JACOBIAN IDEALS AND A THEOREM OF BRIANÇON-SKODA

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1. PRINCIPAL RESULTS

We prove the following generalization of a theorem of Briançon-Skoda (for background cf. [5], [13]):

THEOREM 1. *Let R be a commutative noetherian normal integral domain, and let I_0 be an ideal in R such that the associated graded ring $\bigoplus_{n\geq 0}(I_0^n/I_0^{n+1})$ is regular* (i.e., *all its localizations at prime ideals are regular local rings; for example I_0 could be any ideal such that both R/I_0 and the ring of fractions R_{1+I_0} are regular*). *Let I be an ideal of the form $I = I_0 + (y_1, y_2, \ldots, y_{d+1})R$ $(d \geq 0)$ and let $\lambda \geq 1$ be any positive integer. Then $\overline{I^{d+\lambda}} \subset I^\lambda$ where "$\bar{\ }$" denotes "integral closure" of an ideal.*

Remarks. (1) Some other versions of Theorem 1 are given at the end of this section.

(2) For $d = 0$, Theorem 1 says that *all powers of I are integrally closed.* For more on this situation see Section 4.

Theorem 1 is a corollary of:

THEOREM 1′. *Let R^* be a commutative noetherian normal integral domain and let $0 \neq t \in R^*$ be such that R^*/tR^* is regular. Let $y_1, \ldots, y_{d+1} \in R^*$, let $S = R^*[y_1/t, \ldots, y_{d+1}/t]$ and let $\bar{S}$ be the integral closure of S (in its field of fractions). Then $t^d\bar{S} \subset S$.*

Indeed, take t to be an indeterminate over R, and set

$$R^* = R[t, I_0t^{-1}] = \bigoplus_{n\in\mathbf{Z}} I_0^n t^{-n} \qquad (I_0^n = R \quad \text{if} \quad n \leq 0).$$

Then R^* is normal (because each I_0^n is a valuation ideal) and $R^*/tR^* = \bigoplus_{n\geq 0} I_0^n/I_0^{n+1}$ is regular. Now with I as in Theorem 1, the ring S of Theorem 1′ is the graded ring $S = \bigoplus_{n\in\mathbf{Z}} I^n t^{-n}$ ($I^n = R$ if $n \leq 0$), and so its integral closure is $\bar{S} = \bigoplus_{n\in\mathbf{Z}} \overline{I^n} t^{-n}$. (In fact this is one way to *define* $\overline{I^n}$.). So from Theorem 1′ we conclude that for all n, $\overline{I^n}t^{-n+d} \subseteq I^{n-d}t^{-n+d}$, and setting $n = d + \lambda$ we get Theorem 1.

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(Conversely, Theorem 1′ follows easily from the case $R = R^*$, $I_0 = tR^*$ of Theorem 1.)

Theorem 1′ will be deduced from a stronger result. Note first that to prove Theorem 1′, it suffices to check the assertion after localizing R^* at each maximal ideal containing t; in other words we may assume that R^* is a regular local ring. This being so, we consider, more generally, a *regular* noetherian domain R with fraction field K, a finite separable field extension L of K, and a finitely-generated R-subalgebra S of L. We define the *Jacobian ideal:*

$$J_{S/R} = \begin{cases} \text{0-th Fitting ideal of the } S\text{-module} \\ \text{of Kähler } R\text{-differentials } \Omega_{S/R} \end{cases}.$$

Given any surjective R-homomorphism $f: R[X_1,\ldots,X_n] \to S$ (X_i-indeterminates) with kernel, say, P, we have that $J_{S/R}$ is generated by all the Jacobians

$$f\left(\frac{\partial(g_1,\ldots,g_n)}{\partial(X_1,\ldots,X_n)}\right) \qquad g_1,\ldots,g_n \in P.$$

(It is enough here to use the n-tuples $(g_1,\ldots,g_n)$ coming from some fixed set of generators for P.)

We remark that the *integral closure $\bar{S}$ of S in L is a finite S-module* (so that it makes sense to talk about $J_{S/R}$). Indeed, it is clear that for some $a \neq 0$ in R, $S[1/a]$ is integral over $R[1/a]$; and so $\bar{S}[1/a]$ is a finite $R[1/a]$-module (L being separable over K), hence a finite $S[1/a]$-module. Moreover, for any prime ideal p in S, we have that $\bar{S}_p = \bar{S} \otimes_S S_p$ *is a finite S_p-module:* to see this, we may assume that L is the fraction field of S, and that R is local; now S_p contains a finite R-module R_1 whose fraction field is L, and the completion $\hat{R}_1$ is reduced, since $\hat{R}_1 = R_1 \otimes_R \hat{R} \subset L \otimes_R \hat{R} = L \otimes_K (K \otimes_R \hat{R}) \subset L \otimes_K \hat{K}$ where $\hat{K}$ is the fraction field of the regular local ring $\hat{R}$, and, L being separable over K, $L \otimes_K \hat{K}$ is reduced; so the finiteness of $\bar{S}_p$ over S_p is given by [17, Theorem 1.5]. Finally, we can imitate the proof of (35.3) in [14, pp. 128–129] to see that $\bar{S}$ is finite over S. (In case R is local—a case which is sufficient for proving Theorem 1′—this rather complicated final step is not needed.)

Now we can state the *basic result,* whose proof—involving some Grothendieck duality—will be given in Section 2, and again—without explicit use of duality—in Section 3.

THEOREM 2. *Let $R \subset S \subset L$, $J = J_{S/R}$, $\bar{J} = J_{\bar{S}/R}$ be as above; and assume that L is the fraction field of S. Set $S:J = \{x \in L : xJ \subset S\}$, $\bar{S}:\bar{J} = \{x \in L : x\bar{J} \subset \bar{S}\}$. Then $(\bar{S}:\bar{J}) \subset (S:J)$. In other words, $J(\bar{S}:\bar{J}) \subset \mathfrak{C}_{S/S}$, where $\mathfrak{C}_{S/S}$ is the conductor of $\bar{S}$ in S.*

COROLLARY. *Let $L = K$ and let $S = R[y_1/t_1,\ldots,y_n/t_n]$ $(y_i, t_j \in R)$. Then $(t_1 t_2 \ldots t_n) \in \mathfrak{C}_{S/S}$.*

Proof (of Corollary). Let $f: R[X_1,\ldots,X_n] \to S$ be the R-homomorphism such that $f(X_i) = y_i/t_i$ $(i = 1,2,\ldots,n)$. Then $g_i = t_i X_i - y_i$ is in the kernel of f, and by Theorem 2

$$(t_1 t_2 \ldots t_n) = \frac{\partial(g_1,\ldots,g_n)}{\partial(X_1,\ldots,X_n)} \in J \subset \mathfrak{C}_{S/S}.$$

In particular, for S in Theorem 1′ we have $t^{d+1} \in \mathfrak{C}_{S/S}$. To prove Theorem 1′, however, we need to show that $t^d \in \mathfrak{C}_{S/S}$. For this, in addition to Theorem 2, the following Lemma clearly suffices:

LEMMA. *With notation as in Theorem 2, let $0 \neq t \in R$ be such that R/tR is regular. Assume that S contains an element y/t with $y \in R - tR$. Then $\bar{J} \subset t\bar{S}$.* [*Consequently (by Theorem 2) $Jt^{-1}\bar{S} \subset J(\bar{S}:\bar{J}) \subset S$, i.e., $J \subset t\mathfrak{C}_{S/S}$.*]

Proof. We first note the following multiplicative property of Jacobian ideals: *if $R_0 \subset R_1 \subset R_2$ are noetherian regular domains such that R_2 and R_1 are rings of fractions of finitely generated R_0-algebras, and the fraction field of R_2 is finite over that of R_0, then*

$$J_{R_2/R_0} = J_{R_2/R_1} J_{R_1/R_0}. \tag{1.1}$$

(As before, J is the 0-th Fitting ideal of the relative differential module.) Indeed, we may assume that R_2 and R_1 are local; and then for suitable polynomials

$$f_1, \ldots, f_n \in R_0[X_1,\ldots,X_n] = R_0[X]$$
$$g_1, \ldots, g_m \in R_0[X_1,\ldots,X_n,Y_1,\ldots,Y_m] = R_0[X,Y]$$

we have that

—R_1 is a localization of $R_0[X]/(f_1,\ldots,f_n)$

—R_2 is a localization of $R_1[Y]/(\bar{g}_1,\ldots,\bar{g}_m)$, where $\bar{g}_i$ is the natural image of g_i in $R_1[Y]$

—R_2 is a localization of $R_0[X,Y]/(f_1,\ldots,f_n,g_1,\ldots,g_m)$.

Then one sees that $J_{R_1/R_0}, J_{R_2/R_1}, J_{R_2/R_0}$ are principal ideals, generated respectively by (the appropriate images of)

$$\frac{\partial(f_1,\ldots,f_n)}{\partial(X_1,\ldots,X_n)}, \quad \frac{\partial(g_1,\ldots,g_m)}{\partial(Y_1,\ldots,Y_m)}, \quad \frac{\partial(f_1,\ldots,f_n,g_1,\ldots,g_m)}{\partial(X_1,\ldots,X_n,Y_1,\ldots,Y_m)};$$

and (1.1) results.

Next, to prove the Lemma, we may replace $\bar{S}$ by $R_2 = \bar{S}_p$, where p is an associated prime of $t\bar{S}$, and we may replace R by $R_0 = R_{p \cap R}$. R_2 is then a discrete valuation ring, whose residual transcendence degree over R_0 is $\dim(R_0) - 1$ (since R_0 is regular, hence universally catenary); the same is therefore true of $R_1 = R_0 \cap K$. In view of [1, p. 77, Prop. 4.4] (where the word "algebraic" can be omitted), and (1.1) above, we may replace R_2 by R_1; and we have a finite sequence of regular local rings $R_0 = R_0' < R_1' < R_2' < \ldots < R_e' = R_1$ where each R_{i+1}' $(i \geq 0)$ is the local ring of a closed point on the scheme obtained by blowing up the maximal ideal of R_i'. Note that $e > 0$, because t is a non-unit in R_0 and by assumption R_1 contains an element y/t with $y \in R - tR$.

Set $J_i = J_{R'_{i+1}/R'_i}$ $(0 \le i < e)$. By (1.1) we have then $J_{R_1/R_0} = J_{e-1}J_{e-2} \dots J_0$. To better describe J_i, let $(x_1, \dots, x_{d_i})$ be a minimal generating set for the maximal ideal M_i of R'_i $(d_i = \dim(R_i) \ge 2)$, the labelling being such that R'_{i+1} is a localization of

$$R'_i[x_2/x_1, \dots, x_{d_i}/x_1] = R'_i[X_2, \dots, X_{d_i}]/(x_1X_2 - x_2, \dots, x_1X_{d_i} - x_{d_i})$$

(cf., [6, p. 199. Remark] or [16, Lemma 2.3]). We see then that J_i is generated by the single element

$$\frac{\partial(x_1X_2 - x_2, \dots, x_1X_{d_i} - x_{d_i})}{\partial(X_2, \dots, X_{d_i})} = x_1^{d_i - 1}$$

i.e., $J_i = x_1^{d_i-1}R'_{i+1} = (M_iR'_{i+1})^{d_i-1}$. Thus $J_{R_1/R_0} = \prod_{i=0}^{e-1} (M_i^{d_i-1}R_1)$.

To compare J_{R_1/R_0} with tR_1, let I_i be the *proper transform* of the ideal tR_0 in R'_i, defined inductively by $I_0 = tR'_0$ and

$$I_{i+1} = \begin{cases} (I_iR'_{i+1})(M_iR'_{i+1})^{-1} & \text{if } I_i \ne R'_i \\ R'_{i+1} & \text{if } I_i = R'_i \end{cases}.$$

Clearly $tR_1 = I_e \prod_{i=0}^{e-1} (M_i^{\epsilon_i}R_1)$ where $\epsilon_i = 1$ if $I_i \ne R'_i$, and $\epsilon_i = 0$ if $I_i = R'_i$. In any case, $\epsilon_i \le d_i - 1$, so to finish the proof that $J_{R_1/R_0} \subset tR_1$, it suffices to show that $I_e = R_1$.

Let v_i be the "order" valuation defined by R_i, i.e., the unique valuation such that for $x \in R_i$, $v_i(x) = \sup\{n : x \in M_i^n\}$. Then v_{e-1} is the valuation whose valuation ring is $R'_e = R_1$, so we need only show that $v_{e-1}(I_{e-1}) \le 1$. Let us show, by induction on j, that $v_j(I_j) \le 1$ for all $j < e$. For $j = 0$, this is true by assumption (R_0/tR_0 is assumed regular). Suppose we have shown that $v_i(I_i) \le 1$. If $v_i(I_i) = 0$, then I_j is the unit ideal for $j \ge i$, and $v_j(I_j) = 0$. If $v_i(I_i) = 1$, then we can choose $x_1, \dots, x_{d_i}$ as above, with labelling such that *either* $I_i = x_1R'_i$, in which case $I_{i+1} = R'_{i+1}$ and we are done; *or* $I_i = x_2R'_i$. In the latter case, $I_{i+1} = (x_2/x_1)R'_{i+1}$, and we need to show that $R'_{i+1}/(x_2/x_1)R'_{i+1}$ is regular. But this follows from the fact that $R'_i[x_2/x_1, \dots, x_{d_i}/x_1]/(x_1, x_2/x_1)$ is isomorphic to a polynomial ring in $d_i - 2$ variables over the residue field of R'_i (so that $R'_{i+1}/(x_2/x_1, x_1)R'_{i+1}$ is regular, of dimension $d_{i+1} - 2$). This completes the proof.

We close this Section with two reformulations of Theorem 1.

First of all, in proving Theorem 1, we can replace R by its localizations at maximal ideals containing I_0; i.e., we may assume that R is local (and $I_0 \ne R$). Let m be the unique maximal ideal of R, $m_0 = m/I_0$, and in the graded ring $G = \bigoplus_{n \ge 0} (I_0^n/I_0^{n+1})$ let M be the maximal ideal

$$M = m_0 \oplus I_0/I_0^2 \oplus I_0^2/I_0^3 \oplus \dots = m_0 \oplus G_+.$$

If $(u_1,\ldots,u_r)$ is a minimal generating set of m_0 (in R/I_0), $(v_1,\ldots,v_s)$ is a minimal generating set of I_0 (in R), and $\bar{v}_i$ is the natural image of v_i in I_0/I_0^2 $(1 \le i \le s)$, then the images of $u_1, \ldots, u_r, \bar{v}_1, \ldots, \bar{v}_s$ in $M/M^2 = m_0/m_0^2 \oplus I_0/mI_0$ form a basis of this G/M-vector space; and since by assumption G_M is regular, we see that the sequence of homogeneous (in G) elements $(u_1,\ldots,u_r,\bar{v}_1,\ldots,\bar{v}_s)$ is G_M-regular, hence also G-regular. It follows at once that $(u_1,\ldots,u_r)$ is a regular sequence in R/I_0, so that R/I_0 is regular; and furthermore G_+ is generated by the regular sequence $(\bar{v}_1,\ldots,\bar{v}_s)$ in G, and so $G = G/G_+ \oplus G_+/G_+^2 \oplus \ldots$ is a polynomial ring over $G/G_+ = R/I_0$, i.e., $(v_1,\ldots,v_s)$ is a regular sequence in R; so that R too is regular. (So the "for example" in the statement of Theorem 1 actually covers all possibilities). Thus *it suffices to prove Theorem* 1 *with the seemingly stronger assumption that* R *and* R/I_0 *are regular local rings* (so that I_0 is generated by a subset of a regular system of parameters in R).

With these assumptions on R and I_0, again let m be the maximal ideal of R. To any ideal A in R, we associate two integers $\delta_1(A)$ and $\delta_2(A)$ with the property that if $A \supset A'$ and A is integral over A' (i.e., A' is a *reduction* of A), then $\delta_1(A) = \delta_1(A')$ and $\delta_2(A) = \delta_2(A')$. Namely,

$\delta_1(A) =$ ("analytic spread" of A) $- 1$

$=$ dimension of the closed fibre $f_A^{-1}(\{m\})$ where $f_A : X_A \to \mathrm{Spec}(R)$ is the morphism obtained by blowing up A.

$\delta_2(A) =$ dimension of the R/m-vector space $(A + m^2)/m^2 \cong A/A \cap m^2$.

If A' is a reduction of A, then X_A is finite over $X_{A'}$, and that is why $\delta_1(A) = \delta_1(A')$. It is easily seen that for any such A',

(1.2) $\quad 1 + \delta_1(A') \le \dim(A'/mA') =$ minimum number of generators of A';

and moreover in [15, p. 151] it is shown that if R/m is infinite, then there always exist A' for which equality holds in (1.2).

To show that $\delta_2(A') = \delta_2(A)$, we need to show that $A' + m^2 = A + m^2$. Since $A + m^2$ is integral over $A' + m^2$, it suffices to show that $A' + m^2$ *is integrally closed*. Let y be integral over $A' + m^2$; and let $y_1, \ldots, y_r \in A'$ be such that their images in $(A' + m^2)/m^2$ form a vector space basis. Then $\bar{R} = R/(y_1,\ldots,y_r)R$ is regular, and $\bar{y}$ (the natural image of y in $\bar{R}$) is integral over $\bar{m}^2$ ($\bar{m}$ = maximal ideal of $\bar{R}$); if $\bar{v}$ is the valuation defined by $\bar{v}(x) = \sup\{n : x \in \bar{m}^n\}$ $\quad (x \in \bar{R})$ it follows easily that $\bar{v}(\bar{y}) \ge 2$, i.e., $\bar{y} \in \bar{m}^2$. Hence $y \in A' + m^2$, as desired.

Now let I be as in Theorem 1, and set $\delta = \delta(I) = \delta_1(I) - \delta_2(I)$. Since I_0 is generated by a subset of a regular system of parameters, we have $\delta(I) \le d$ (cf. (1.2)). So the inclusion $\overline{I^{d+\lambda}} \subset I^\lambda$ of Theorem 1 is *implied by* the inclusion

(1.3) $$\overline{I^{\delta+\lambda}} \subset I^\lambda.$$

On the other hand, Theorem 1 *implies* (1.3) for any ideal I in a regular local ring R (with maximal ideal, say, m). For, standard techniques allow us to assume that the residue field of R is infinite; and then, taking I' to be a reduction of

I generated by $1 + \delta_1(I)$ elements, say $I' = (x_1,\ldots,x_{\delta_2})R + (y_1,\ldots,y_{\delta+1})R$ with $\delta_2 = \delta_2(I) = \delta_2(I')$ and $x_1, \ldots, x_{\delta_2}$ such that their natural images in $I'/m^2 \cap I'$ form a basis of this R/m-vector space, and applying Theorem 1 (with $I_0 = (x_1,\ldots, x_{\delta_2})R$) to I', we see that indeed $\overline{I^{\delta+\lambda}} = \overline{(I')^{\delta+\lambda}} \subset (I')^\lambda \subset I^\lambda$. In fact, then, if I^* is *any* reduction of I, we have $\overline{I^{\delta+\lambda}} = \overline{(I^*)^{\delta+\lambda}} \subset (I^*)^\lambda$. We see easily now that Theorem 1 is equivalent to:

THEOREM 1″. *Let R be a regular noetherian ring and let I be an ideal in R. Set $\delta = \delta(I) = \sup_m \{\delta_1(I_m) - \delta_2(I_m)\}$ where m runs through all maximal ideals containing I, and $I_m = IR_m$. If I^* is any reduction of I, then for every integer $\lambda > 0$ we have $\overline{I^{\delta+\lambda}} \subset (I^*)^\lambda$.*

Remark. In the conclusion of Theorem 1″, we can even replace δ by

$$\delta_\lambda^* = \sup_p \{\delta_1(I_p) - \delta_2(I_p)\}$$

where p runs through all associated primes of $(I^*)^\lambda$. (It is easily checked that $\delta_\lambda^* \leq \delta$).

Here is the second reformulation:

THEOREM 1‴. *With notation as in Theorem 1″, let X be the scheme obtained from $\mathrm{Spec}(R)$ by blowing up I, and let $\pi: \bar{X} \to X$ be its normalization. Then $I^\delta(\pi_* \mathcal{O}_{\bar{X}}) \subset \mathcal{O}_X$.*

To get the equivalence of Theorem 1‴ with Theorem 1, note first that Theorem 1′ is a corollary of Theorem 1‴ (as previously noted, in Theorem 1′ we may assume that R^* is a regular local ring; now apply Theorem 1‴ with $R = R^*$ and $I = (t, y_1,\ldots,y_{d+1})R^*$, so that the ring S of Theorem 1′ is $\Gamma(U, \mathcal{O}_X)$ for some affine open subset U of X, and $t^d \bar{S} = \Gamma(U, I^d(\pi_* \mathcal{O}_{\bar{X}}))$; moreover $\delta(I) \leq d$.) As we saw before Theorem 1′ implies Theorem 1, which implies Theorem 1″. Finally, in Theorem 1‴, $X = \mathrm{Proj}\left(\bigoplus_{\lambda \geq 0} I^\lambda\right)$, $\bar{X} = \mathrm{Proj}\left(\bigoplus_{\lambda \geq 0} \overline{I^\lambda}\right)$, and Theorem 1″ gives

$$I^\delta\left(\bigoplus_{\lambda \geq 0} \overline{I^\lambda}\right) \subset \bigoplus_{\lambda \geq 0} \overline{I^{\delta+\lambda}} \subset \bigoplus_{\lambda \geq 0} I^\lambda$$

whence $I^\delta(\pi_* \mathcal{O}_{\bar{X}}) \subset \mathcal{O}_X$; i.e., Theorem 1″ implies Theorem 1‴.

In Summary. Theorems 1, 1′, 1″, 1‴ are equivalent statements, and they all follow from Theorem 2, which we will now prove.

2. PROOF OF THEOREM 2

I. Let $f: R[X_1,\ldots,X_n] \to S$ (X_i-indeterminates) be a surjective R-homomorphism, with kernel, say, P. Let $\Omega = \Omega_{R[X]/R}$ be the module of Kähler R-differentials of $R[X] = R[X_1, \ldots, X_n]$; Ω is a free $R[X]$-module, with basis $dX_1, \ldots, dX_n$. Set $\Omega^n = \Lambda^n \Omega$, the n-th exterior power of Ω. Our first remark is that there is an *isomorphism of S-modules*

$$H = \mathrm{Hom}_S(\Lambda^n(P/P^2), \Omega^n/P\Omega^n) \xrightarrow{\sim} S:J. \tag{2.1}$$

Indeed, $\Omega^n/P\Omega^n$ is isomorphic to $R[X]/P = S$, so H is a torsion-free S-module and the natural map $H \to H \otimes_S L$ is injective. Now

$$H \otimes_S L = H \otimes_S (S \otimes_R K) = \mathrm{Hom}_L(\Lambda^n(P_K/P_K^2), \Omega_K^n/P_K\Omega_K^n)$$

where P_K is the kernel of

$$f \otimes 1 : K[X] = R[X] \otimes_R K \to S \otimes_R K = L$$

and

$$\Omega_K = \Omega \otimes_R K = \Omega_{K[X]/K}.$$

Since L is a field, P_K is a maximal ideal of $K[X]$ and P_K/P_K^2 is an n-dimensional L-vector space; hence $H \otimes_S L$ is a one-dimensional L-vector space. Moreover, since L is *separable* algebraic over K, the natural map

$$d_0^n : \Lambda^n(P_K/P_K^2) \to \Omega_K^n/P_K\Omega^n,$$

which is the n-th exterior power of the L-linear map $d_0 : P_K/P_K^2 \to \Omega_K/P_K\Omega_K$ induced by the universal derivation $d : K[X] \to \Omega_K$, is *not the zero-map* (because the cokernel of d_0 is $\Omega_{L/K} = (0)$). Thus $H \otimes_S L = Ld_0^n$. We leave it as an exercise to show that the image of the composition $H \to H \otimes_S L = Ld_0^n \to L$ (where the last map takes xd_0^n to x) is $S:J$; whence the isomorphism (2.1).

II. Next, we recall the "fundamental local homomorphism"

$$\phi : \mathrm{Ext}^n_{R[X]}(S, \Omega^n) \to H = \mathrm{Hom}_S(\Lambda^n(P/P^2), \Omega^n/P\Omega^n). \tag{2.2}$$

[This is obtained by combining the natural maps

$$\mathrm{Ext}^n_{R[X]}(S,\Omega^n) \to \mathrm{Ext}^n_{R[X]}(S,\Omega^n/P\Omega^n) \to \mathrm{Hom}_S(\mathrm{Tor}_n^{R[X]}(S,S), \Omega^n/P\Omega^n)$$

with the map $\Lambda^n(P/P^2) \to \mathrm{Tor}_n^{R[X]}(S,S)$ arising from a canonical isomorphism $P/P^2 \xrightarrow{\sim} \mathrm{Tor}_1^{R[X]}(S,S)$ plus the natural anticommutative graded S-algebra structure on $\bigoplus_{n \geq 0} \mathrm{Tor}_n^{R[X]}(S,S)$.] It is not hard to see that ϕ operates as follows (cf. [7, p. 149-06]): let $\ldots \to E_n \to E_{n-1} \to \ldots \to E_0 \to S \to 0$ be an $R[X]$-projective resolution of S, and let $\alpha \in \mathrm{Ext}^n_{R[X]}(S,\Omega^n)$ be represented by a map $\alpha : E_n \to \Omega^n$; then for $g_1, \ldots, g_n \in P$, the map $\phi(\alpha) : \Lambda^n(P/P^2) \to \Omega^n/P\Omega^n$ takes $\bar{g}_1 \wedge \ldots \wedge \bar{g}_n$ ($\bar{g}_i$ = natural image of g_i in P/P^2) to $\psi_{g,\alpha}(1)$, where $\psi_{g,\alpha}$ is found as follows: the Koszul complex $\mathbf{K}_g$ over $R[X]$ determined by the sequence $g = (g_1, \ldots, g_n)$ is a projective $R[X]$-complex augmenting to S, so there is a homotopy-unique map of complexes $\mathbf{K}_g \to (\ldots \to E_n \to \ldots \to E_0)$ over the identity map of S; in particular, we have, in degree n, $R[X] = (\mathbf{K}_g)_n \to E_n$, and $\psi_{g,\alpha}$ is the (uniquely determined!) composition

$$R[X] \to E_n \xrightarrow{\alpha} \Omega^n \xrightarrow{\text{natural}} \Omega^n/P\Omega^n.$$

From this description, it is clear that ϕ "commutes with localization on $R[X]$". Moreover, if Q is a prime ideal in $R[X]$ containing P, and if $PR[X]_Q$ is generated by a *regular* $R[X]_Q$-sequence $(g_1,\ldots,g_n)$ then the localization

$$\phi_Q: \mathrm{Ext}^n_{R[X]_Q}(S_Q,\Omega^n_Q) \to \mathrm{Hom}_S(\Lambda^n(P/P^2),\Omega^n_Q/P\Omega^n_Q)$$

is an *isomorphism* (because the Koszul complex over $R[X]_Q$ determined by $(g_1,\ldots,g_n)$ is an $R[X]_Q$-projective resolution of S_Q).

III. We will use "pre-duality" theory to establish two facts ((A) and (B) below).

(A). *If S is normal, then $\mathrm{Ext}^n_{R[X]}(S,\Omega^n)$ is a reflexive S-module.*

(A) is really a local statement: it is enough to know that if A is a regular local ring, B is a homomorphic image of A which is normal, and $n = \dim A - \dim B$, then $E = \mathrm{Ext}^n_A(B,A)$ is a reflexive B-module of rank one; i.e., E is isomorphic to a non-zero B-submodule of the fraction field of B, and $E = \bigcap_p E_p$, where p runs through all height one primes of B. This can be seen, for example, by using the Cousin complex of A as an injective resolution of A to calculate E (cf. e.g. [8, p. 239]).

As a corollary of (A), we claim that *if S is normal, then the composition of* (2.1) *and* (2.2), viz.

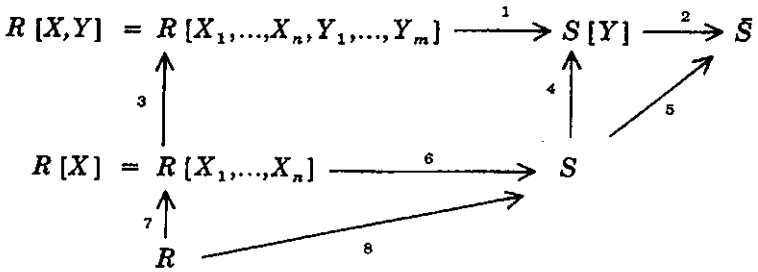
$$\mathrm{Ext}^n_{R[X]}(S,\Omega^n) \to S:J \tag{2.3}$$

is an isomorphism.

Indeed, since both modules in (2.3) are reflexive, we can check our assertion after localizing at height one primes q in S. Lifting such a q back to a prime $Q \supset P$ in $R[X]$, we have that $PR[X]_Q$ is generated by a regular $R[X]_Q$-sequence of length n (since $R[X]_Q$ and $S_Q = S_q$ are both regular local rings), so that by preceding remarks, the localization of (2.3) at q is an isomorphism.

Now, without assuming S to be normal, let $\bar{S}$ be the integral closure of S in L. We have a commutative diagram:

(2.4)



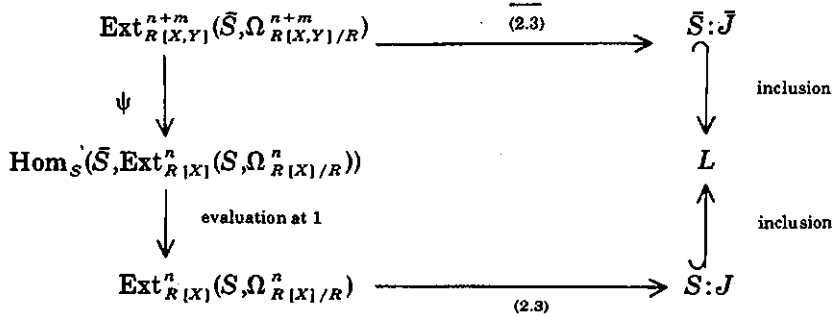
Here the maps labelled 3, 4, 5, 7, 8 are inclusions; 6 is what we have been calling f; 2 is some surjective S-homomorphism; and 1 is the unique $R[Y]$-homomorphism such that $1 \circ 3 = 4 \circ 6$.

As above, we can use the surjective R-homomorphism $2 \circ 1$ to construct an *isomorphism*

$$\overline{(2.3)} \qquad \mathrm{Ext}^{n+m}_{R[X,Y]}(\bar{S},\Omega^{n+m}_{R[X,Y]/R}) \xrightarrow{\sim} \bar{S}:\bar{J}.$$

We then have:

(B) *There is an isomorphism ψ making the following diagram commute:*



Since $\overline{(2.3)}$ is an isomorphism, it is clear that Theorem 2 follows from (B).

IV. (B) can be proved by means of [8, p. 190, Theorem 8.7]. We give an outline of the main steps in the argument.

Given any homomorphism $g: A \to B$ of noetherian rings, making B into a finitely generated A-algebra, we denote by $g^!$ the functor on derived categories (of modules) $D^+(A) \to D^+(B)$ described in *loc. cit.* Very roughly speaking, $g^!$ is uniquely determined by the following three properties:

(i) If B is a finite A-module, $C^\bullet$ is a complex of A-modules, bounded below, and $C^\bullet \to D^\bullet$ is a map of complexes inducing homology isomorphisms, where $D^\bullet$ is bounded below and injective, then $g^!C^\bullet = \mathbf{R}\mathrm{Hom}_A(B,C^\bullet) = \mathrm{Hom}_A(B,D^\bullet)$.

(ii) If B is a polynomial ring in n variables over A, and $C^\bullet$ is any complex of B-modules, bounded below, then $g^!C^\bullet = C^\bullet \otimes_A \Omega^n_{B/A}$ shifted n places.

(iii) For a composition $A \xrightarrow{g} B \xrightarrow{h} B'$, with g as above and h making B' into a finitely generated B-algebra, we have $(h\circ g)^! = h^!\circ g^!$ (up to canonical isomorphism).

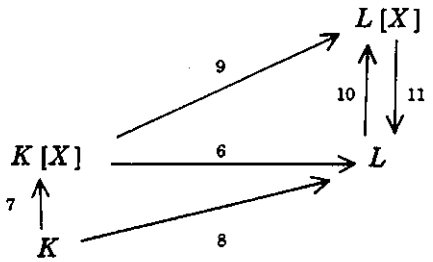
The isomorphism ψ in (B) is constructed, with reference to (2.4), as follows (where "=" denotes natural isomorphism, as in (iii) preceding):

$$\begin{aligned}
\mathrm{Ext}^{n+m}_{R[X,Y]}(\bar{S},\Omega^{n+m}_{R[X,Y]/R}) &= H^0(2^!1^!3^!7^!R) \xrightarrow{\sim} H^0(2^!4^!6^!7^!R) \\
&= H^0(5^!6^!7^!R) = H^0(\mathbf{R}\mathrm{Hom}_S(\bar{S},6^!7^!R)) \\
&= \mathrm{Hom}_S(\bar{S},H^0(6^!7^!R)) \\
&= \mathrm{Hom}_S(\bar{S},\mathrm{Ext}^n_{R[X]}(S,\Omega^n_{R[X]/R})).
\end{aligned}$$

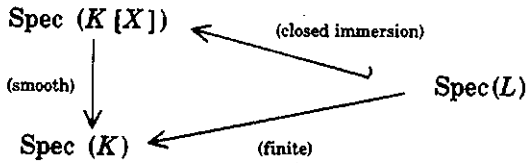
Finally, to check commutativity in (B), we can apply $\otimes_R K$ to everything in sight. This reduces us to showing that (with R replaced by K, and both S and $S:J$ by L) the map (2.3) is identical with the canonical isomorphism $H^0(6^!7^!K) \xrightarrow{\sim} H^0(8^!K)$ where the one-dimensional L-vector space

$$H^0(8^!K) = \mathrm{Hom}_K(L,K)$$

is identified with L by choosing the *trace map* as a basis. This can be done by playing with the diagram



[where 6, 7, 8 are as in (2.4) (with $R = K$, $S = L$); 9 and 10 are inclusions; and 11 is the unique L-homomorphism such that $11 \circ 9 = 6$] in accordance with considerations in [8, Chapter III, Section 8]. In fact it is nothing but the formula $(R6)$ of [loc. cit. p. 198], applied to the simple situation



Details are left to the reader.

Remark. Let R be regular and let $f: Y \to \mathrm{Spec}(R)$ be a proper birational map, with Y normal. The fact that (2.3) above is an isomorphism when S is normal leads to the conclusion that the sheaf ω_Y of [13, Section 4] is $\mathcal{O}_Y : \mathcal{J}$ where $\mathcal{J}$ is the $\mathcal{O}_Y$-ideal which sheafifies our Jacobian ideal. In view of the "Corollary of (iii)" in *loc. cit,* we have then another proof that *regular local rings are pseudo-rational.*

(This proof can replace the proof of (iv) in *loc. cit;* however that proof came up again in this paper in the proof of the lemma in Section 1 above; and this lemma will be used in Section 4 below to show that certain other local rings are pseudo-rational.)

3. IN WHICH THE PROOF IS BROUGHT DOWN TO EARTH

The duality theory used in the preceding proof of (B) is quite rich, but also quite abstract; so that after working through Section 2, one may still be left with the feeling of having a less than complete understanding of Theorem 2. This, at least, is what happened to us. So in this Section we give two concrete versions of the proof, that is, in essence, we give "understandable" descriptions of the map ψ in (B).

The first version—involving some restrictions, viz. assumption (3.1)—is based on a comparison of certain Kähler and Dedekind differents (Theorem 2′) via a form of "Lagrange interpolation" (Lemma 3.4). It covers the case of algebraic

varieties over a perfect field k, where Theorem 2 was first discovered. In this case the Ext's in (B) can be realized, via the *residue isomorphism* [8, p. 185], as canonical dualizing modules of "holomorphic differentials" in the sense of Kunz [10], [11]; in this way the composition (evaluation at 1)$\circ\psi$ in (B) becomes identified with the inclusion map of holomorphic differentials on $\bar{S}$ into holomorphic differentials on S. However this realization, illuminating as it is, does not form part of our proof, and a reasonably simple exposition—which does not seem to exist in the literature—would take us too far afield; so we leave it at that.

The second version is valid in full generality. As written, it uses no homological algebra; but in an appendix we will show how it ties in with Section 2.

FIRST PROOF

Notation remains as in Theorem 2. We make the following

Assumption (3.1). There exists a normal noetherian ring $R' \subset S$ an invertible R-module $\lambda \subseteq L$ such that S is a ring of fractions of a (module-) finite R'-algebra $S' \subset S$, and such that $J_{S/R} = \lambda J_{S/R'}$, $J_{S/R} = \lambda J_{S/R'}$.

We will see below (Example (3.5)) that (3.1) holds whenever R is a ring of fractions of a finitely generated algebra over a perfect field k.

Let K' be the fraction field of R'; then L, being the fraction field of S', is finite over K'. Since J commutes with localization and $J_{L/K} = L$ (because $\Omega_{L/K} = (0)$) therefore $J_{S/R'} \neq 0$, whence $J_{S/R'} \neq 0$, $J_{L/K'} = L$, and so $\Omega_{L/K'} = (0)$, that is, L is separable over K'. We consider the *trace map* $\tau: L \to K'$, and the complementary module $\mathscr{C}_{S'/R'} = \{x \in L : \tau(xS') \subset R'\}$. Let $\bar{S}'$ be the integral closure of S' in L. Clearly $\mathscr{C}_{\bar{S}'/R'} \subset \mathscr{C}_{S'/R'}$; and Theorem 2′ below gives

$$\bar{S}' : J_{\bar{S}'/R'} = \mathscr{C}_{\bar{S}'/R'} \subset \mathscr{C}_{S'/R'} \subset S' : J_{S'/R'}. \tag{3.2}$$

Since "integral closure" and "Jacobian ideal" both commute with localization, we conclude from (3.1) and (3.2) that

$$\begin{array}{ccc} \bar{S} : J_{S/R'} & \subset & S : J_{S/R'} \\ \| & & \| \\ \lambda(\bar{S} : J_{S/R}) & & \lambda(S : J_{S/R}) \end{array}$$

and Theorem 2 results.

Let us then prove:

THEOREM 2′. *Let R be an integral domain with fraction field K, and let $S \supseteq R$ be an integral domain whose fraction field L is finite and separable over K, and such that S is a finitely generated R-algebra. Then, with the complementary module $\mathscr{C}_{S/R}$ defined as above, we have $\mathscr{C}_{S/R} \subset S : J_{S/R}$; and equality holds if R is noetherian and normal and S is the integral closure of R in L.*

Proof. In the terminology of [3], $J_{S/R}$ is the 0-th *Kähler different* of S/R, and $S : \mathscr{C}_{S/R}$ is the *Dedekind different*. The equality statement in Theorem 2′ is

given then by [ibid, p. 36, Satz 4]. (Basically, by localizing at height one primes of R one reduces to the case where R is a discrete valuation ring, in which case the equality of the Kähler and Dedekind differents is a generalization of the Hurwitz formula for algebraic curves. Cf. also [12, bottom of p. 178].)

In [12, p. 179], Kunz shows that the Kähler different $J_{S/R}$ is contained in the *Noether different* $d_N(S/R)$, which can be described as follows:

Represent S as $S = R[X_1,\ldots,X_n]/P = R[x_1,\ldots,x_n]$; then, with $X = (X_1, \ldots,X_n)$, $x = (x_1,\ldots,x_n)$, one has

$$d_N(S/R) = \{G(x) \mid G(X) \in PS[X] : (X_1 - x_1,\ldots,X_n - x_n)\}.$$

For convenience to the reader we include the proof:

Let $(g_1(X),\ldots,g_n(X))$ be any sequence of polynomials in P. It is enough to show that the jacobian $(\partial g_i(X)/\partial X_j) = g_X(X)$, say, satisfies $g_X(x) \in d_N(S/R)$. Write the usual (first order) Taylor-series expansion

$$g_i(X) = g_i(X) - g_i(x) = \sum_j h_{ij}(X)(X_j - x_j)$$

where the coefficients $h_{ij}(X) \in S[X]$ exist and are unique modulo

$$(X_1 - x_1,\ldots,X_n - x_n).$$

By Cramer's Rule it follows that $H(X) = \det(h_{ij}(X))$ satisfies

$$H(X)(X_j - x_j) \in (g_1(X),\ldots,g_n(X)) \subset PS[X].$$

Thus $H(x) \in d_N(S/R)$. On the other hand it is clear that $H(x) = g_X(x)$, hence the conclusion follows.

We will prove even more than Theorem 2', namely that

$$\mathscr{C}_{S/R}\, d_N(S/R) \subset S \tag{3.3}$$

that is, *the Noether different is contained in the Dedekind different.*

Remark. (3.3) is proved in [2, Prop. 3.1] when S is finite over R, a case which suffices for the present proof of Theorem 2. But anyway the following simple proof covers the general case.

LEMMA (3.4) (Lagrange Interpolation). *Let $\tau_X : L[X] \to K[X]$ be the map obtained by applying the trace $\tau : L \to K$ coefficientwise. Fix*

$$G(X) \in PS[X] : (X_1 - x_1,\ldots,X_n - x_n). \tag{3.4.1}$$

For any $v \in L$, write: $G_v(X) = \tau_X(vG(X))$. Then we have: $G_v(x) = vG(x)$.

In particular, if $v \in \mathscr{C}_{S/R}$, then the coefficients of $G_v(X)$ lie in R, so $vG(x) \in S$, proving (3.3).

To prove (3.4) let (θ_1 = identity, $\theta_2,\ldots,\theta_r$) be a complete set of K-isomorphisms of L into some fixed algebraic closure, so that $\tau = \sum_{j=1}^{r} \theta_j$. Let G_j be the polynomial obtained by applying θ_j to the coefficients of G. From (3.4.1) we obtain $G_j(x)(x_i - \theta_j(x_i)) = 0 \qquad (1 \le i \le n)$ whence, for $j > 1$, $G_j(x) = 0$. Thus

$$G_v(x) = \sum_{j=1}^{r} \theta_j(v)\, G_j(x) = vG(x).$$

Example (3.5). Suppose that the regular ring R is a ring of fractions of a finitely generated algebra over a perfect field k. By normalization [14, (39.11)] the finitely generated R-algebra S is a ring of fractions of a finite R'-algebra, where $R' = k[t'_1,\ldots,t'_m]$ is a polynomial ring with $t'_1, \ldots, t'_m$ a separating transcendence basis of L/k. Since R is regular we have $\Lambda^m \Omega_{R/k} = \lambda dt'_1 \wedge \ldots \wedge dt'_m$, with λ an invertible R-module. We claim that *R' and λ satisfy assumption* (3.1).

Without harm we may assume further that R is local. Then the $\Omega_{R/k}$ is free. We have a natural exact sequence

$$\Omega_{R/k} \otimes_R S \xrightarrow{\alpha} \Omega_{S/k} \to \Omega_{S/R} \to 0$$

and since $\Omega_{R/k} \otimes_R S$ is a free S-module and $\alpha \otimes_S L$ is an isomorphism, therefore α is *injective.* Similarly we have an exact sequence

$$0 \to \Omega_{R'/k} \otimes_{R'} S \to \Omega_{S/k} \to \Omega_{S/R'} \to 0.$$

So our claim is obtained by applying the following generalization of [4, p. 65, Prop. 15] to $M = \Omega_{S/k}$, $E = \Omega_{R/k} \otimes_R S$, and $E' = \Omega_{R'/k} \otimes_{R'} S$:

PROPOSITION. *Let A be an integral domain, with fraction field, say, F. Let M be a finitely generated A-module, and let E, E' be two free submodules of M such that M/E and M/E' are torsion* (i.e., *annihilated by a non-zero element of A). Let J and J' be the 0-th Fitting ideals of M/E and M/E' respectively. Then $J = \lambda J'$ where $0 \ne \lambda \in F$ is as follows: let $(e_1,\ldots,e_m)$ be a free basis of E and let $(e'_1,\ldots,e'_m)$ be a free basis of E'; then in the m-dimensional F-vector space $E \otimes_A F = M \otimes_A F = E' \otimes_A F$ we have (uniquely)*

$$e_i = \sum_{j=1}^{m} a_{ij} e'_j \qquad (a_{ij} \in F;\ 1 \le i, j \le m),$$

and we set $\lambda = \det(a_{ij})$.

Proof. Choose a free A-module H with basis $\eta_1, \ldots, \eta_m, \eta'_1, \ldots, \eta'_m, \xi_1, \ldots, \xi_n$, and a surjective map $f: H \to M$ with $f(\eta_i) = e_i$, $f(\eta'_i) = e'_i$ $(1 \le i \le m)$. Let C be the kernel of f. We have then an exact sequence of F-vector spaces $0 \to C_F \to H_F \xrightarrow[f_F]{} M_F \to 0$ where $C_F = C \otimes_A F$, etc., and hence a natural isomorphism of one-dimensional F-vector spaces $\phi: \Lambda^m M_F \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda^{m+n} C, \Lambda^{2m+n} H_F)$ satisfying, for $c_1, \ldots, c_{m+n}$ in C, $h_1, \ldots, h_m$ in H_F, and $\tilde{h}_i = f_F(h_i)$:

$$[\phi(\bar{h}_1 \wedge \ldots \wedge \bar{h}_m)](c_1 \wedge \ldots \wedge c_{m+n}) = h_1 \wedge \ldots \wedge h_m \wedge c_1 \wedge \ldots \wedge c_{m+n}.$$

Using the presentation $M/E = H/C + A\eta_1 + \ldots + A\eta_m$ one checks that *the image of the map* $\phi(e_1 \wedge \ldots \wedge e_m)$ *is*

$$\text{Im } [\phi(e_1 \wedge \ldots \wedge e_m)] = J\eta \wedge \eta' \wedge \xi \qquad (\eta = \eta_1 \wedge \ldots \wedge \eta_m, \text{ etc.}).$$

Similarly

$$\text{Im } [\phi(e_1' \wedge \ldots \wedge e_m')] = J'\eta' \wedge \eta \wedge \xi = J'\eta \wedge \eta' \wedge \xi.$$

Since $e_1 \wedge \ldots \wedge e_m = \lambda\, e_1' \wedge \ldots \wedge e_m'$ we conclude that $J = \lambda J'$, as asserted.

SECOND PROOF

I. *Preliminaries.* As in Theorem 2, we consider noetherian domains $R \subseteq S$ such that S is a finitely generated R-algebra, and such that the fraction field L of S is finite and separable over the fraction field K of R. The only additional assumption we make throughout is that R is *locally Cohen Macaulay* (an assumption which is of course satisfied if R is regular). Equivalently [14, (25.6)], *if an ideal* I *in* R *of height* $n \geq 0$ *is generated by* n *elements, then every associated prime of* I *has height* n. Note that an ideal $I = (g_1,\ldots,g_n)R$ has height n if the sequence $(g_1,\ldots,g_n)$ is R-regular (i.e.,

$$(g_1,\ldots,g_i)R : g_{i+1} = (g_1,\ldots,g_i)R$$

for $0 \leq i < n$, and $(g_1,\ldots,g_n)\,R \neq R$).

We represent S as $S = R[X_1,\ldots,X_n]/P = R[X]/P$, so that

$$L = K[X]/P_K \qquad (P_K = PK[X]),$$

and then $R[X]_P = K[X]_{P_K}$ is a regular local ring of dimension n. Set $\Omega_* = \Omega_{R[X]_P/R}$ so that Ω_* is a free $R[X]_P$-module with basis $dX_1, \ldots, dX_n$. Put $M = PR[X]_P$. Since $L = R[X]_P/M$ is separable over K (i.e., $\Omega_{L/K} = (0)$) therefore the natural map $M/M^2 \to \Omega_*/M\Omega_*$ is an isomorphism. For a sequence $g = (g_1,\ldots,g_n)$ in P, set $g_X = \partial(g_1,\ldots,g_n)/\partial(X_1,\ldots,X_n)$. From the preceding, we conclude in a straightforward way that

$$(3.6) \qquad (gR[X]_P = PR[X]_P = M) \Leftrightarrow (g_X \notin P).$$

LEMMA (3.7). *Let* $S = R[X]/P$ *be as above. For any* z *in* $R[X]$, *let* $\bar{z}$ *be its natural image in* S. *Then* $J = J_{S/R}$ *is generated by all* $\overline{g_X}$ *with* $g = (g_1,\ldots,g_n)$ *an* $R[X]$*-regular sequence in* P *such that* $g_X \notin P$ *(and such* g *exist!).*

Proof. By (3.6), and since $R[X]_P$ is regular and n-dimensional, we can find a sequence $g = (g_1,\ldots,g_n)$ in P such that $g_X \notin P$. Then, we claim, there is a *regular* sequence $g' = (g_1',\ldots,g_n')$ with $g_i' - g_i \in P^2$ (whence: $g_X' - g_X \in P$, i.e.,

$\overline{g'_x} = \overline{g}_x$). Indeed, $R[X]$ is locally Cohen Macaulay [14, (25.10)], so we can apply the following Lemma, with $I = P^2$:

LEMMA (3.8). *Let I be an ideal of height $\geq n$ in a locally Cohen Macaulay ring T, and let $g = (g_1,\ldots,g_n)$ be a sequence in T such that $I + gT \neq T$. Then there exists a T-regular sequence $g' = (g'_1,\ldots,g'_n)$ with $g'_j - g_j \in I$ $(1 \leq j \leq n)$.*

Proof. Suppose inductively that for some $i \leq n$ we have found a regular sequence $(g'_1,\ldots,g'_{i-1})$ with $g'_j - g_j \in I$ $(j < i)$. Let $Q_1, \ldots, Q_r$ be those associated primes of $(g'_1,\ldots,g'_{i-1})R$ which contain g_i, and let $Q_1^*, \ldots, Q_s^*$ be those which don't. Since the Q and Q^* all have height $i - 1 < n$, there is an element h_i such that $h_i \in I \cap Q_1^* \cap \ldots \cap Q_s^*$, $h_i \notin Q_1 \cup \ldots \cup Q_r$. Then $g'_i = g_i + h_i$ does not lie in any Q or Q^*, and $I + (g'_1,\ldots,g'_i)T \subset I + gT \neq T$, so the sequence $(g'_1,\ldots,g'_i)$ is regular. Proceeding in this way, we get the desired g'.

II. *Main steps in the proof.* We now give an outline of the proof; details will follow.

First we choose a regular sequence $g = (g_1,\ldots,g_n)$ in P such that

$$gR[X]_P = PR[X]_P;$$

that is, $g_X \notin P$ (cf., (3.6), (3.7)). Denoting by (g) the ideal $gR[X]$, we then define a homomorphism of S-modules $\phi_g: ((g):P)/(g) \to L$ by

$$\phi_g(u + (g)) = \bar{u}/\overline{g_x} \tag{3.9}$$

(where, as in (3.7), "$\bar{\ }$" denotes natural image in $S = R[X]/P$).

Remark (3.10). Examining the primary decomposition of (g), one finds that $(g) = P \cap ((g):P)$. i.e., ϕ_g *is injective.*

LEMMA (3.11). *The image* $\operatorname{Im}\phi_g$ *of the map* ϕ_g *does not depend on the choice of g. Consequently, by* (3.7), $\operatorname{Im}\phi_g \subset S:J$.

The proof will be given below. (Actually, in the appendix we will see that ϕ_g is just the map (2.3) in disguise; but this will not be used in the proof.)

Next, for any prime ideal Q containing P, let $\bar{Q}$ be its image in S and let $\phi_{g,Q}$ be the natural extension of ϕ_g to the localization at $\bar{Q}$:

$$\phi_{g,Q}: \frac{(gR[X]_Q : PR[X]_Q)}{gR[X]_Q} = \frac{(g):P}{(g)} \otimes_S S_{\bar{Q}} \to L.$$

$\phi_{g,Q}$ is also given by (3.9), *mutatis mutandis;* and it is easily checked that

LEMMA (3.12). *If* $PR[X]_Q = gR[X]_Q$ *then* $JS_{\bar{Q}} = J_{S_{\bar{Q}}/R} = \overline{g_X}\, S_{\bar{Q}}$ *and the image of* $\phi_{g,Q}$ *is* $(S_{\bar{Q}} : JS_{\bar{Q}}) = (S:J)_{\bar{Q}}$.

The second fact (equivalent to (A) of Section 2) whose proof is given below is:

LEMMA (3.13). *If S is normal, then the S-module $((g):P)/(g)$ is reflexive.*

From this we obtain:

COROLLARY (3.14). *Suppose that S is normal, and that for every height one prime q in S, the local ring $R_{q\cap R}$ is regular. Then $\phi_g:((g):P)/(g)\to S:J$ is an isomorphism.*

Proof of (3.14). We have already seen in (3.10) that ϕ_g is injective. Because of (3.13), it follows from [4, p. 52, Prop. 7] that every associated prime ideal of the cokernel of ϕ_g has height less than or equal to 1, so that ϕ_g is surjective if and only if all its localizations at height one primes of S are surjective. But if q is such a height one prime, then, replacing R by $R_{q\cap R}$ in (3.12), and S by $S\otimes_R R_{q\cap R}$, and choosing Q so that $\bar{Q}=q$, we find that the localization of ϕ_g at q is indeed surjective. (The hypothesis of (3.12) is satisfied for some q because $R_{q\cap R}$ and S_q are both regular local rings.)

Finally, in view of (3.11) and (3.14), Theorem 2 will be proved if we show, for some representation $\bar{S}=R[Y_1,\ldots,Y_m]/P'$ of the integral closure $\bar{S}$ of S, and some regular sequence $g'=(g'_1,\ldots,g'_m)$ in P' with $g'_Y\notin P'$, that

$$\operatorname{Im}\phi_{g'}\,(=\bar{S}:J_{S/R})\subset\operatorname{Im}\phi_g\,(\subset S:J_{S/R}).$$

We assume here that $\bar{S}$ is a finite S-module (cf. remarks preceding Theorem 2 in Section 1). It will be enough, then, to proceed "one generator at a time," as follows:

Let $S=R[X]/P$ be as before. Let $S^*=S[y]\subset L$ be an integral extension of S, generated by a single element y. Extend the natural map $R[X]\to S$ to an epimorphism $f^*:R[X][Y]\to S^*$ ($Y=$ one indeterminate) with $f^*(Y)=y$. Then the kernel P^* of f^* contains an element of the form $\alpha Y-\beta$ ($\alpha,\beta\in R[X]$, $\alpha\notin P$) and also a monic polynomial

$$h_{n+1}=Y^m+r_1Y^{m-1}+\ldots+r_m\qquad(r_i\in R[X];\ m>1).$$

If $g=(g_1,\ldots,g_n)$ is, as above, a regular sequence in P with $g_X\notin P$, then clearly the sequence $h=(g_1,\ldots,g_n,h_{n+1})$ is regular, and the Jacobian $h_{X,Y}$ satisfies $h_{X,Y}=g_X(\partial h_{n+1}/\partial Y)$. For any v in $R[X,Y]$, write $\bar{v}=f^*(v)$. Replacing h_{n+1} by $h_{n+1}+\alpha Y-\beta$ if necessary, we may assume that the polynomial

$$\bar{h}_{n+1}=(Y^m+\bar{r}_1Y^{m-1}+\ldots+\bar{r}_m)\in L[Y]$$

is not divisible by $(\bar{\alpha}Y-\bar{\beta})^2$; then it follows that $\partial h_{n+1}/\partial Y\notin P^*$, and hence that $\overline{h_{X,Y}}\neq 0$. So, denoting by (h) the ideal $hR[X,Y]$ we may define as above a map $\phi_h:((h):P^*)/(h)\to L$ by $\phi_h(v+(h))=\bar{v}/\overline{h_{X,Y}}=\bar{v}/\overline{g_X(\partial h_{n+1}/\partial Y)}$. To complete the proof, we will show:

LEMMA (3.15). $\operatorname{Im}\phi_h\subset\operatorname{Im}\phi_g$. *In other words, for every $v\in R[X,Y]$ with $vP^*\subset(h)$, there is a $u\in R[X]$ with $uP\subset(g)$ such that*

$$v\equiv u(\partial h_{n+1}/\partial Y)\ (\text{mod. } P^*).$$

It remains to prove (3.11), (3.13), and (3.15).

III. *Proofs of the Lemmas.*

Proof of (3.11). Let $h = (h_1,\ldots,h_n)$ be a regular sequence of elements of P with $h_X \neq 0$. We are claiming that $\operatorname{Im}\phi_g = \operatorname{Im}\phi_h$, i.e., $\operatorname{Im}\phi_{g,Q} = \operatorname{Im}\phi_{h,Q}$ for every prime $Q \supseteq P$, cf. remarks following (3.11); so we may replace $R[X]$ by $R' = R_Q$. (We do this, following a suggestion of E. Kunz, to be able to permute elements in a regular sequence without destroying regularity.)

Define a "distance" between two sequences $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$ in a commutative ring by

$$\rho(x,y) = \min\{s : xE_1 = (x'_1,\ldots,x'_s,w_{s+1},\ldots,w_n) \text{ and } yE_2 = (y'_1,\ldots,y'_s,w_{s+1},\ldots,w_n) \text{ for some invertible matrices } E_1, E_2\}.$$

We will prove the claim by induction on $\rho(g,h)$. The claim is easily checked when $\rho(g,h) = 0$. Hence when $\rho(g,h) = 1$, we may assume that

$$g = (g_1,w_2,\ldots,w_n) \qquad h = (h_1,w_2,\ldots,w_n).$$

For $u \in (g):P$ write $uh_1 = vg_1 + \sum_{i=2}^{n} \lambda_i w_i$. It is easy to see that $u + (g) \to v + (h)$ is a well defined isomorphism $((g):P)/(g) \xrightarrow{\sim} ((h):P)/(h)$; and that $\bar{u}\overline{h_X} = \bar{v}\overline{g_X}$. Consequently $\operatorname{Im}\phi_g = \operatorname{Im}\phi_h$.

Now for the inductive step let $\rho(g,h) = m > 1$.

As before, without loss of generality we may assume that $g_i = h_i$ for $i > m$. Since the ideal $(h) = (h_1,I)$ with $I = (h_2,\ldots,h_n)R'$ has height n we can find, as in the proof of (3.8), an $h'_1 = h_1 + \sum_{i=2}^{n} \mu_i h_i$, such that for all $i = 1, 2, \ldots, m$, h'_1 does not belong to any associated prime of the ideal $(g_1,\ldots,g_{i-1},g_{i+1},\ldots,g_n)R'$ (all such primes being of height $n-1$); in other words all the sequences $g^*_{(i)} = (g_1,\ldots,g_{i-1},h'_1,g_{i+1},\ldots,g_n)$ $(1 \leq i \leq m)$ are regular. Since $(g)R'_P = PR'_P$, we can find an element $u \in (g):P$, $u \notin P$, and then we can write $uh'_1 = \sum_{i=1}^{n} \lambda_i g_i$.

Now some λ_i with $i \leq m$ is not in P, since otherwise we would have

$$uh'_1 \in P^2 + (g_{m+1},\ldots,g_n)R' = P^2 + (h_{m+1},\ldots,h_n)R'$$

contradicting the fact that $(h'_1,h_2,\ldots,h_n)$ is a minimal basis of PR'_P. If, say, $\lambda_i \notin P$, then with $g^* = g^*_{(i)}$ as above, we have $g^*R'_P = gR'_P = PR'_P$ so ϕ_{g^*} is well-defined; and since $\rho(g,g^*) \leq 1$ and $\rho(h,g^*) < m$, the induction hypothesis gives

$$\operatorname{Im}\phi_g = \operatorname{Im}\phi_{g^*} = \operatorname{Im}\phi_h.$$

Proof of (3.13). Since R' is locally Cohen Macaulay and the sequence g is regular, therefore the ring $A = R'/(g)$ is also locally Cohen Macaulay; and $p = P/(g)$ is a minimal prime ideal in A. There is an obvious isomorphism

$$\frac{(g):P}{(g)} \cong \mathrm{Hom}_A(A/p,A).$$

So (3.13) is given by the following Lemma:

LEMMA (3.16). *Let A be a noetherian ring satisfying the Serre condition* (S_2) (i.e., *if $a = 0$ or if a is not a zero divisor in A, then the ideal aA has no embedded prime divisors). If p is a minimal prime ideal in A such that A/p is a normal domain, then the A/p-module* $\mathrm{Hom}_A(A/p,A)$ *is reflexive.*

Proof. Let T be the total ring of fractions of A. Let $\mathscr{P}$ be the set of height one prime ideals in A. Since A satisfies (S_2), it follows easily that the natural sequence $0 \to A \to T \to \bigoplus_{q \in \mathscr{P}} (T/A)_q$ is *exact.* Hence so is the derived sequence

$$0 \to \mathrm{Hom}_A(A/p,A) \to \mathrm{Hom}_A(A/p,T) \to \mathrm{Hom}_A(A/p, \oplus (T/A)_q).$$

Now $\mathrm{Hom}_A(A/p,T)$ can be identified with the set of elements in T annihilating the maximal ideal pT, and so $\mathrm{Hom}_A(A/p,T)$ is a finite-dimensional vector space over T/pT, which is the fraction field of A/p. Moreover, it is easily checked that every associated prime of the A/p-module $\mathrm{Hom}_A(A/p, \oplus (T/A)_q)$ is of the form q/p $(p \subset q \in \mathscr{P})$, and so has height 1. The conclusion follows then from [4, p. 50, Théorème 2].

Proof of (3.15). It will be convenient to prove a slightly more general fact:

LEMMA (3.17). *Let T be a commutative ring, Y an indeterminate, and P^* an ideal in $T[Y]$ such that P^* contains a monic polynomial h, of degree say m, and also P^* contains an element $\alpha Y - \beta$ where $\alpha, \beta, \in T$ and $P^* : (\alpha) = P^*$. Let $G \subseteq P^* \cap T$ be a T-ideal. Then for every $v \in T[Y]$ such that $vP^* \subset (h,G)T[Y]$ there is a $u \in T$ such that $u(P^* \cap T) \subset G$ and such that*

$$v \equiv u h_Y \pmod{P^*} \qquad (h_Y = \partial h / \partial Y).$$

Proof. We may replace T by T/G; in other words, we may assume that $G = (0)$. We have

$$v = hw + v_1 Y^{m-1} + v_2 Y^{m-2} + \ldots + v_m \qquad (w \in T[Y], v_i \in T)$$

and since $v(P^* \cap T) \subset vP^* \subset (h,G)T[Y] = hT[Y]$ we see that

$$v_i(P^* \cap T) = (0) \qquad 1 \leq i \leq m-1. \tag{3.17.1}$$

Moreover, $v(\alpha Y - \beta) \in vP^* \subset hT[Y]$ whence

$$(v - hw)(\alpha Y - \beta) = v_1 \alpha h. \tag{3.17.2}$$

Differentiating (3.17.2) with respect to Y, we get $(v - hw)\alpha \equiv v_1 \alpha h_Y \pmod{P^*}$. Since $P^* : (\alpha) = P^*$ and $h \in P^*$, we have $v \equiv v_1 h_Y \pmod{P^*}$ which, by (3.17.1), gives the desired conclusion.

This completes the proof of Theorem 2.

APPENDIX. CONNECTING SECTION 3 TO SECTION 2

LEMMA (A.1). *With the notation of the second proof, there is an isomorphism* η_g *making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Hom}_{R[X]}(S, R[X]/(g)) = \dfrac{(g):P}{(g)} & \xrightarrow{\eta_g} & \mathrm{Ext}^n_{R[X]}(S, \Omega^n) \\ \Big\downarrow{\scriptstyle \phi_g} & & \Big\downarrow{\scriptstyle (2.3)} \\ L & \xleftarrow[\text{inclusion}]{} & S:J \end{array}$$

(Note, by the way, that (A.1) gives another proof of (3.11).)

Once we have proved (A.1), then (3.13) gives (A) of Section 2; and to prove (B) of Section 2, the key point is:

Observation (A.2). With notation as in Lemma (3.15), the element u is uniquely determined mod. (g) by v (cf., (3.10)); and if $v \in (h)$ then we can take $u = 0$; so that in fact we have a map

$$\psi_{h,g}: \frac{(h):P^*}{(h)} \to \frac{(g):P}{(g)}$$

such that $\phi_h = \phi_g \circ \psi_{h,g}$; and therefore (via η_h and η_g) we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^{n+1}_{R[X,Y]}(S^*, \Omega^{n+1}_{R[X,Y]/R}) & \xrightarrow{(2.3)} & S^*:J_{S^*/R} \\ \Big\downarrow{\scriptstyle \psi'} & & \Big\downarrow{\scriptstyle \text{inclusion}} \\ & & L \\ & & \Big\uparrow{\scriptstyle \text{inclusion}} \\ \mathrm{Ext}^n_{R[X]}(S, \Omega^n_{R[X]/R}) & \longrightarrow & S:J \end{array}$$

From ψ' one gets the map ψ in (B) (Section 2), with S^* in place of $\tilde{S}$. A proof that ψ is an isomorphism (using, for example, the description of $\psi_{h,g}$ provided by the proof of (3.15)) is left to the interested reader.

Proof of (A.1). Let A be any commutative ring, and $g = (g_1, \ldots, g_n)$ a regular sequence in A. Then the sequence consisting of the Koszul complex K^g_A over A determined by g, augmented by the natural surjective map $A \to A/(g)$:

$$(*) \qquad 0 \to A = K_n \to K_{n-1} \to \ldots \to K_0 = A \to A/(g) \to 0$$

is *exact,* and hence determines an element $\xi \in \mathrm{Ext}^n_A(A/(g), A)$. (One may think of ξ as an equivalence class of exact sequences; or one may use K^g_A as a projective

resolution of $A/(g)$, and then ξ is the homology class of the n-cocycle in $\mathrm{Hom}_A(K_A^g, A)$ given by the *identity* map: $A = K_n \to A$.) For any $A/(g)$-module M, and any $\theta \in \mathrm{Hom}_A(M, A/(g))$ let $\theta^* : \mathrm{Ext}_A^n(A/(g), A) \to \mathrm{Ext}_A^n(M, A)$ be the corresponding map. Then we have an A-homomorphism $\eta_M : \mathrm{Hom}_A(M, A/(g)) \to \mathrm{Ext}_A^n(M, A)$ given by $\eta_M(\theta) = \theta^*(\xi)$ $(\theta \in \mathrm{Hom}_A(M, A/(g)))$.

(A.1) is a direct consequence (details left to the reader!) of the following more or less well-known lemma:

LEMMA (A.2). (i) *η_M is an isomorphism.*

(ii) *Let P be an ideal in A containing the regular sequence g, let $B = A/P$, and let $\pi : A/(g) \to B$ be the natural map. Let $\phi : \mathrm{Ext}_A^n(B, A) \to \mathrm{Hom}_A(\Lambda^n(P/P^2), B)$ be the fundamental local homomorphism* (cf. Section 2). *For any $\theta \in \mathrm{Hom}_A(B, A/(g))$, set $\tilde{\theta} = \phi\eta_B(\theta)$; and for any $g \in P$, let $\bar{g}$ be its natural image in P/P^2. Then*

$$\tilde{\theta}(\bar{g}_1 \wedge \bar{g}_2 \wedge \ldots \wedge \bar{g}_n) = \pi\theta(1).$$

Proof. (i). It is easily seen that the map η_M is the iterated connecting homomorphism associated with (*). In other words, (*) breaks up into short exact sequences $0 \to L_{i+1} \to K_i \to L_i \to 0$ $(0 \le i < n)$ where $L_n = K_n = A$ and $L_0 = A/(g)$; and η_M is the composition of the corresponding connecting homomorphisms

$$\mathrm{Ext}_A^i(M, L_i) \xrightarrow{\delta_i} \mathrm{Ext}_A^{i+1}(M, L_{i+1}).$$

Now, by induction on n, we have an isomorphism

$$\mathrm{Ext}_A^i(M, A) \cong \mathrm{Hom}_A(M, A/(g_1, \ldots, g_i)A) = (0) \quad (i < n)$$

where the vanishing is due to the fact that $g_{i+1}M = (0)$ and

$$((g_1, \ldots, g_i)A) : g_{i+1} = (g_1, \ldots, g_i)A.$$

Since each K_i is a free A-module, we have

$$\mathrm{Ext}_A^i(M, K_i) = (0) \qquad (i < n), \qquad \mathrm{Ext}_A^{i+1}(M, K_i) = (0) \qquad (i < n - 1)$$

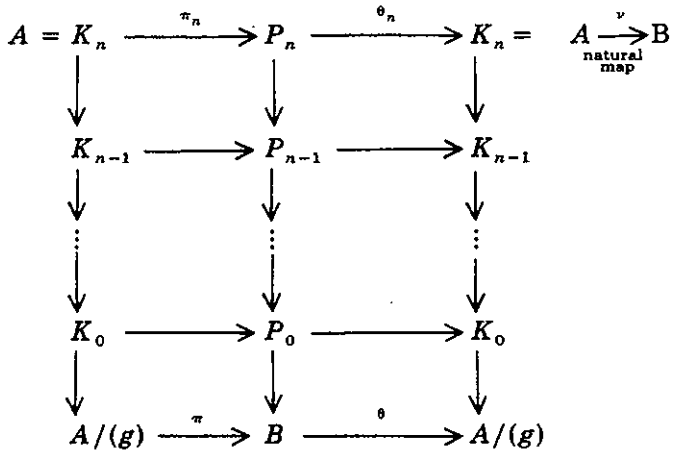
so that the kernel of δ_i vanishes for $i < n$, and the cokernel vanishes for $i < n - 1$. Thus we are left with an exact sequence

$$0 \to \mathrm{Hom}_A(M, A/(g)) \xrightarrow{\eta_M} \mathrm{Ext}_A^n(M, A) \xrightarrow{\sigma} \mathrm{Ext}_A^n(M, K_{n-1}).$$

But σ, which comes from the boundary map $(K_n \to K_{n-1}) \in g\,\mathrm{Hom}_A(K_n, K_{n-1})$ is the *zero-map*, because $gM = (0)$; so η_M is an isomorphism.

(ii). Let $\ldots \to P_n \to P_{n-1} \to \ldots \to P_0 \to B \to 0$ be an A-projective resolution of B. Lifting the maps θ and π, we get a commutative diagram

(**)



By the definitions,

$$\tilde{\theta}(\bar{g}_1 \wedge \ldots \wedge \bar{g}_n) = \nu\theta_n\pi_n(1). \tag{A.2.1}$$

Now the composition of the maps in (**) is an A-homomorphism $K_A^e \to K_A^e$ lifting the map $\theta\pi$. But so is multiplication by α, where α is any element in A such that $\alpha + (g) = \theta(1)$. By homotopy uniqueness of liftings, $\theta_n\pi_n(1) \equiv \alpha \pmod{(g)}$ whence

$$\nu\theta_n\pi_n(1) = \nu(\alpha) = \pi\theta(1). \tag{A.2.2}$$

(A.2.1) and (A.2.2) give the desired conclusion.

4. REMARKS ON THE CASE $d = 0$ OF THEOREM 1

Recall from the discussion preceding Theorem 1″ in Section 1 that in proving Theorem 1 we may assume both R and R/I_0 to be regular local rings. So part (i) of the following proposition contains the case $d = 0$ of Theorem 1. (We are grateful to W. Heinzer for some illuminating discussion about this result). A special case of part (ii) was proved by Viehweg [18, Proposition 2].

PROPOSITION. (i) *Let R be a commutative noetherian ring, let $(x_1,\ldots,x_r)$ be a regular sequence in R such that $R/(x_1,\ldots,x_r)R$ is normal, and let $y \in R$. Set $I_0 = (x_1,\ldots,x_r)R$ and $I = I_0 + yR$. Then all the powers I^λ $(\lambda \geq 1)$ are unmixed and integrally closed in R* (i.e., $\overline{I^\lambda} = I^\lambda$).

(ii) *Assume furthermore that R and R/I_0 are both regular. Let X be the scheme obtained from* Spec(R) *by blowing up I. Then all the singularities of X are pseudo-rational, of multiplicity two.*

Proof of (i). I is unmixed because R/I_0 is normal. For $\lambda > 1$, assume inductively that $I^{\lambda-1}$ is unmixed, i.e., every associated prime of $I^{\lambda-1}$ is a minimal prime of I. If $c \in R$ is such that its image in R/I is not a zero-divisor, and $x \in R$ is such that $cx \in I^\lambda$, then $x \in I^{\lambda-1}$ (since $cx \in I^{\lambda-1}$ and $I^{\lambda-1}$ is unmixed). But I is

generated by a regular sequence, so $\bigoplus_{\lambda \geq 1} (I^{\lambda-1}/I^{\lambda})$ is polynomial ring over R/I; it follows at once that $x \in I^{\lambda}$. Thus I^{λ} is unmixed.

Now to test the inclusion $\overline{I^{\lambda}} \subset I^{\lambda}$ we can localize at the associated primes of I^{λ}, i.e., at the minimal primes of I. For such a prime p, R_p/I_0R_p is normal of dimension less than or equal to 1, hence regular; and therefore R_p is regular. By Theorem 1 (or, without much difficulty, directly) we see then that all the powers $I^{\lambda}R_p$ are indeed integrally closed.

Proof of (ii). First of all, by (i), $X = \operatorname{Proj}\left(\bigoplus_{n\geq 0} I^n\right) = \operatorname{Proj}\left(\bigoplus_{n\geq 0} \overline{I^n}\right)$ is *normal.* Also, the local rings of points on X are analytically unramified [17, Theorem 1.6].

We assume that $y \notin I_0$, since otherwise X is regular. Thus $(x_1,\ldots,x_r,y)$ is a regular sequence. To examine the singularities of X, we must look at localizations at prime ideals P in rings of one of two types:

(a) $R' = R[x_1/y,\ldots,x_r/y]$

(b) $R' = R[y/x_i,x_1/x_i,\ldots,x_r/x_i] \qquad (1 \leq i \leq r)$.

Furthermore we may assume that R is local, with maximal ideal $m = P \cap R \supset I$, because e.g. R'_P is a localization of $R_{\phi^{-1}(P)}[x_1/y,\ldots,x_r/y]$ (where $\phi: R \to R'$ is the natural map), and if $I \not\subset \phi^{-1}(P)$, then R'_P is a localization of the regular ring R.

We first consider case (a). If $y \in m - m^2$, then y can be interchanged with one of the x_i, so that R' is of the type to be considered below in case (b). So we assume that $y \in m^2$. In this case, R'_P is actually *regular* for any prime $P \supset m$.

The proof is by induction on r. For $r = 1$, since (x_1,y) is a regular sequence, we have $R[x_1/y] = R[T]/yT - x_1$ (T-indeterminate) (cf. [16, Lemma 2.1]). If M is any maximal ideal in $R[T]$ containing m, then $(yT - x_1) \notin M^2$, since otherwise $x_1 \in M^2$, contradicting the fact that $R[T]/x_1 = (R/x_1)[T]$ is regular; hence $R[x_1/y]_P$ is regular for any prime $P \supset m$.

For $r > 1$, set $R^* = R[x_r/y]$, $P^* = P \cap R^*$. As we have just seen, $R^*_{P^*}$ is regular; and moreover, $(x_1,\ldots,x_{r-1})$ is part of a regular system of parameters in $R^*_{P^*}$, since for $1 \leq i \leq r-1$, $R^*_i = R^*_{P^*}/(x_1,\ldots,x_i)$ is a localization of

$$R/(x_1,\ldots,x_i)[T]/\bar{y}T - \bar{x}_r$$

(where "$-$" denotes "natural image in $R/(x_1,\ldots,x_i)$)," and so the case $r = 1$ applied to the regular local ring $R/(x_1,\ldots,x_i)$ gives the regularity of R^*_i (and note that $0 \neq \bar{x}_{i+1}$ = image of x_{i+1} in R^*_i). Now R'_P is a localization of $R^*_{P^*}[x_1/y,\ldots,x_{r-1}/y]$ at a prime ideal containing the maximal ideal of $R^*_{P^*}$, so by induction we conclude that R'_P is indeed regular.

Next we consider case (b). The ring $R^* = R[x_1/x_i,\ldots,x_r/x_i]$ is regular; and so is R^*/x_i, which is a polynomial ring in $r-1$ variables over the regular ring R/I_0. So we are reduced to the case $r = 1$. As before

$$R' = R[y/x_1] = R[T]/(x_1T - y)$$

and so the Jacobian ideal $J' = J_{R'/R}$ is $x_1 R'$. Now R' is normal, and so (2.3) (with $S = R'$) is an isomorphism. It follows that if, as in [13, Section 4], we take $\omega_R = R$, then if R_0 is any localization of R', for the R_0-module ω_{R_0} we should take $\omega_{R_0} = R_0 : J' = x_1^{-1} R_0$. Note that R_0 is a complete intersection, hence Cohen-Macaulay. From the Lemma in Section 1 above, with $t = x_1$, (and again since (2.3) is an isomorphism when S is normal), and from the corollary of (iii) in Section 4 of [13], it follows that R_0 is in fact pseudo-rational (because if $f: Y \to \operatorname{Spec}(R_0)$ is proper and birational, with Y normal, and S is the local ring of a closed point s on Y, then the stalk of ω_Y at s is $S : J_{S/R}$, which contains $x_1^{-1} S = \omega_{R_0} S$).

Finally the multiplicity of R_0 is less than or equal to 2. To see this, it is enough to check, for any maximal ideal M of $R[T]$ containing the maximal ideal m of R, that $x_1 T - y \notin M^3$. But if $x_1 T - y \in M^3$, then applying the derivation $\partial / \partial T$ we get $x_1 \in M^2$, which contradicts the fact that $R[T]/x_1 = (R/x_1)[T]$ is regular.

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