Addenda to the introduction of Volume 4 of Oscar Zariski's Collected papers

§1: Branch loci

The connection between the geometry of branch loci and differential equisingularity is now better understood, thanks to the use of the local polar varieties (see the addendum to §4 below); in particular the numbers μ^i associated to the isolated singularities of hypersurfaces have been generalized as *polar multiplicities*, stressing their connection with multiplicities of discriminant loci. A recent result on the geometry of the local polar varieties due to Henry and Merle has provided a clear picture of Zariski equisingularity in codimension 2.

Concerning the connection between Zariski equisingularity and resolution, a very significant example was found by Luengo: it is a one parameter family of surfaces, with equation $z^7 + y^7 + ty^5x^3 + x^1 0 = 0$; the discriminant of a general projection is equisingular along the image of the t-axis T, but there is no equisingularity along a stratum mapping onto T.

Bibliography:

J.P.G. Henry et M. Merle: Fronces et doubles plis, *preprint,* École Polytechnique, 91928 Palaiseau, 1989.
Ignacio Luengo: A counterexample to a conjecture of Zariski, *Math. Annalen,* **267**, (1984), 487–494.

§2: Saturation

A definition of saturation in positive characteristic was introduced and studied by Campillo and his students.

Bibliography:

Antonio Campillo, On saturations of curve singularities (any characteristic), *Proc. Symposia in Pure Math.,* Vol. 40, part 1, 211–220.

§3: Simultaneous resolution of singularities

Several canonical processes for resolving singularities of algebraic varieties in characteristic zero, or of complex-analytic spaces, have now been discovered. The first method was announced by Abhyankar, but has not yet been completely written up. A paper providing a constructive resolution was published by Villamayor. A preprint by Moh treats the hypersurface case. Another approach is being developed by Youssin. Most recently, a very elegant proof of canonical resolution has been found by Bierstone and Milman (in preparation).

Concerning simultaneous resolution, a breakthrough was made by Laufer, who proved numerical criteria for (weak and strong) simultaneous resolution of a family of isolated Gorenstein *surface* singularities, in the complex analytic framework. The resolution is not, however, obtained in general by blowing up nonsingular permissible centers. Then Vaquié proved a somewhat more general result in algebraic geometry, extending to a criterion of simultaneous resolution for rational surface singularities; and finally Kollár and Shepherd-Barron proved another similar result using recent advances in the theory of three-dimensional algebraic varieties. The general case remains completely open.

Bibliography:

S. S. Abhyankar, Weighted Expansions for Canonical Desingularization, *Lecture Notes in Math.* No. 910, Springer-Verlag, New York (1982).
Edward Bierstone and Pierre Milman, paper in preparation. Henry Laufer, Weak simultaneous resolution for deformations of Gorenstein surface singularities, *Proc. of Symposia in pure Math.*, Vol. 40, part 2, A.M.S. 1983, 1–30.
Henry Laufer, Strong simultaneous resolution for surface singularities, *Advanced studies in Pure Mathematics,* **8**, 1987, 207-214.
T. T. Moh, Canonical resolution of hypersurface singularities of characteristic zero, *preprint.*
Nick Shepherd-Barron and János Kollár, Threefolds and deformations of surface singularities, *Inventiones math.*, **91**, (1988), 299–308.
Michel Vaquié, Résolution simultanée des surfaces normales, *Ann. Institut Fourier*, **35**, (1985), 1–38.
Orlando Villamayor, Constructiveness of Hironaka's resolution, *Ann. sci. École normale supérieure,* **22**, (1989), 1–32.
Boris Youssin, Canonical formal uniformization, in preparation.

§4: Differential equisingularity

An algebraic characterization of differential equisingularity for reduced complex spaces and algebraic varieties over **C** was given by Teissier, who also answered affirmatively the question of Zariski in [88] on the analyticity of the locus of points of a subspace where the Whitney conditions are not satisfied. The key tool is the collection of local polar varieties introduced by Lê and Teissier, and the main result is a characterization of the differential equisingularity of the stratum X°= (smooth part of a reduced complex space X which is purely of dimension d) along a nonsingular space $Y \subset X$ by the local constancy along Y of the sequence $M^*_{X,y} = (m_y(P_0(X)), \dots, m_y(P_{d-1}(X)))$, where $m_y(P_k(X))$ is the multiplicity at y of the (general) polar variety of codimension k of X. Since $P_0(X) = X$, this contains the result of Hironaka mentioned above. The paper of Merle and Henry mentioned in §1 can be deemed to generalize this criterion to a criterion for Zariski equisingularity in codimension 2.

The topological invariance of hypersurface multiplicity is still an open question, but affirmative answers have been given in special cases (see the paper of Laufer mentioned below and its references, and also Cor. 4 in the paper of Yau and Xu); and the C^1-invariance of multiplicity in any codimension has been proved by Gau and Lipman.

Bibliography:

Yih-Nan Gau and J. Lipman: Differential invariance of multiplicity in algebraic varieties. *Inventiones math.* **73**, (1983), 165–188.

Henry B. Laufer: Tangent cones for deformations of two-dimensional quasi-homogeneous singularities, *Contemporary Mathematics,* Vol. 90, A.M.S., 1989, 183–197.

Dũng Tráng Lê et Bernard Teissier, Variétés polaires locales et classes de Chern des variétés singulières, *Annals of Math.* **114**, 1981, 457–491.

Bernard Teissier, Variétés polaires 2, *Algebraic Geometry, La Rábida 1981,* Springer Lecture Notes No. 961.

Stephen Yau and Yijing Xu, Classification of topological types of isolated quasi-homogeneous two dimensional hypersurface singularities, *Manuscripta math.* **64**, 1989, 445–469.