hom}_k(H^n(\mathcal{L}), k)$ of coherent $\mathcal{O}_F$-modules $\mathcal{L}$, cf. [**4**, Chapter III, §7]. Also, $\omega_F = H^{-n}(\mathcal{R}_F)$, where $\mathcal{R}_F$ is a *residual complex* on F [**3**, Chapter VI].) Since $\mathcal{O}_{\mathbf{P}^{n+1}}(-n-2)$ is a dualizing sheaf on $\mathbf{P}^{n+1}$,

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the "adjunction formula" gives

$$\omega_F \cong \mathcal{E}xt^1_{\mathbf{P}^{n+1}}(\mathcal{O}_F, \mathcal{O}_{\mathbf{P}^{n+1}}(-n-2)) \cong \mathcal{O}_F(d-n-2).$$

There is a natural *injective* trace map

$$\tau : f_*(\omega_{\bar{F}}) \to \omega_F,$$ [2]

and the adjoint ideal $\mathcal{A} \subseteq \mathcal{O}_F$ may be defined to be the annihilator of the cokernel $\mathcal{C}$ of τ, i.e.

$$\mathcal{A} = \mathcal{H}om_{\mathcal{O}_F}(\omega_F, f_*(\omega_{\bar{F}})) = \mathcal{H}om_{\mathcal{O}_F}(\mathcal{O}_F(d-n-2), f_*(\omega_{\bar{F}})) = (f_*(\omega_{\bar{F}}))(-d+n+2).$$

Let $\pi : \mathcal{O}_{\mathbf{P}^{n+1}} \to \mathcal{O}_F$ be the natural map, and let $\mathcal{A}' = \pi^{-1}(\mathcal{A})$, so that we have an exact sequence

(*) $$0 \to (\text{kernel of } \pi) = \mathcal{O}_{\mathbf{P}^{n+1}}(-d) \to \mathcal{A}' \to \mathcal{A} \to 0.$$ [3]

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[2] τ can be obtained from [3, p. 369, Theorem 2.1]. In more down-to-earth terms, τ can be described (sketchily) as follows. Factor f as $\bar{F} \xrightarrow{g} F_1 \xrightarrow{h} F$, where F_1 is the normalization of F. Over any affine open set $U \subseteq F$, the sections $\Gamma(U, \omega_F)$ can be identified with certain meromorphic n-differentials on F, cf. [6] from which one also sees that

$$\Gamma(U, h_*\omega_{F_1}) \subseteq \Gamma(U, \omega_F).$$

Furthermore, $\Gamma(U, h_*\omega_{F_1})$ consists of meromorphic n-forms having no polar divisors in $h^{-1}(U)$, while $\Gamma(U, f_*\omega_{\bar{F}})$ consists of meromorphic n-forms without poles in $g^{-1}h^{-1}(U)$; thus

$$\Gamma(U, f_*\omega_{\bar{F}}) \subseteq \Gamma(U, h_*\omega_{F_1}) \subseteq \Gamma(U, \omega_F),$$

and τ is just the resulting inclusion map.

[3] So the geometric genus $p_g(\bar{F}) = H^0(f_*(\omega_{\bar{F}}))$ satisfies

$$p_g(\bar{F}) - 1 = H^0(\mathcal{A}(d-n-2)) - 1 = H^0(\mathcal{A}'(d-n-2)) - 1$$

(cf. (*)), which is the *actual* dimension of the linear system of hypersurfaces of degree $d-n-2$ adjoint to F.

What (#) says then is that

$$
\begin{aligned}
(-1)^n[\chi(\mathcal{O}_{\tilde F}) - 1] &= \chi(\mathfrak{A}'(d-n-2)) \\
&= \chi(\mathfrak{A}(d-n-2)) + \chi(\mathcal{O}_{\mathbf{P}^{n+1}}(-n-2)) \\
&= \chi(\mathfrak{A}(d-n-2)) + (-1)^{n+1} \\
&= \chi(f_*(\omega_F)) + (-1)^{n+1};
\end{aligned}
$$

and finally Serre duality gives

$$(-1)^n\chi(\mathcal{O}_{\tilde F}) = \chi(\omega_{\tilde F}).$$

Thus, (#) simply says that

$$\text{(\#\#)} \qquad \chi(\omega_{\tilde F}) = \chi(f_*(\omega_{\tilde F})).$$

But (##) holds for *any* reduced pure n-dimensional variety F proper over k, with $n \leq 2$ when k has characteristic > 0 (for such F a desingularization $f:\tilde F \to F$ always exists [5], [1]). For, the Leray spectral sequence for f gives

$$\chi(\omega_{\tilde F}) = \sum_{i=0}^{n-1} (-1)^i \chi(R^i f_*(\omega_{\tilde F}));$$

and the vanishing theorem says that $R^i f_*(\omega_{\tilde F}) = 0$ for $i > 0$. (In fact the spectral sequence degenerates, so that $H^i(\tilde F, \omega_{\tilde F}) = H^i(F, f_*\omega_{\tilde F})$ for all i). Q.E.D.

Remarks. (1) The forms of degree j adjoint to F form a finite dimensional k-vector space

$$A_j = H^0(\mathbf{P}^{n+1}, \mathfrak{A}'(j)) \subseteq H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(j)) = S_j \qquad (0 \leq j < \infty)$$

and $A = \bigoplus_{j=0}^{\infty} A_j$ is an ideal in the polynomial ring

$$k[X_0, X_1, \ldots, X_{n+1}] = \bigoplus_{j=0}^{\infty} S_j.$$

The dimension of A_j is given, for sufficiently large j, by the polynomial

$$H_A(j) = \chi(\mathfrak{A}'(j));$$

and so (#) says that

$$p_a(\tilde{F}) = H_A(d - n - 2).$$

All this begs the question: how to calculate A, or at least H_A?

In general, there is no easy answer. In case $\tilde{F}$ is the normalization of F (for example in the classical situation where F is a generic projection of $\tilde{F}$ into $\mathbf{P}^{n+1}$) we have

$$\begin{aligned} f_*(\omega_{\tilde{F}}) &= \mathcal{H}om_{\mathcal{O}_F}(f_*(\mathcal{O}_{\tilde{F}}), \omega_F) \\ &= [\mathcal{H}om_{\mathcal{O}_F}(f_*(\mathcal{O}_{\tilde{F}}), \mathcal{O}_F)](d - n - 2) \end{aligned}$$

and τ is "evaluation at 1" (cf. [**3**, p. 319, (c)]; or use [**6**] as in footnote 2 above). Hence

$$\mathfrak{A} = \mathcal{H}om_{\mathcal{O}_F}(f_*(\mathcal{O}_{\tilde{F}}), \mathcal{O}_F)$$

which is the *conductor* of $\mathcal{O}_{\tilde{F}}$ in $\mathcal{O}_F$. Thus, in this case, adjoints coincide with *subadjoints*, which are more or less calculable [cf. [**10**, pp. 71–72]).

(2) For any reduced pure m-dimensional Cohen-Macaulay variety F, proper over k, by Serre-Grothendieck duality $\chi(\mathcal{O}_F) = (-1)^m\chi(\omega_F)$. So, given a desingularization $f:\tilde{F} \to F$, we have, using Grauert-Riemenschneider as above,

$$\begin{aligned} p_a(F) - p_a(\tilde{F}) = (-1)^m(\chi(\mathcal{O}_F) - \chi(\mathcal{O}_{\tilde{F}})) &= \chi(\omega_F) - \chi(\omega_{\tilde{F}}) \\ &= \chi(\omega_F) - \chi(f_*\omega_{\tilde{F}}) \\ &= \chi(\mathcal{C}). \end{aligned}$$

($\mathcal{C}$ is the cokernel of the above injective map τ). This generalizes formula (**) on p. 153 of [**7**].

If, in particular, $F \subseteq \mathbf{P}^{n+1}$, and F is such that the dualizing sheaf ω_F is $\mathcal{O}_F(D - n - 2)$ for some D (for example if the vertex of the projecting cone over F is Gorenstein; in particular if F is a complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n+1-m}$, we can take $D =$

$d_1 + \cdots + d_{n+1-m}$) then, proceeding as above, we have

$$\mathcal{C} = \mathcal{O}_{\mathbf{P}^{n+1}}(D - n - 2)/\mathfrak{A}'(D - n - 2)$$

so that

$$(\#)' \; p_a(F) - p_a(\tilde{F}) = \chi(\mathcal{C})$$
$$= \chi(\mathcal{O}_{\mathbf{P}^{n+1}}(D - n - 2)) - \chi(\mathfrak{A}'(D - n - 2))$$
$$= \binom{D-1}{n+1} - \chi(\mathfrak{A}'(D - n - 2)).$$

(If F is a hypersurface of degree D then

$$p_a(F) = \binom{D-1}{n+1},$$

and $(\#)'$ reduces to $(\#)$.)

(3) Let F be as in (2), with $m \leq 2$ if k has characteristic > 0. If F is normal and has only isolated singularities, then $\mathcal{C}$ has zero-dimensional support (viz. on the singular points), and

$$\chi(\mathcal{C}) = \sum_{x \text{ singular}} \dim_k(\mathcal{C}_x).$$

Here is another "dual" description of $\dim_k(\mathcal{C}_x)$. Let U be an affine neighborhood of a singular point x, containing no other singular point, let $\tilde{U} = f^{-1}(U)$, $E = f^{-1}(x)$, and assume (without real loss of generality) that f induces an isomorphism $\tilde{U} - E \xrightarrow{\sim} U - x$. We have an exact sequence

$$0 \to H^0_E(\omega_{\tilde{F}}) \to H^0(\tilde{U}, \omega_{\tilde{F}}) \to H^0(U - x, \omega_{\tilde{F}})$$
$$\to H^1_E(\omega_{\tilde{F}}) \to H^1(\tilde{U}, \omega_{\tilde{F}}) = 0$$

(the last equality by Grauert-Riemenschneider). Now

$$H^0(U - x, \omega_{\tilde{F}}) = H^0(U - x, \omega_F) = H^0(U, \omega_F)$$

because, F being normal, ω_F is the sheaf of meromorphic m-forms with no polar divisors on F.

We conclude that

$$\mathcal{C}_x \cong H^1_E(\omega_{\tilde{F}}).$$

But by [7, p. 188], $H^1_E(\omega_{\tilde{F}})$ is dual to the stalk $R^{m-1}f_*(\mathcal{O}_{\tilde{F}})_x$. Thus

$$\mathcal{C}_x \cong \mathrm{Hom}_k(R^{m-1}f_*(\mathcal{O}_{\tilde{F}})_x, k)$$

and so

$$\dim_k(\mathcal{C}_x) = \dim_k(R^{m-1}f_*(\mathcal{O}_{\tilde{F}})_x).$$

(This is essentially Theorem A in [**9**].)

PURDUE UNIVERSITY

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